Introduction to Time-Frequency Analysis and Wavelet Transforms
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Lecture - 6.1
Wigner-Ville Distributions

Hello friends, welcome to lecture 6.1. With this lecture we are going to start our journey on Wigner-Ville distributions. In the previous unit we discussed at length the concept of short-time Fourier transform, the spectrogram, its properties, its limitations, its uses and also how you could use a spectrogram for denoising.

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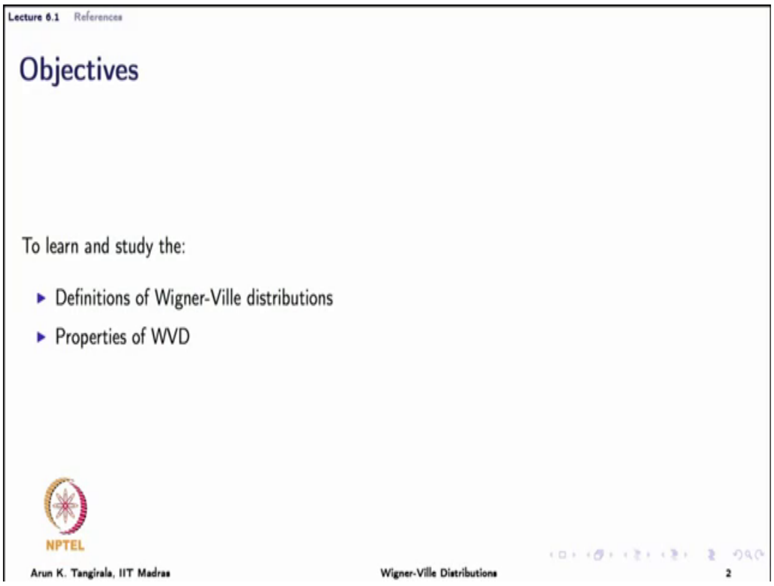

In this unit, we are going to focus on the Wigner-Ville distribution and its variants. And in this lecture what we are going to do is, we are going to look at the definition of Wigner-Ville distributions. Although I say properties here, primarily what I am going to do is illustrate how the Wigner-Ville distribution works out for different classes of signals, and towards the end we will show that the Wigner-Ville distribution results in interference, which is something that we will work on in the next lecture in 6.2.

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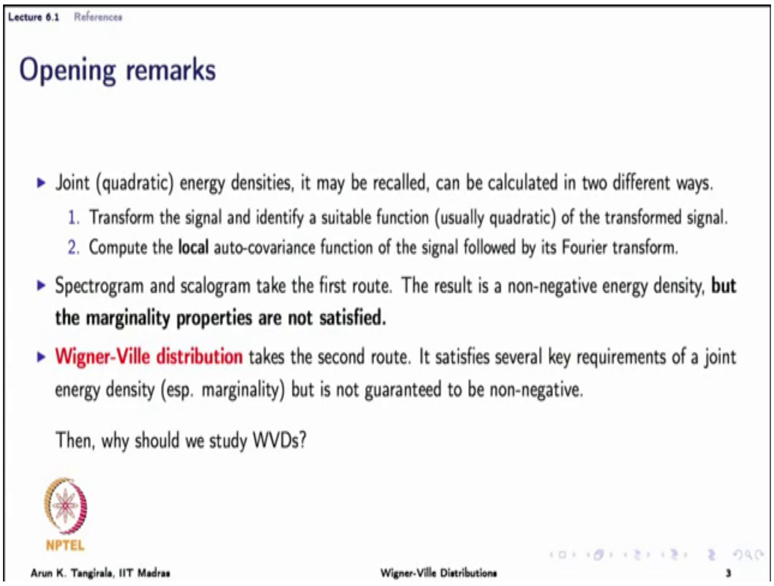

So, let us get going. Before we look at the definition of Wigner-Ville distribution, it is useful to recall that there are fundamentally two ways of calculating the energy density. Whether it is a joint energy density or just the energy spectral density that we calculate using Fourier transform, there are two routes. The first route being transform the signal, and then identify some kind of functional operation, typically quadratic magnitude square to get the energy density. Of course, justification has to be provided why that functional will return, result in energy density. The second route is to compute the auto-covariance functions if you are looking at purely energy spectral density. But if you are looking at joint energy density, we are interested in computing the local auto-covariance function, and then take a Fourier transform of the same, which is essentially an extension of the ((Refer Time: 02:16)) to the time-frequency plane.

Now, as we have noted on several occasions, spectrogram and scalogram belong to the first category. In spectrogram, you take the short-time Fourier transform and then compute the squared magnitude. And in a scalogram, the story is the same, compute the wavelet transform and then take this squared magnitude of the wavelet transform.

The good thing about adapting this strategy for constructing the joint energy density is that you have a non-negative energy density, which is one of the key requirements that a joint energy density should satisfy to make some sense. However we lose out on the marginality properties and we have mentioned this even in the case of spectrogram. And early on when we talked about the basis of time-frequency analysis owing to Wigner's result who showed that there exists no positive quadratic distribution that can satisfy the marginality property.

So, what about Wigner-Ville distributions? Wigner-Ville distributions takes a second role and this again we have noted earlier as well, that is it computes a local auto-covariance functions and then takes a Fourier transform. As a result, by virtual of Wigner's own result it satisfy several key requirements, especially marginality, but unfortunately it is not guaranteed to be non-negative for all signals in the entire time-frequency plane. For a specific class of signals, may be the WVD can turn out to be non-negative, but not necessarily for all signals. So, why should we really study Wigner-Ville distributions is a question if it is point to be non-negative.

Well, what happens is, although it does not guarantee a non-negative energy density or distribution, the good thing about it is, it serves as a prototype for several other distributions that came along. The moment Wigner-Ville distribution came about in a, in the limelight, several other distributions as variance of Wigner-Ville distribution in an attempt to improve its properties or retain its properties have come about and the unification of spectrogram, scalogram and the Wigner-Ville distribution came about. So, that is a fantastic result. Therefore, it is good for pedagogical reasons at least to start off with Wigner-Ville distribution and then gradually see how we can make improvements to this. And one path of improvement leads to spectrogram. The other path of improvements leads to scalogram that is the idea.

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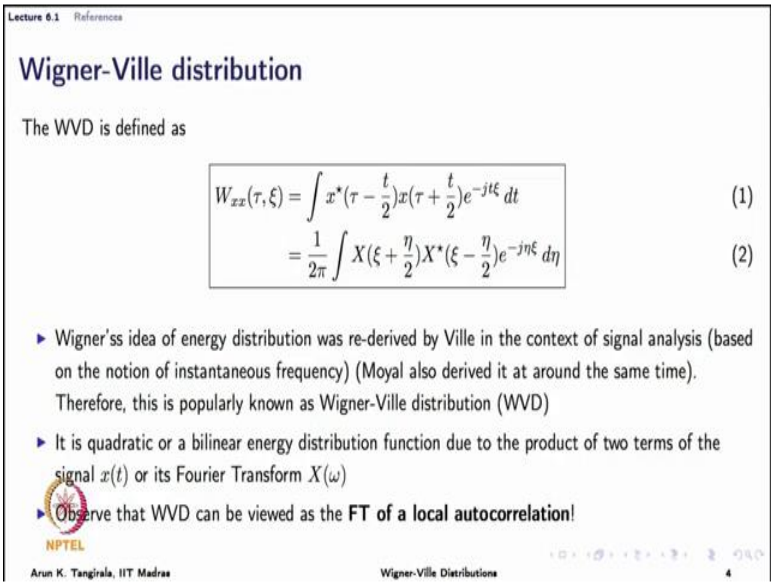

Now, let us move on to the definition of Wigner-Ville distribution. A quick historical background, Wigner conceived this distribution in about, in around 1930s in the context of quantum thermodynamics. And Ville, who was interested in signal processing and in signal analysis, about 15 years later, roughly rediscovered this in a context of signal analysis where he was interested in estimating the instantaneous frequencies from the joint energy density.

In fact, we will see later on, that this Wigner-Ville distribution gives us a nice estimate of instantaneous frequency in a sense, that the average frequency that you compute from the Wigner-Ville distribution is nothing but the instantaneous frequency itself, whereas for the spectrogram and scalogram you would have to compute ridges. So, now you understand why the name Wigner-Ville, although their discoveries for 15 inventions are 15 years apart.

So, look at equation 1, it gives you the definition of the Wigner-Ville distribution. It is a single integral. Unlike the short-time Fourier transform, here the integral directly gives me the energy density. Again, compare in spectrogram, you will have to compute the short-time Fourier transform and then take the squared magnitude. Here you do not, do you, you compute the energy density in a single step. What you are doing here is, you have a product of x of tau plus t by 2 times x star of tau minus t by 2 and Fourier transform of that is being evaluated, right.

And what I give you in equation 2 is the frequency domain version of the same results. So, you can compute Wigner-Ville distribution, either in time domain if you know the time domain expression or if you know the frequency domain expression, you can compute that in frequency domain. Again, this is in a theoretical definition. Later on, we look at the practical aspects of Wigner-Ville where I will give you expression for computing Wigner-Ville based on sample data.

Now, what is happening here, that is, that is very important to understand rather than just looking at integral or getting intimidated by it. First point to observe is, that this is a quadratic or bilinear energy distribution because you are involving a product of signal with itself. So, there is a product of two terms x of tau plus t by 2 times x star of tow minus t by 2 and I use a subscript x, x to indicate that you are evaluating this energy density or auto energy auto joint energy density for the signal. You could therefore, conceive of a cross joint energy density as well. So, that is the first thing.

And secondly, that which is very important is that Wigner-Ville distribution is a Fourier transform of a local auto-covariance or autocorrelations. If you talk to a signal processing person, you would like to call this as autocorrelation. Strictly speaking, by our definition this is actually a local auto-covariance. Look at the similarity of this expression here with the definition of auto-covariance that we had for the energy signals, aperiodic energy signals. The only difference is, this is the local one, instantaneous; you can think of an instantaneous auto-covariance also.

So, let us understand what is happening here. Physically, when I am computing the Wigner-Ville distribution, I need to understand what is happening like in the short-time Fourier transform. I know what I am doing. I am essentially breaking up the signals into a bunch of sine waves and determining how much each sine wave is present in the signal and at what time.

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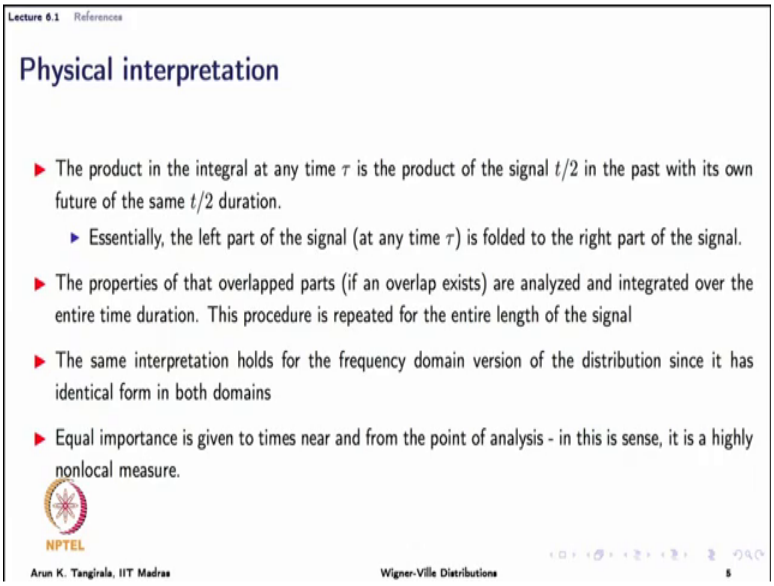

Here what we are doing is, if you look at a product in the integral what we are doing is, at any point in time I am looking to the left and to right of the signal. Assume, that the signal is real, therefore x star is nothing but x itself. So, for real valued signals, I am looking to the left and right of the signal over the same duration and then folding, that is, I am actually computing what is the extent of overlap between these left and right segments. And I am not just looking at one segment, I am looking at all possible segments with respect to tau. So, let me show this to you or illustrate this to you on the board.

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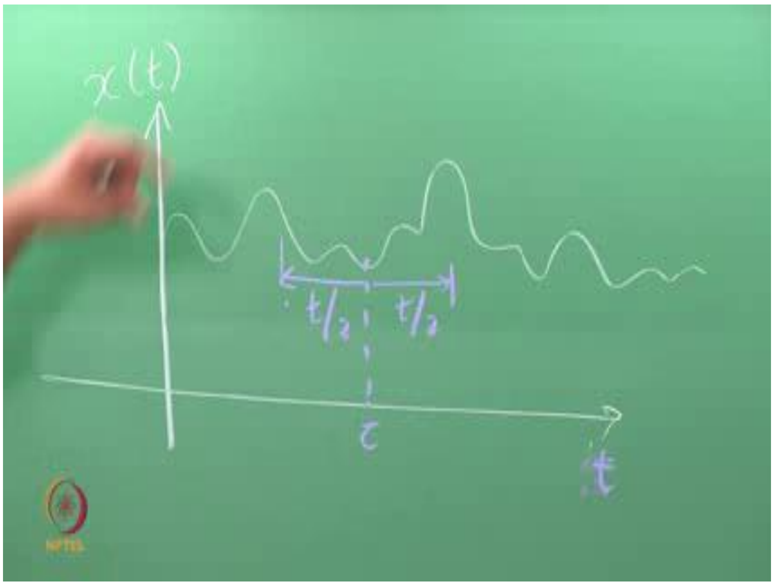

Let us take the signal here. So, let us say I have some signal, need not be periodic. What is happening in Wigner-Ville is, that let us say, I am standing at tau, this is the tau and looking to the left, let us say this length is t by 2 and I look to the right or for the same length. This is t by 2 and at this point. So, you can say that our dummy variable right now is, of course, t.

So, I am standing at time tau and I am looking to the left and right and asking, what is the extent of overlap between this segment here and this segment here. If there is any similarity, it will show up, even though that property may not be holding this tow. So, that is one of the drawbacks of the Wigner-Ville distribution as we will see later on, which does not give it the finite support property and so on. But that is the point you are essentially looking at, similarities between the segments to the left and right and over the entire duration. Of course, that entire duration makes it also non-local. Look at the integral. So, we have this integral. If you look at it carefully you realize, that it is essentially looking at all possible segments and it is giving uniform weighting to all the segments. That should not be the case if I am standing at tau. I should give more importance to the local properties. If I truly want to get the local properties at tau, I should give more importance to the segments here and less importance to the segments that are far away from the tau. But the original Wigner-Ville distribution did not do that.

It essentially, gave uniform importance to uniform weighting to all the segments and that kind of spoils some of the properties of Wigner-Ville distribution.

Later on, we will study what is known as a pseudo Wigner-Ville distribution, which applies a window on top of the Wigner-Ville distribution so that the window function will give more importance to the segments close up to tau and less importance to the segments for away from tau. So, that is the physical interpretation that we have and this interpretation can be applied in the frequency domain as well. If you look at the equation 2, the arguments are more are less the same, despite some of these drawbacks it has some beautiful properties as we will learn today and that is in this lecture and also the next lecture.

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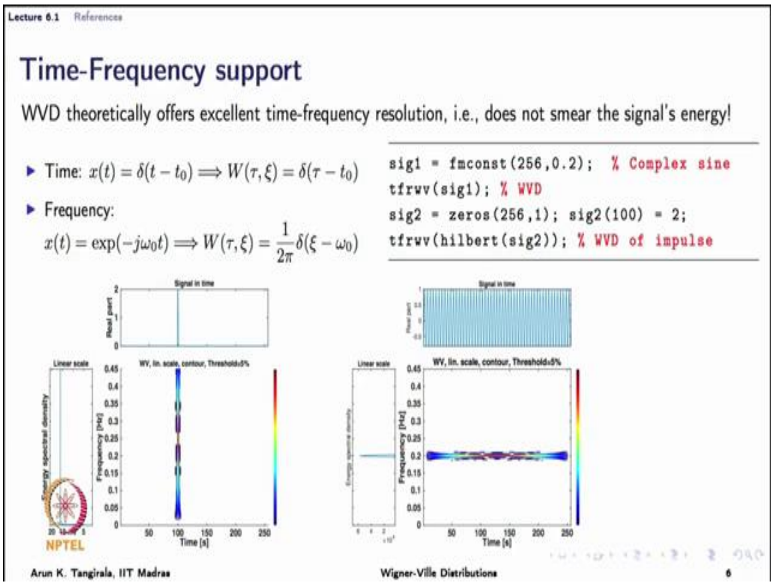

So, let us get a feel of what is happening. Remember, the purpose of studying this distribution is to get the local features of the signal in the time-frequency plane. So, we would like to know what is happening in the time-frequency plane, what cap, what features does it capture, how does it capture, is there a, is there a smearing of the energy like we saw in the spectrogram or is there not.

Now, as I show this to, on the slide, theoretically Wigner-Ville distribution of, as excellent time-frequency support, which means it does not smear the signals energy in time or frequency and you can show this mathematically. If I use this signal here, x of t as a, as a drag, located center at t naught, then the Wigner-Ville is also exactly centered at t naught. It is also an impulse. The most important thing is, not just centered around t naught, at t naught, but that it is also an impulse.

So, an impulse in the signal domain translate, transfers to an impulse in the Wigner-Ville or the joint energy time-frequency plane as well and I show this to you as a way of illustration here, by way of illustration and the MATLAB code is given to you. I am using the time-frequency tool box once again. F m constant generates signal. Of course, here first I am showing the sine wave of the complex exponential, but let us look at the later part of the code and generating an impulse here. And I am computing the Wigner-Ville distribution of the analytic representation of the signal. It is important to actually use analytic representation if you want to use this routine from the time-frequency toolbox.

So, you can see clearly, that WVD has an excellent time localization. Compare this situation with what we had with the spectrogram. This, the time resolution of the spectrogram was limited by the window. Any windowing method will suffer from the drawback, whereas Wigner-Ville distribution does not. In fact, there is something else that learn by going back to the integral. Here, we are not definitely applying any windowing technique to the signal. We are not windowing the signal and then transforming. We are directly computing the energy density.

Now, early on, when we talked about time-frequency analysis and basis functions and so on, we said there are two classes of basis functions, fixed basis and adaptive basis for the purpose of signal analysis. Although Wigner-Ville distribution does not really involve a transform, sometimes it is argued, that there is transform involve of the signal. So, let us assume, let us do a change of variable here. You can set tau plus t pi 2 as some other dummy variable t prime. Then, you can think of this Wigner-Ville distribution as a transform of the x of t plane with a basis e to the minus j t zee times x star of return dummy variable here. So, a basis functions is not just e to the minus j tau zee, but also times this.

Now, this factor in the basis functions comes from the signal and therefore, Wigner-Ville distribution sometimes conceived as a transform of the signal with an adaptive basis. The adaptive term coming from the fact, that the basis is derived from the signal itself unlike in Fourier transforms where the basis is fixed a priory that gives certain advantages to the Wigner-Ville distribution. Whereas, in the case of Fourier transform or wavelet transform or short time Fourier transform, the basis are fixed, and there are some disadvantages. But of course, advantage is, you know the basis a priory, you know the properties. Therefore, once you break up the signal into the difference components, it is easier to interpret them. Whereas, in the Wigner-Ville distribution you do not know the basis a priory, and therefore you have to interpret everything after performing the transform and keeping the signal in mind.

So, the point here is Wigner-Ville distribution offers excellent time resolution compared with the spectrogram. And likewise, in the frequency domain as well I have a complex sine wave here, plug that complex sine wave into the theoretical definition, you will be able to show, that the Wigner-Ville is once again at ((Refer Time: 17:43)) in the frequency domain centered at omega not where omega naught is the frequency of the signal itself.

So, I have on illustration for you of this fact. I have a sine wave. Once again I am using the time-frequency toolbox and I show you to Wigner-Ville distribution, it is highly localized. Think of this, obtaining this with the short-term Fourier transforms. I can do this, but I have to use very wide window. In fact, only may be Fourier transforms will give you such kind of resolution exactly what Fourier transforms gives you theoretically. So, here there is no concept of windowing. I do not have to break my head on determining what is an appropriate window length, and so on. It is all built in.

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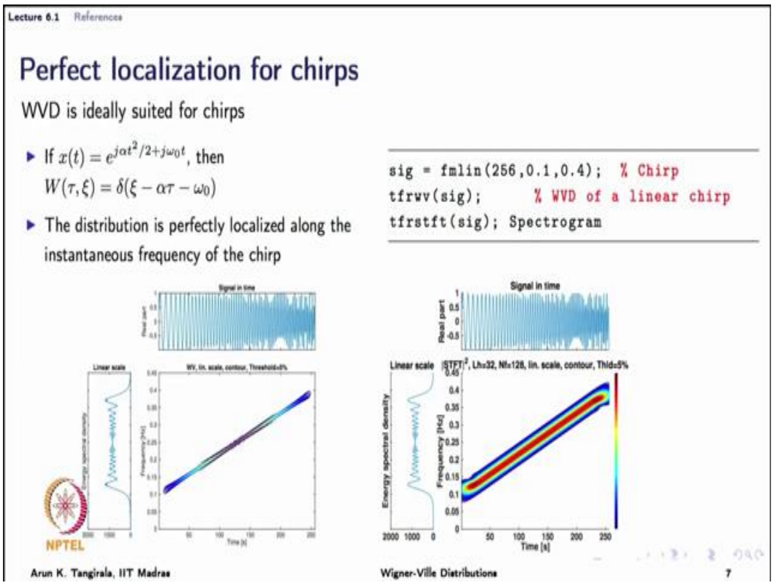

And the main point to remember is, that the Wigner-Ville is ideal for chirps. Once again we can verify this by writing the expression, the theoretical expression for linear chirps. You should be able to see, that is, the linear chirps once again because the phase of the signal is a quadratic function of time. And the Wigner-Ville of this linear chirp is a dirac. In what sense it is a dirac? Not in the regular sense, it is a dirac along the line of an instantaneous frequency. Earlier when I had a sine, it was a dirac in the frequency itself, regular Fourier frequency. Now, it is a dirac in the instantaneous frequency.

Note, that the instantaneous frequency of the signal is beta t, sorry, alpha t plus omega naught. So, that exactly what is showing up here and the distribution is therefore, perfectly localized along the instantaneous frequency. Once again I show you the, I illustrate this fact for you using the time frequency toolbox. The fmlin generates the complex linear chirps and therefore, you do not have to take in a hilbert transform or construct an analytical representation.

I have the Wigner-Ville distribution here. It just looks beautiful and very attractive. It is very nice, right; it is too sweet, in fact. Compare this with what we have for the spectrogram or in fact, I would recommend that also you try this with wavelet, that is, generate a scalogram. Either way, there is a heavy smearing of the energies. In fact, you should construct a scalogram, you should expect that smearing will be higher in the high frequency region because wavelets have poor frequency localization in the high frequency region and good frequency localization in the low frequency region. So, I leave that as an exercise all you have to do is generate the signal and run tfrscalo and you will be able to generate the scalogram great.

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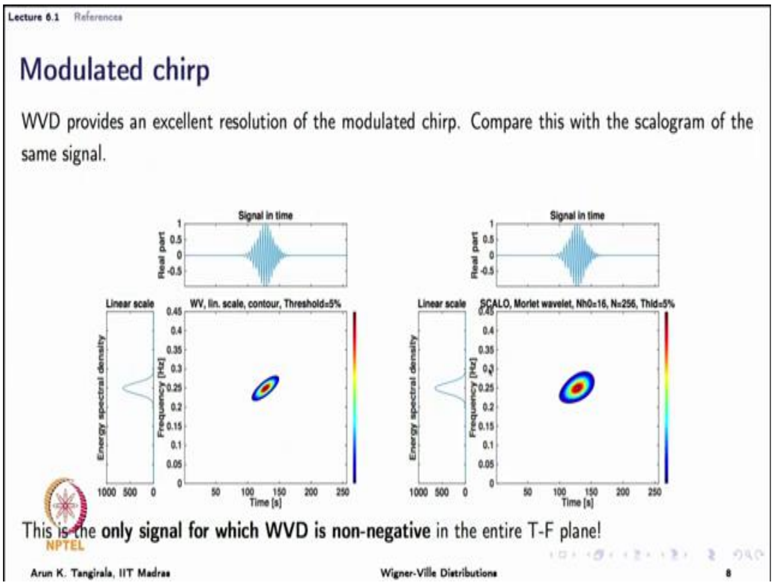

So, now what about a modulated chirp? Again, the modulated chirp is special, first of all because the WVD provides an excellent resolution of the modulated chirp. What we mean by the modulated is amplitude modulated chirps. I have taken this chirp signal, for example, here and multiplied this with a Gaussian wave, Gaussian amplitude modulation that only exits over this period of time as result of which I have this amplitude modulated chirp.

The Wigner-Ville looks very nice because it has perfectly localized the frequency as well as the time. Everything is nicely captured. Compare this with what scalogram computed with the modulated wavelet provides for you. In this scalogram as well there is nothing to clear on as far as windowing is concerned. This spectrogram, at least, you may have to play around with the window. But again, I invite you to compute the spectrogram with different window lengths and see if you can get as fine as an energy density, that Wigner-

Ville gets for you, that I show here right now. The specialty, the second specialty of this modulated chirps within the context of Wigner-Ville is that this is the only signal for which Wigner-Ville is non-negative in the entire time frequency, great.

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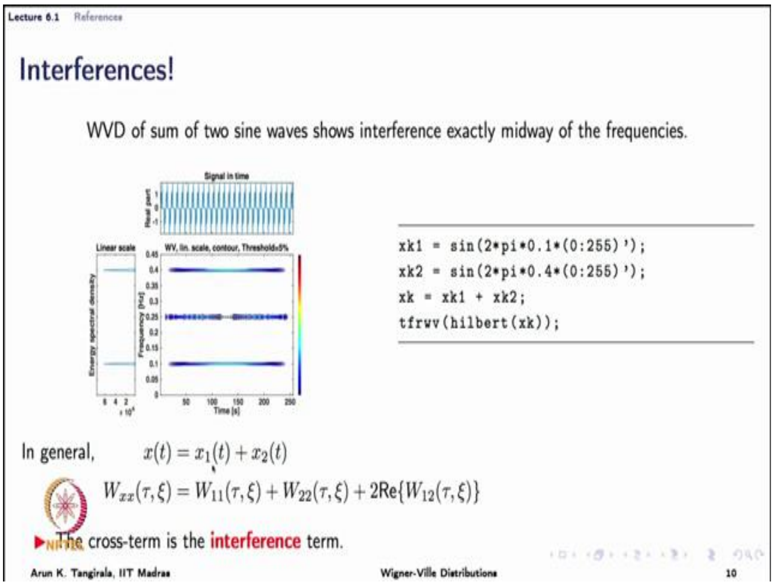

So, we have lot of things to boast about Wigner-Ville, but now you have this beast called interference. So, the Wigner-Ville is sum of two sine waves, shows interferences exactly mid-way of the frequencies. Now, that is both the good and bad, I tell you what is the bad news? The bad news is, of course, the interference. Once again I illustrate this for you first and then we will just quickly talk about the theory and wind up.

So, I have generated two sine waves of frequency 0.1 cycles for sample and 0.2 cycles for sample. I have added them up, computed the Wigner-Ville. You can see this interference stem coming, appearing exactly at 0.25 frequency, sorry, cycles per sample, whereas a spectrum does not contain it. So, it is always useful to plot the spectrum by the side and the signal on the top so that you can make meaningful interpretations from the energy density plot, joint energy density plot.

Now, why does this happen? Because in general, if I have two signals, then the super position of the Wigner-Ville of a, super position of these two signals involves the sum of the Wigner-Ville's respective components and a cross term, which involves the cross Wigner-Ville term. In fact, two times the real part of that. Therefore, Wigner-Ville is always real. Although I have not mentioned that early on, the Wigner-Ville is always real. It does not mean it is non negative, but it is real value.

This term here, the third term, the 2 times real part of the W 12 is the one that causes the interference. The good news is, that I know exactly where the interference occurs. It is not going to occur arbitrary, at arbitrary locations. I will also show you this small animation in, in the coming lectures where we try to remove the interference stamp to show you, that the interference always occurs mid-way, whether in time or in frequency depending on how they add up and that gives me the key to eliminating the, to an idea of how to eliminate the interference stamp. That is the key, alright.

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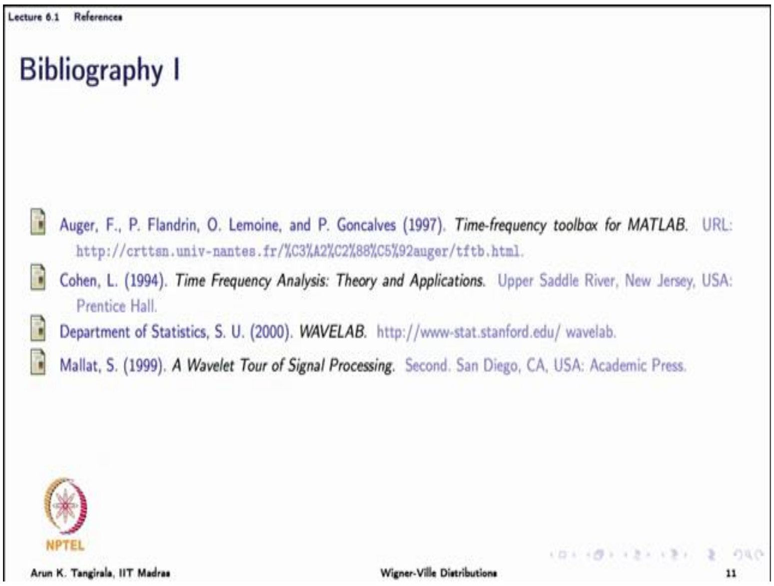

So, with this slide we will draw this lecture to a close, and once again I have a few references for you. These are more or less the same references that you have been seeing. So, what we have learnt is, what is the definition of Wigner-Ville, how it is different philosophically from spectrogram and even scalogram, and that it has certain very nice features. We have not studied thoroughly all the properties. In the next lecture, we will examine the joint energy density for the all the properties that an energy density should have. It has some beautiful properties, it is ideal for chirps and so on, modulated chirps, but there are going to be interferences and we have to see how to get rid of this interferences from the Wigner-Ville distribution. So, hope you enjoyed lecture. We will meet again in lecture 6.2.

Thanks.