Introduction to Time-Frequency Analysis and Wavelet Transforms

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Lecture – 5.1
Short-Time Fourier Transform (STFT)

Hello friends, welcome to the first lecture on the topic of Short Time Fourier Transforms. So, this lecture is numbered 5.1 as per our convention. In this lecture we are going to exclusively talk about the definition of short time Fourier transform. And also look at the reconstruction property of the short time Fourier transform. Primarily will keep this lecture theoretical. But, somewhere during the lecture or towards end of the lecture I will show you, how to implement short time Fourier transform in mat lab using the time frequency tool box.

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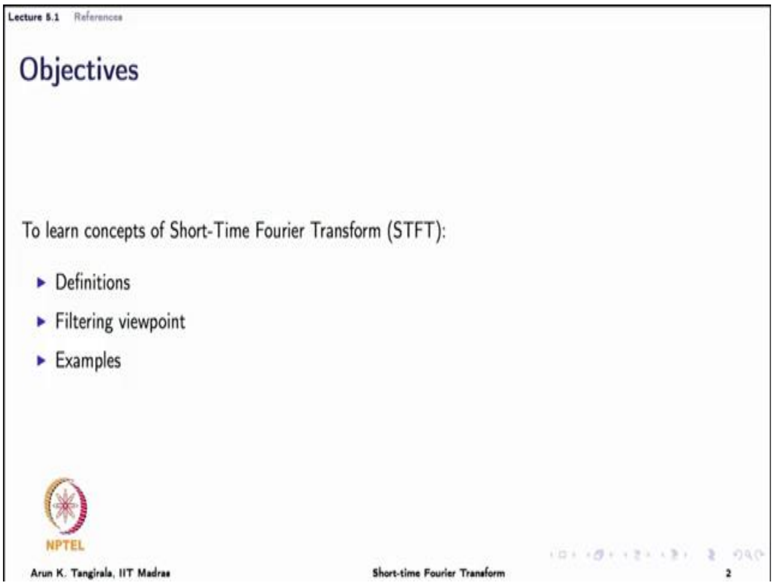

So, we will look primarily at definitions and the filtering view point. And then, look at a few theoretical examples. Towards end I will have an illustration of the short time Fourier transform an impulse like signal or a sin wave and so on.

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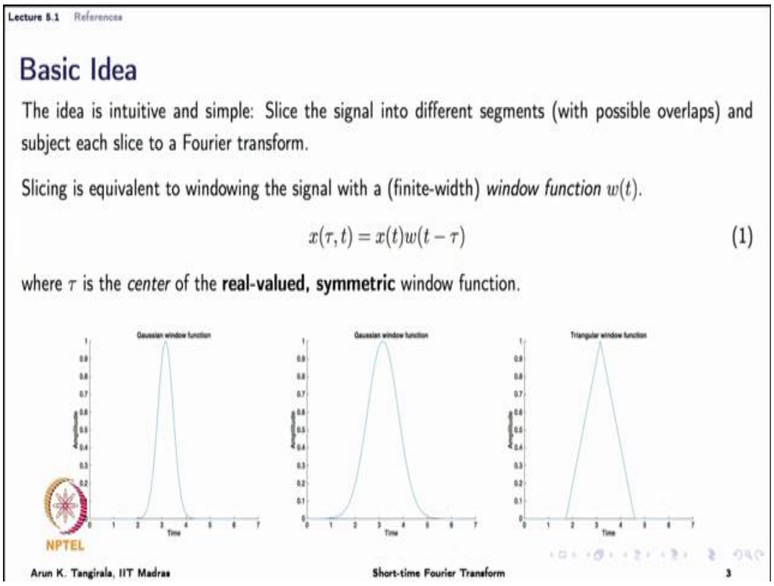

So, let us gets started I think we are familiar with the basic idea of the short time Fourier transform. We have discussed this in the introductory lecture as well. The main idea is to slice the signal into different segments so, as to gain time localization. And then, subject each segment to a Fourier transforms, that remains the basic idea the rest are all flavors of this idea.

Now, how I segment, what kind of window I choose, whether I am going to have overlapping or non overlapping, dictates what properties I get for the short time Fourier transform, particularly in terms of spectral leakage or orthogonal representation or redundant representation and so on. In this lecture, we are not going to study the effects of different windows or window lens and so on. We just going to touch based on those ideas.

There is the separate lecture which will talk about the practical aspects of the short time Fourier transform where, in we will also talk about the discrete time short time Fourier transform. So, this and the next lecture will primarily be theoretical. But of course, give us lot's of insides into how the short time Fourier transform behaves for a different windows and different signals and so on. So, let us get in to the math here. First of all slicing is equivalent to windowing the signal with a finite width window function.

That is a key you, now we are we require this window functions to be of finite width. Obviously, because we want time localizations. You can imagine the regular Fourier transform to be the short time Fourier transform. In fact, it is a long time Fourier transform, the window functions are infinite in length. So, I can represent now this sliced segment of the signal as x of tau comma t, t is a general function of time that we have been using.

Now, tau is the center of the window function that I am going to use. At the bottom of the slide I show you three examples of window functions. The first two are both Gaussian window functions, while the last one that is on the bottom right is the triangular window function. The difference between the first two is fairly obvious. The first one is a narrow window function. That is of shorter duration, then the second one.

And in the third one I have a triangular window just to show you, that the difference between triangular and Gaussian is this abrupt beginning and end of the window function. Whereas, with a Gaussian window function I have smooth beginnings and ends. And typically these are the kinds of the window functions that are preferred. If you look at the literature on the window functions. You will find the discussions on window functions both in the short time Fourier transform literature and more, so in the Fourier transform literature.

Where, the window functions are used to mitigate spectral leakage. Recall this concept that we discussed in the context of Fourier transforms. There you will find that these windows are preferred simply because of their smooth paperings at the beginning and end. The sharp are the abrade beginning and end introduces a some kind of artificial discontinuity. So, you want to be kind to the signal at the beginning and the end.

And we will talk more about this when we talk about window functions. One of the key requirements of this window function is that, it should be real value and symmetric. Why symmetricity and so on will talk about it late one. But, we want it to be symmetric. Now, notice that again tau is the center of the window function. So, what we are essentially going to do is, pick any of these window functions and going to let us say pick the first one.

I am going to place this window function in such a way, that is at the beginning of my analysis. In such a way that the center of this window will co-inside with the start of the signal. So; that means, initially tau is 0. And then, I traverse along the length of the signal. In practice I am going to have sample data. So, the question that arises is how am

I going to march forward it time? Am I going to march ahead one sample at a time, two samples at a time and so on that dictate. So, there I am going generate a redundant representation or an orthogonal representation and so on.

If I want a dense representation or a dense short time Fourier transform, then I would march one step ahead, one sample at each time. Then, my tau will increment in the same way as a time instant itself. In this lecture we are not going to talk particularly about sample data. So, we will postpone the discussion on what values tau will take in the discrete time. But, as of now tau will run from minus infinity to infinity for example. So, that is just a theoretical discussion or if you signal begin from 0, then it runs from 0 to infinity.

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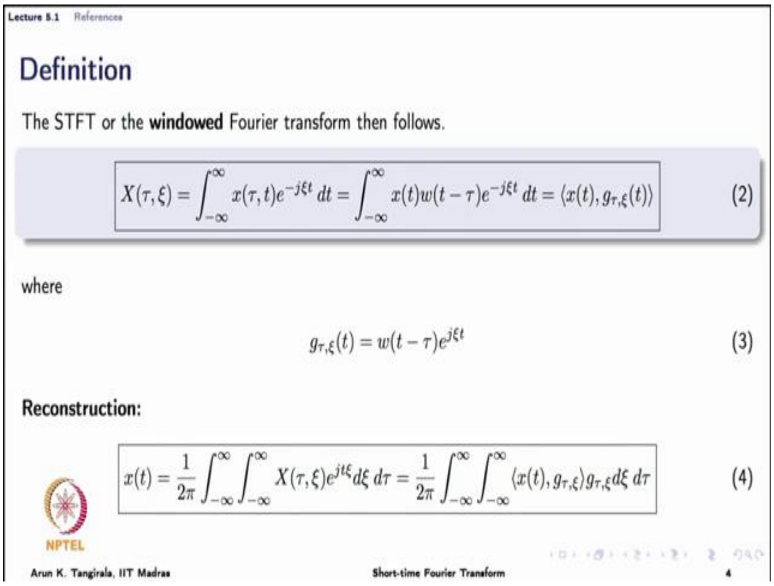

So, now having mathematically define the slice signal. I can define the short time Fourier transform. Which is nothing but, the Fourier transform of the sliced segment. This is one view point, very soon we will talk about an alternative view point of the short time Fourier transform. So, the only difference between the classical Fourier transform. And the short time Fourier transform is this windowed segment, that is being used in the short time Fourier transform. We saw the original signal in the classical one.

Now, also in place of omega I have the symbol xi. This xi is nothing but, the center frequency of the window function. And I will talk about it soon I will show you also what the center frequency means. So, essentially tau is a center of the window in time.

Or you can say the mean location of the window function in time. And xi is the mean frequency of the window. So, what we are doing here is and taking a Fourier transform of the slice segment around a center frequency xi.

Now, my time access of course, runs from minus infinity to infinity. Because, I am looking at theoretical problem. So, now, I have substituted for x of tau comma t here have expanded it, now my window function appears. The nice thing about these integral is I can write this as an inner product between x of t and a function g which subscripts tau comma xi. And that g is the function of time t. What is this g? This g is nothing but, the product of the window and the complex sin wave that I have.

In fact, once again you can realize or recognize that I recover the classical Fourier transform. If I replace this g simply with the complex exponential itself. So, that is the difference again between the short time Fourier transform and the classical Fourier transform, will talk about this g more in detail in the next slide. So, this g now is call what is known as the short time Fourier transform atom at the time frequency atom. And the entire short time Fourier transform is believe to now be based on and a decomposition on this time frequency atoms.

In the Fourier transform the premise is that I can decompose my signal on sinusoid atoms. That is complex sine wave atoms. Now, I have this what is known as a time frequency atom. I can reconstruct my signal, in other words here this is my analysis equation. And here, now I have my synthesis equation given the short time Fourier transform I can always recover the signal using this expression here. Pretty much, this is the two dimensional extension of the one dimensional recovery expression that I have for the continuous time Fourier transform.

Once again, what I have done here is, I replaced this short time Fourier transform with again the result that I have it is an inner product between the signal and the time frequency atom or the short time Fourier transform atom, that I am using. Now, notice notations here I have lower case for the signal and the upper case for the transform itself. And once again, the short time Fourier transform is the function of two parameters tau and xi.

Whereas, the classical Fourier transform or the function of a single variable which is frequency itself. This tau and xi have the units of time in frequency, they are now the center in time of the window and center in frequency of the window. So, they are actually centers and not necessarily just time and frequency themselves.

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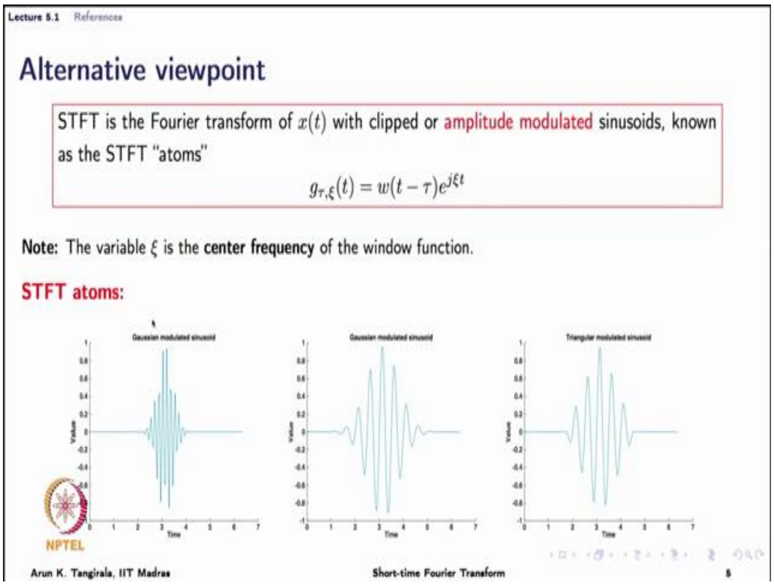

So, let us take an alternative view point of the short time Fourier transform, based on this concept of short time Fourier transform atom that we just discussed. What is this alternative view point? In this definition in equation 2, we set that the short time Fourier transform is the Fourier transform of the windowed segment. The alternative view point is, that the short time Fourier transform is the transform of the signal itself. But, now with the clipped sinusoid.

So, you should appreciate both these view points, in one view point we are saying. And going to the clip the signal and then take the Fourier transform of the clip signal. The other viewpoints is now, I am going to keep the signal as is, but I am going to clip the basis or I am going to clip the analyzing function. That gives us some different insides into the how the short time Fourier transform works.

So, this clip sinusoid is what we have introduced as g earlier, it is a function of tau and xi. That is, it basically is the function of where the windows located in time and where it is Fourier transform is centered in frequency. Now, earlier I showed you window functions, that is I showed you what this double use can look like. Now, I am showing you here how the g's can look like. So, all I have done is I taken those three window functions that I shown you earlier. And multiplied them with the sin waves and just going to show and going to show you only the real parts here.

So, the Gaussian window earlier that we saw of narrow or width is multiplied by this complex sin wave. And likewise this Gaussian window and the triangular window as well. So, these are my analyzing functions. I am going to really measure what is there in the signal using these atoms. When, the local feature, that is a segmented part of the signal matches or I can say when the signal itself matches with this atom.

Remember now, the view point is that I am going to analyze the original signal. I am not now looking at a clip signal, I am going to analysis the entire signal with this atom here. So, for the purpose of discussion assume that your signal exist as long as I have shown here for the window function itself or the atom itself. Then, imagine that I bring a signal here and analyze that signal with this atom. Obviously, where the analyzing function the atom is 0, it will not be able to detect what is happening in the signal in the corresponding time portions.

It will only be able to analyze the signal, where the atom is active. Where the analyzing function is atom. That naturally gets me the time localization are it gets me the local features of the signal in time. And question is now what about the local features in frequency? So, it is clear now how these atoms are able to get me the local features of the signal in time.

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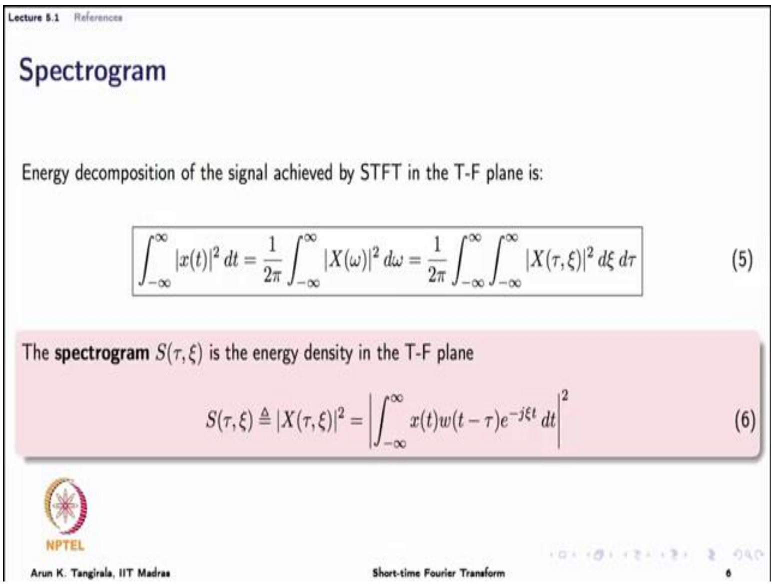

For that we will discuss the duration and bandwidth of the atom itself shortly. Before we do that, it is nice to talk about the spectrogram. Because, that gives me the energy density of the signal in the time frequency plane. How do I define the spectrogram? Earlier I have define told you this in the previous lectures. That the spectrogram is nothing but, this squared magnitude of the short time Fourier transform. Now, the equation 5 here gives me the basis for defining the spectrogram as a square magnitude of the short time Fourier transform.

So, I have here on the left hand side the energy of the signal define in time. And by virtual of parseval's relation, you have seen this expression before. This is the energy in terms of frequency domain variables or Fourier transforms. And now by virtual parseval's relation, I can also show that the energy is preserved in this manner. So, the two dimensional area under the surface of the magnate square magnitude of x of tau comma xi gives me the total energy.

And x of tau comma xi is a continuous function. Notice that, tau and xi are continuous functions right, the continuous valued variables right now. Therefore, the squared magnitude of the short time Fourier transform qualifies to be an energy density in the time frequency plane. Main requirement is the area under the spectrogram should give me the energy. And that is exactly what equation 5 is establishing.

So, we will use this spectrogram extensively to analyze signals and that is how you to it has been used in the literature for analyzing the frequency content of the signal as a function of time.

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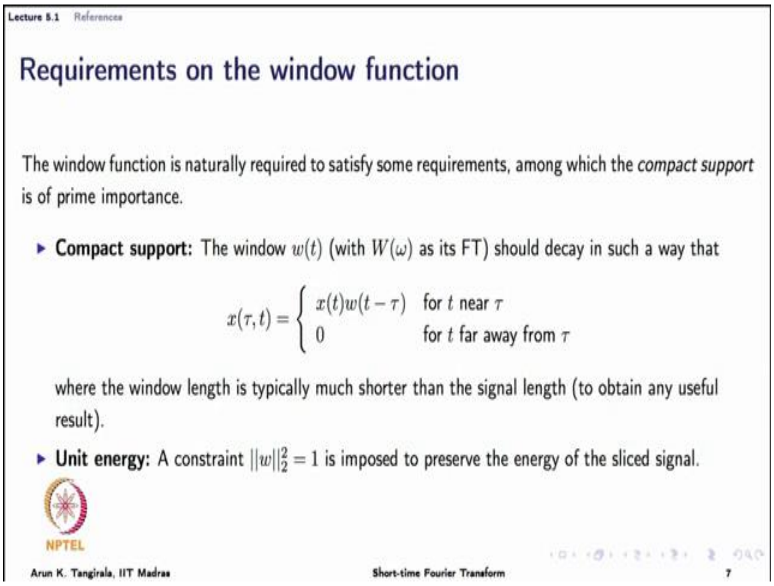

So, let us briefly talk about some theoretical requirements on the window function. We have already set this I cannot use any window function that I like by window itself it gives us an idea, that it has finite duration. And that is exactly what we want, technically we say that the window should have compact support. This is a technical term to indicate, that the window exist only over a finite period of time and vanishes outside this interval.

So, the window should decay in such a way, that x of tau comma t should be this for t near tau. T hat is near the center of the window and 0 for t faraway. Basically the says that I can have my window function decay exponentially it need not approperly go to 0 also. So, such window functions are also being accommodate. And; obviously, I want the window length or the width of the window to be much shorter than that of the signal.

Otherwise, the entire purpose of computing the short time Fourier transform is defeated. If the window is as long as the signal, then it is nothing but, the Fourier transform. So, that is more of a common sense requirement. And typically, we normalize the window functions to have unit energy. So, that I have energy is preserve. So, that is important, not you should indeed verify that the routine that your using is in fact normalizing the windows to have unit energy.

If you are writing your own routine to compute the short time Fourier transform. This should be one of the primary steps before you evaluate the short time Fourier transform.

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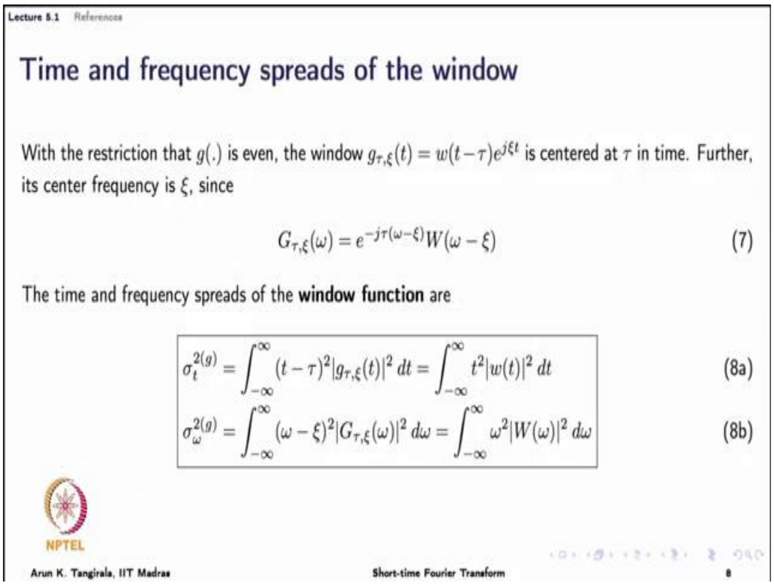

So, let us return now to the discussion that we had on the time frequency spreads of the window. We said that tau and xi represent the centers of the window in time and frequency. And why is that? So, the tau of it was obvious why tau is the center of the window and time. Now, let us look at this atom here time frequency atom, all though it says spreads of the window, ideally and talking about the spread of the atom itself. But, please note the correction.

So, this is the atom that I have the time frequency atom. And it is Fourier transform is given by this, it is e to the minus j tau times omega minus xi times the Fourier transform of the window shifted by xi. So, this w big W is nothing but, the Fourier transform of the window function. Big G is the Fourier transform of the atom itself. So, the only difference between the Fourier transform of the atom and that of the window is this factor here. That as for as a magnitude is concern, they are the same.

Now, it is clear from this expression here in 7. That the Fourier transform of the time frequency atom is centered around xi. This expression itself clearly tells me, that the center frequency is xi. Therefore, now I can compute the time and frequency spreads of the window function. Why do I want to compute the time and frequency spreads? Because, the roll of this window function is to get us the local features of the signal in time and frequency.

Now, I want to know what is mean by local? Local is very qualitative term, for a given choice of window function. I want to see what is the neighborhood of the window function. In the time and frequency plane, is it very worst is it going to get me this features of the signal in a small neighborhood or a large neighborhood. What is exactly the neighborhood of the signal? There it is looking at in both time and frequency.

And that is why we are defining or we are analyzing the time and frequency spreads of the atom here. Rather than, other set instead of window function think of the atom. So, this is the duration or the squared duration of time frequency atom. And this comes from the definition itself by definition, we have recall the definitions of duration that we had for any signal. So, you can think of the time frequency atom is another signal itself. Whose center is tau and whose energy density in time is given by this.

Now, by a simple change of variables and by substituting for g this expiration here. I can simply rewrite this integral as integral t square modules w of t square. Why did I do this to show and to the recognize, that the duration is independent of the center. That is the spread of the window in time as got nothing to do with where it is located, where it is center is located in time. So, as I am traversing in time, it has the same window spread it has a same time spread.

Like vice now for the spread of the atom in frequency domain, I invoke the definition. And then, once again I rewrite this in terms of the window function itself. It terms out that, it is just because I have chosen the atom this way. I can rewrite this, it is not there it is true for any time frequency atom. Only for the short time Fourier transform I can write it this way. Once again I see, that the spread in frequency of the window of the time frequency atom is independent of where it is in the frequency plane.

That means, it is going to actually extract the signal features in the frequency plane, over the same neighborhood. Regardless of whether you are looking at low frequencies or mid frequencies or high frequencies.

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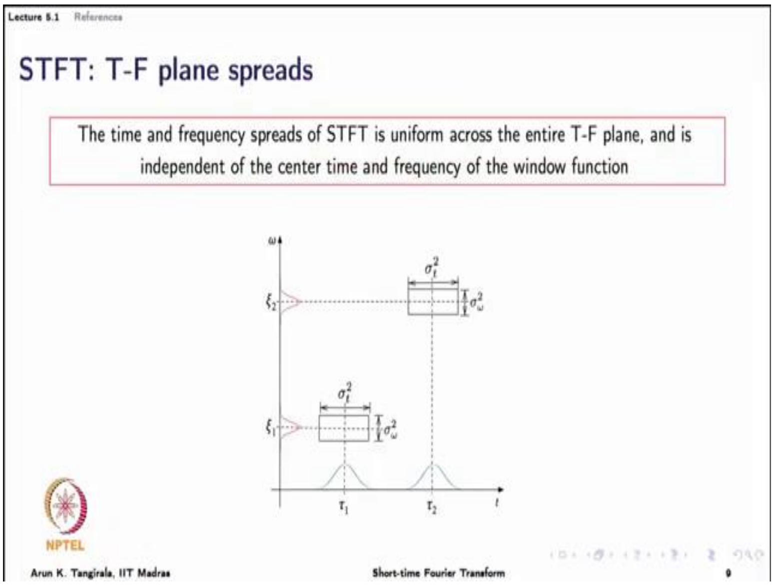

So, let me illustrate this to you, this is something that you will also seen mallet's book. We will now henceforth frequently refer to mallet's book as well. This kind of diagram schematic illustration of the time and frequency spreads of short time Fourier transform is fairly common in the literature. The main message that we get from the previous analysis is four going analysis. This is the time and frequency spreads of the short time Fourier transform is uniform, across the time frequency plane and is independent of the center time and frequency of the window function.

So, this is the time frequency plane. The x axis or the horizontal access is time vertical axis is frequency. I have this window function, just a symbolic one. In time which has the certain width and it is width is characterize by sigma square t. Remember the width of the window function is not sigma square t, it is only sigma square t is only a measure of the width. And by the duration bandwidth principle. I know if I choice wide window such as this, it is going to have a narrow bandwidth. That is in narrow of frequency spread, which is measured by sigma square omega.

So, when I move to analyze the signal in time domain, this way with this window function. Where ever I go does not matter, let us say I move along frequency axis instead of time. So, I want to get the local features, I want to extract the local features of the signal in the frequency domain. So, I am asking what is the frequency content in the high low frequency region and the high frequency region? Even I say move up words, the same window spread in frequency domain and in time domain exists.

It is not adjusting itself like the way it does the wavelet transforms. Remember, the wavelets are specially designed in such a way, that when you are looking at the low frequency region, you use wide wavelets. And when you are looking at the high frequency region. That is when you filter the high frequency content of the signal, you would use narrow wavelets. But, that is not necessarily the case with short time Fourier transform.

The reason being look at how the short time Fourier transform is evaluated. I window the signal. And then I match this windowed signal with all frequencies possible that I know. But, then simple practicality or common sense tells me, that if I have windowed, if I have chosen a slice of the signal. Then, there is a limit to which I can detect the frequency content. The limit being that at least the that particular slice should have completed one cycle.

So, if I am choosing a very narrow window, then I should not be really testing examine the signal for the low frequency content. Because, if there was a frequency component it would not have had the opportunity to complete one cycle in that narrow segment of the window. So, a judicious choice would be to only examine, then if I have a narrow segment for the high frequency content. And if I chosen a wide segment I would choose two analyze only the low frequency content.

But, that is not what is happening in short time Fourier transform. And analyzing does not matter, regardless of the width of the segment that I have chosen. I am going to analyze it for all frequencies. And that is what is the consequence of this uniform spread in time and frequency across the entire time frequency plane? And this will become more obvious. That is you will able to appreciate this better when we talk about the wavelet transform as well.

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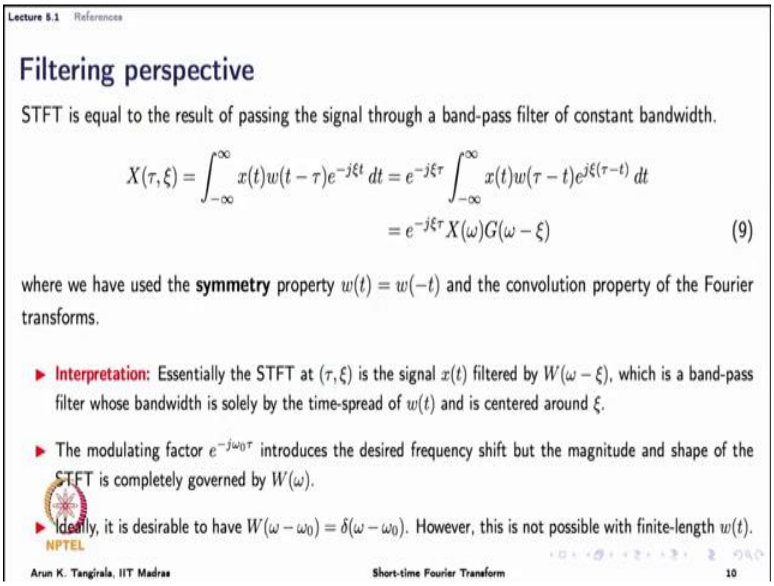

So, this also leads us to the filtering perspective of the short time Fourier transform. And in fact, it will reinforce what we have just discussed. What we are doing here is we use a definition of the short time Fourier transform. And rewrite it deliberately this way, there is nothing that we have lost, we have just a multiplied and divide it by this factor e to the minus xi tau. And now, we can rewrite this integral in this why.

Remember, that this note this integral is in the convolution form, it is a convolution of x with the time frequency atom. The time frequency atom being w of t times e to the xi tau. So, it is a convolution of the signal with this w of t times e to the xi tau. And therefore, x of tau comma xi is nothing but, the product of x of omega times g of omega minus xi times of course, e to the minus j xi tau. So now, all of this is mathematics, but what is interpretation that I have here.

Well, short time Fourier transform, what it gets me is actually the local features of x in the frequency domain, centered around is xi. And that it does not just get me x of omega centered around xi. This x of omega is being multiplied by g of omega minus xi by this function, which is not unity in the vicinity of xi. It will have it is own shape. And that is the problem, that I have with the short time Fourier transform. And but also it a nice thing it tells me, that the short time Fourier transform acts as a filter. So, let me just illustrate this to you on the board.

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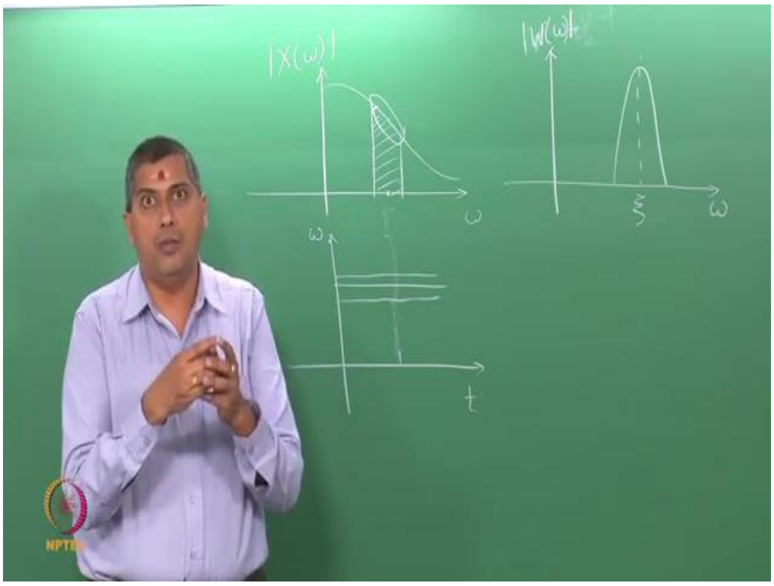

So, what is happening is let us assume that x of omega or the magnitude of x of omega, x of omega in general is a complex number. Let us say it looks like this. And let us say I am interested right now I am looking at some value of xi which is here. So, this is what is the x of omega, what happens is ideally, I would like to get exactly x of omega around this center frequency as is whatever is a magnitude of the Fourier transform of the original signal in around this neighborhood.

In this neighborhood of xi unfortunately this result tells me, that I am not going to exactly get that. It s going to be multiplied with a window which will actually look like this, if it is centered around xi. So, this is the magnitude let us a magnitude of the window of the g that I have or the window itself. There is no difference between the magnitude of g and w we have seen that earlier. So, I am at as well say this is the window itself.

So, what I have is, typically I choose a window that has finite width, we have discuss this earlier. That is a purpose of short time Fourier transform. As a result of which, the window function will have some spread in frequency domain. So, let us say it has this kind of spread in frequency. Let me just write the w of omega itself. And this is let us say centered here, it has a center frequency of xi.

So, what is happening is this x of omega here is being multiplied with w of omega minus xi as a result of which I am not going to exactly get the local features of x of omega. It is going to be distorted by this factor. And that is a distortion we have seen and we have talked about earlier. And we will see numerous examples. Remember I said, if I well to analyze a sin wave using the short time Fourier transform. Ideally, the sin wave we know has a peak in frequency if this is time, this is omega.

Ideally I should be able to get this kind of a line in the time frequency plane for a pure sin wave. But, if I were analyze the sin wave with a short time Fourier transform. Then, I would see a band instead of a line. And this band is essentially because of that. So, what is happening is the short time Fourier transform is filtering with a constant bandwidth. But, it is this bandwidth ideally is not the one that is suited for a sin wave it is this bandwidth is coming about, because I have chosen a window of finite width.

So, the interpretation is that essentially the short time Fourier transform a tau comma xi is a signal filtered by this w or g. In fact I use g and w interchangeably. Because, magnitude vice they are one and the same. By the way in deriving this equation 9, we have use the symmetric property and the convolution property of the Fourier transform. So, ideally it is desirable to have this w as a direct. So, that I get even though I pick a small segment of x of t, I just want whatever is truly the frequency domain characteristic of x of t in that window function.

But, this result in nine tells me, I am not going to get that. I am going to get a distorted version of what I see locally. And that is again by virtual of the duration bandwidth principle. So, the two key thinks that we want analysis, that is the primary property that we want to analyze is a time frequency resolution. How well does is s t f t resolve the features of the signal in time and frequency plane. For this we take two test signals, which is the direct delta function.

That is if I had an impulse, how would the short time Fourier transform behave. Well, I just put this impulse through the definition of short time Fourier transform. And I notice, that the short time Fourier transform is nothing but, w of tau minus t naught times e to the minus j xi t naught.

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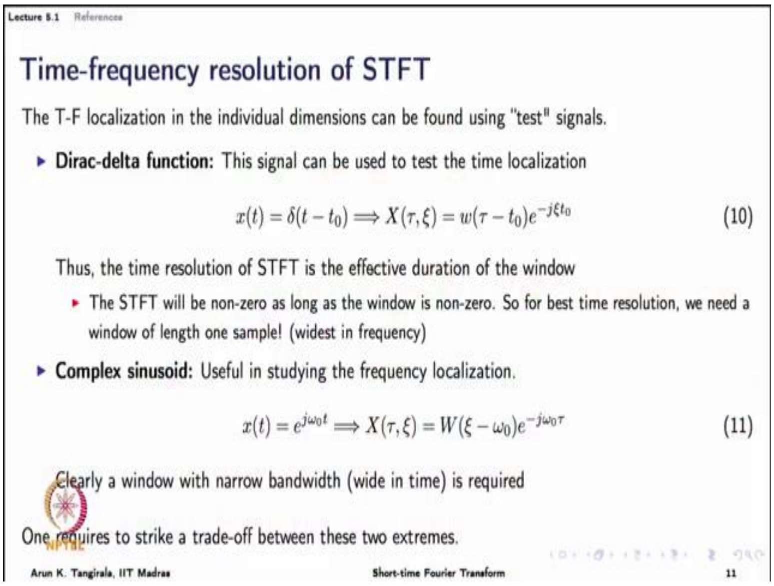

Now, thus the time resolution is essentially the effective duration of the window. If I choose the wide window, then I will be able to resolve accordingly. If I choose a narrow window, then I will be able to resolve accordingly. If I have to resolve an impulse, common sense tells me I have to use a window of very, very narrow width ideally of one sample. But, that is not possible even if... So, what happens if I do that, I have problems resolving complex sinusoids.

Signals are not going to be present with a single feature, they are going to have a mix of features. So, these are the two extreme features the direct delta is extremely localized in time. And sinusoid is highly localized in frequency. In the case of sin wave as I just discuss on the board. When I plug in the sin wave in to the definition of short time Fourier transform. I get this expression x of tau comma xi is w of xi minus omega naught, omega naught is the frequency of the sin wave that I am using.

So, what is this expression tell me? It tells me that the sin wave manifest in short time Fourier transform, depends on the Fourier transform of the window itself. And we know, that this window is not going to be direct in the frequency. It is going to have some finite width. Which means, my sin wave is going to be distorted in the frequency domain. So, the impulse is also going to be distorted and the sin wave is also going to be distorted.

If I do not want the impulse to be distorted in short time Fourier transform. I have to chose a very narrow window. But, what would happen? If I choose a very narrow window, it spread in frequency will be very large. As a result of which I will not be able to localize the sin wave very well. If I want to localize the sin wave, I need a window which is highly localized in frequency domain. By that virtual of the duration bandwidth principle.

If I have the signal which is highly localized in frequency, it should have infinite spread in time. Then, I will be unable to get the local features of the signal in time. So, this is a trade of that is involve.

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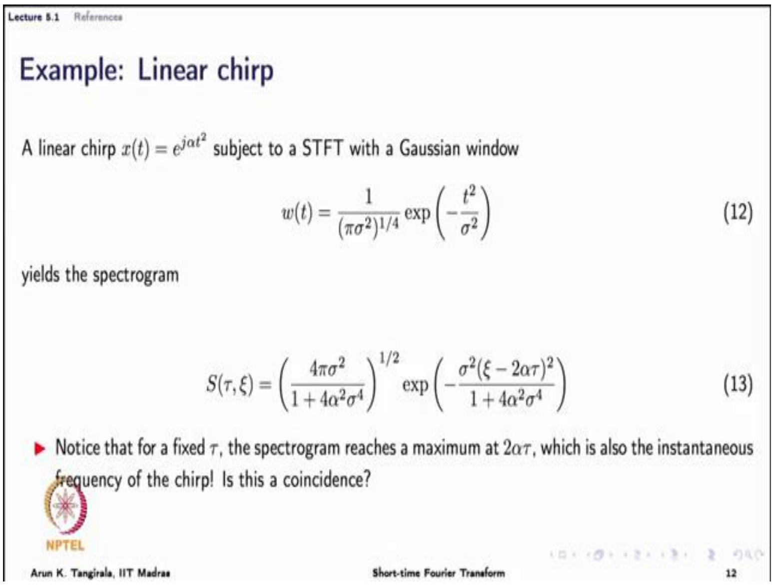

So, simple example that I want to take is for a linear chirp. If discuss to extremes let us look at the linear chirp. And this is an example taken from mallet's book. This is the chirp, you can see there it is a linear chirp. Because, the phase is the quadratic function of time t. And I am using a Gaussian window, this is exactly the window that mallet uses in his book. And if you work out the math, then the spectrogram terms out to be having this expression here.

Now, the reason for picking this example is to illustrate a point, that we will discuss more in detail later on. Look at this expression for the spectrogram for a linear chirp. At a fixed time tau, tau is a center of the window. But, that is also the time at a fix time tau, this spectrogram reaches a maximum, when xi is 2 alpha tau. Alpha is a feature of the signal here, tau is a time that you are looking at and xi is the frequency at which you are situated in the time frequency plane.

So, it says the spectrogram reaches a maximum in the time frequency plane at a given time tau, when the frequency is 2 alpha tau. Now, this 2 alpha tau is also the derivative of this face here. Which means it is nothing but, the instantaneous frequency of the signal. Recall the definition of instantaneous frequency, it is the derivative of the phase. So, what is happening here is tat the maximum at any given time tau, whatever the frequency the spectrogram reaches a maximum, it happens to be the instantaneous frequency.

The question is, if this is a coincident or in general is this true that the maxima of the spectrogram will give me information on the instantaneous frequency, will give me estimates of the instantaneous frequency. He turns out that this is not a coincidence, there is a very nice result which states that the maxima, local maxima of the spectrogram in the time frequency plane gives and estimate a good estimate of the instantaneous frequency of the signal. And this local maxima or known as ridges will talk about it later on.

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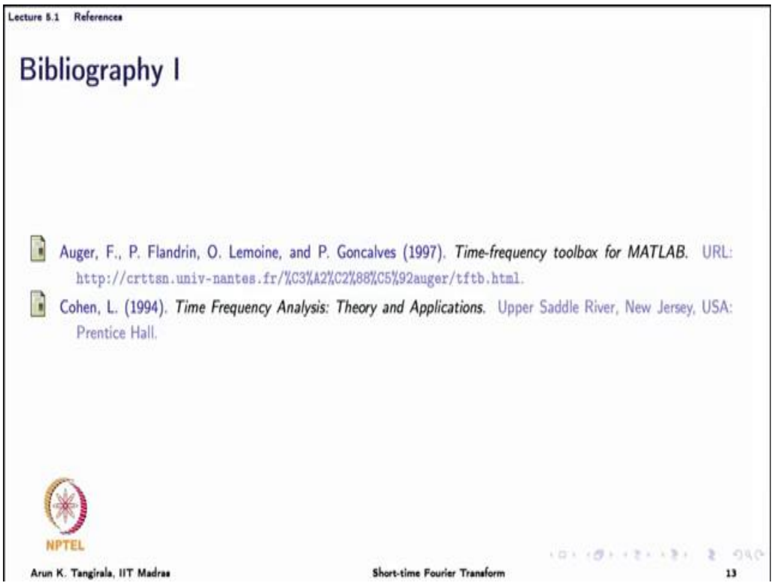

So, with this we come to a closer of this lecture.