Introduction to Time-Frequency Analysis and Wavelet Transforms
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Lecture – 4.4
Duration-Bandwidth Principle

Hello friends; welcome to the lecture 4.4 in the course on introduction to time-frequency analysis and wavelet transforms. In the previous lecture, we learnt the concept of analytic signals instantaneous frequencies and also studied the limitations of instantaneous frequencies. The concept of instantaneous frequency is good. In general, the physical concept is good, but the mathematical definition itself has the limitation. So, you should not get confused between a concept and the definition. The definition that we had for the instantaneous frequency as a change – the derivative of the phase; has its limitations in the sense that, if a signal has more than one frequency at a given time, then the definition gives me observed results. And, it gives me very good results when the signal has a single frequency at a given time instance. The frequencies are allowed to vary from time to time. But, at any given time instant, it should have only single frequency and we call such signals as mono component signals. We are going to at a later stage learn the technique of empirical mode decomposition; which is also known as a Hilbert-Huang transform.

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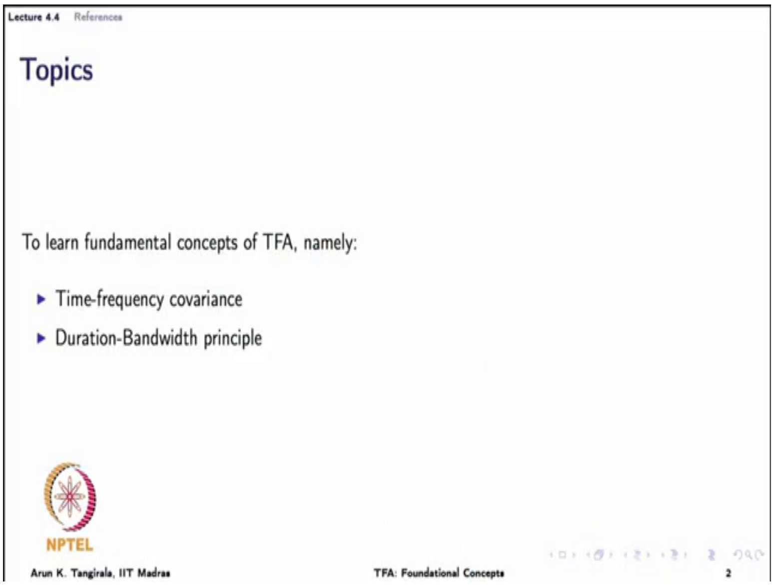

But, for now, to keep the flow of things intact, we will move on to the duration-bandwidth principle, which is a very fundamental result as we have seen in the previous lectures as well in the form of illustrations. Today, we are going to formally prove the duration-bandwidth principle. And, in order to derive the duration bandwidth principle, we have to introduce a concept called time-frequency covariance. Now, this is not surprising; again, recalling the analogy of the energy densities with the probability densities. In fact, constantly, if you keep reminding yourself of this close analogy with the probability theory or the random variables theory, then a lot of these definitions become obvious and easy to understand.

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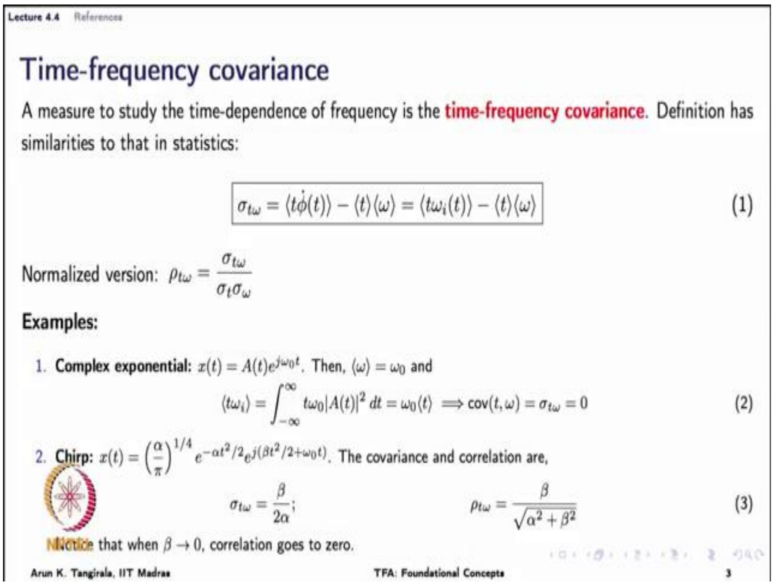

So, when we talk of time-frequency covariance, we are essentially talking about the covariance of frequency and time in a sense that, we are asking how frequency of a signal changes with time. So, we are not really asking how this concept of frequency or this general frequency, dimension of frequency changes with time and so on. What we are asking is how the frequency of a signal changes with time; whether there is a dependence of the frequency of the signal on time. Even before we delve into the definitions of time-frequency covariance and so on, intuitively itself, since time-frequency covariance is going to measure how dependent the frequency is on time for a constant frequency signal, you should expect the covariance to be 0; which means frequency is independent of time. And, we will see that when we discuss a couple of examples here at the… At the bottom, I am given you a couple of examples here. So, the definition of time-frequency covariance is fairly straightforward; it is along the same lines as the covariance in probability theory or covariance for random variables.

Once again, I should tell you that, you should not treat time and frequency as random variables and so on. So, we denote the covariance with sigma subscript t omega denoting that, we are computing the time-frequency covariance as in the case of random variables; the covariance is defined as the average of the product less the product of averages. Let me bring up the analogy of random variables for you here; that is, the definition of covariance for random variables.

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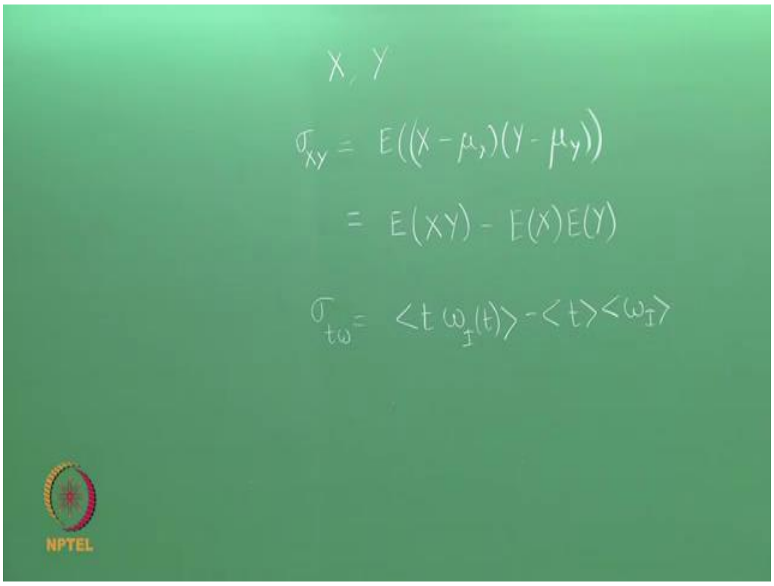

When I have two random variables – continuous valued random variables: X and Y; and, when I want to measure the dependence of Y on X or X on Y, whichever way I am looking at it; then, a standard measure of dependence is the covariance. We have learnt this in the context of deterministic signals as well. When we spoke of cross-covariance function, we did remarked there that, the cross covariance function is a measure of linear dependence; and that, it is based on the concept of covariance, which measures linear dependence between two variables. Of course, there we talked about deterministic signals; now, we are looking at random variables. So, how is the covariance between two random variables defined as? It is defined as expectation of X minus mu X times Y minus mu Y; where, mu X and mu Y are the expected values of X and Y. And, it is fairly easy to show that, this expression can be rewritten as the expectation of the product less the product of expectation, so that whenever the x and y are 0 mean variables, the expectation of the product itself is the covariance. So, this is the definition that you see on the slide as well here.

Instead of omega I, I have phi dot of t in the equation. Remember we are supposed to evaluate the dependence of frequency on time or the influence of time on frequency. Therefore, I have to use instantaneous frequency. Very often when I write omega of t, it should be understood that, I am looking at instantaneous frequency. So, the I need not be there all the time. So, this analogy really helps, but again you should tell yourself that, here I am looking at random variables; these are different. Here time and frequency are not random variables or you should not draw any such observed conclusions. So, the normalized version is often used; but, the only difference between the normalized and the un-normalized version is that, it is independent of the choice of units that you have for time and omega. That is the advantage of working with normalized version.

Now this is also the practice in random variables, where I use correlation instead of covariance. There also correlation is defined in an exactly similar manner. However, in the theory of random variables, the correlation can be shown to be bounded in magnitude above by unity. That is not necessarily the case here. So, the advantage of working with correlation in random variables is that, one – it is independent of the choice of units for X and Y – random variables; and, two – that is the bounded measure. Whereas, here the advantage is that, it is independent of the choice of units that you make for t and omega; but, otherwise, it need not be bounded. So, that is it. So, that is the basic definition of time-frequency covariance. Again, the purpose of this measure is to tell me how dependent the frequency of the signal is on time.

Let us look at these two examples here. I have a complex exponential. Once again, this expression is familiar to us. And, it is fairly straightforward to see that, the frequency of the signal is constant; and that, it is equal to omega naught. Therefore, the average also is omega naught; average frequency is also omega naught. And, the average of the product can be worked out here by evaluating the integral, which is fairly straightforward to evaluate. I get here the average to be… that is, the average of the product to be omega naught times average time; and, plugging in this result into the definition of covariance gives me 0, because here I have average of the product being omega naught times average of time t; and, average of omega itself is omega naught. So, both these terms are identical. And therefore, covariance evaluates to 0, which is what we expect; that is, there is no dependence of frequency on time for this signal.

Now, when will look at a chirp, you should recognize that this chirp has a linear modulation. This is the example that we have looked at in the previous module as well. The phase has a quadratic dependence on time. Therefore, the instantaneous frequency has a linear dependence on time. Hence, we say that, it is a linearly frequency modulated signal. And then, of course, you have an amplitude modulation as well. In this case, you will have to work out the integral; you will have to do a bit of algebra and evaluation of the integrals that finally leads to these values of expressions for covariance and correlations. So, covariance is given by beta by 2 alpha; and, the correlation itself is given by beta by square root of alpha square plus beta square. So, if you look at the correlation, when beta goes to 0… What does it mean when beta goes to 0? That means I am taking of this quadratic dependence on time from the phase. The moment I take off this quadratic dependence, then it goes back to the case of no-frequency modulation. Remember – when phase is simply linear function of time, then the frequency is going to be constant. So, when beta goes to 0, the correlation goes to 0; once again indicating clearly that, there is no dependence of frequency on time. So, this is the beauty of this measure.

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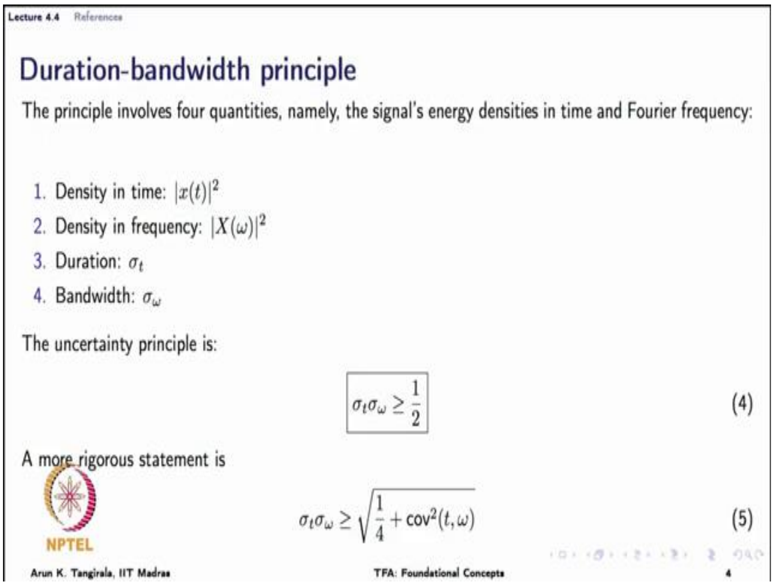

So, let us look at the central topic of this lecture, which is a duration-bandwidth principle. The duration bandwidth principle is something that we have seen earlier as well; and, it places fundamental limitations on what we can do in time-frequency analysis, because it says that, if I try to localize the energy of a signal in time, then the localization of energy in frequency is affected. Fundamentally, what it says is I cannot really localize the energy of a signal in time and in frequency with arbitrary fineness. So, that is the basic problem that I have in the time-frequency analysis. Now, here we have a formal statement of the duration-bandwidth principle, which states that, the product of the duration – sigma t and the bandwidth – sigma omega is bounded below by half clearly telling me that, whenever the duration of the signal becomes small, then its bandwidth increases. And, we have seen this through few illustrations in the previous lectures as well. But, it is very important to understand the duration-bandwidth principle, the assumptions that go into it, the setting in which it is derived, so that we do not have any misconceptions or misinterpretations.

Quite often, this duration-bandwidth principle is called the uncertainly principle for signals, because the expression here in equation 4 has a very strong similarity with the one that you see in quantum mechanics in the form of Heisenberg's uncertainty principle. There the term uncertainty is appropriate, because the Heisenberg's uncertainty principle is derived in a probabilistic framework. There is nothing probabilistic here; pretty much like what we said earlier for time-frequency covariance. The expressions for the covariance look very similar, but that does not make time and frequency random variables; or, that does not mean that, we are working in a probabilistic framework. Nevertheless, this term has stuck on and people have been using widely this phrase called uncertainty principle for signals. So, you should remember that, there is nothing uncertain about things here; it is all about the product of duration and bandwidth; and that, there are four quantities involved in deriving this relation. Two – being the densities in time and frequency; and, the other two – being the duration and bandwidth. And, very importantly, this is not some frequency; it is a Fourier frequency; which means this bandwidth itself is defined with respect to a Fourier transform. If you are working with some other transform, this duration-bandwidth principle or any of these things may not even arise at all. So, more rigorous statement of the duration-bandwidth principle is given in equation 5 that involves a covariance between time and frequency. I have given… We have already discussed this covariance between time and frequency in the previous slides.

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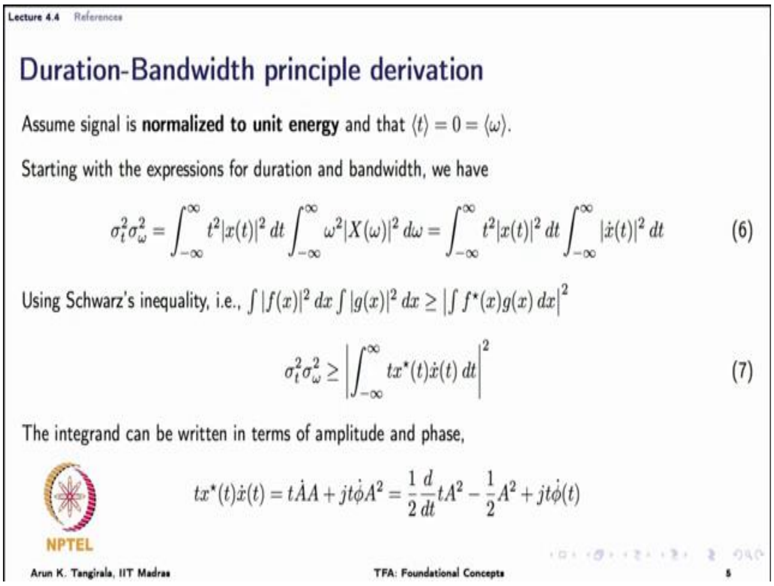

So, let us see how to derive this duration-bandwidth principle. The derivation is fairly straightforward. The fundamental result that we use is that of Schwarz's inequality. And, we apply it to the product of the square duration and square bandwidth. We know from definition, sigma square t, sigma square omega are given by this product of integrals assuming that, the average-time and average-frequency is 0. This is not a major assumption that is going to spoil the result here. So, without loss of generality, you can assume this. It is just that, the math becomes convenient by assuming this; otherwise, you can also derive this by assuming nonzero mean time and mean frequencies.

So, let us get back here. So, I have the product of this square duration and square bandwidth given by these integrals. And, I can rewrite this integral here – omega square times modulus x of omega square d omega as integral x dot of t square dt. And, that is using the property of Fourier transforms. You can refer to the table of properties of Fourier transforms in any standard book. And, it will tell you that, multiplication of X of omega with j omega will correspond to taking the derivative in time. Using that property, we have rewritten this integral in terms of time-domain variables. And now, we invoke the Schwarz's inequality, which says that, the product of integrals – which integrals here? Integral mod f of x square dx and integral mod g of x square dx; that is greater than or equal to the squared modulus of integral f star of x g of x dx. This is a very standard inequality that is prevalent everywhere in functional analysis and so on.

Now, we apply this to the product of the square duration and bandwidth, and we get this inequality. So, on the left-hand side here, if I replace f of x with t times x of t and x itself with t; please note that, x here is a dummy variable; do not get confused with the x here and x of t here. And, I apologize for any confusion that you may have. That is it. So, you apply the left-hand side to the problem of interest; where, now, we have sigma square t times sigma square omega. And, on the right-hand side, now, I have t times x star of t. Remember it says f star of x; and, our f of x here is t times x of t. So, time is a real valued quantity. So, there is no conjugate; conjugate and the number itself are identical; variable itself are identical. So, I have t times x star of t in place of f star of x. And, in place of g of x, I have x dot of t. So, that is what I have here.

Now, what remains is the evaluation of this integral. The integrand itself now can be written in terms of amplitude and phase; where, we are invoking the complex representation for x of t. Remember – x of t is written as a of t times e to the j phi of t. So, I substitute that representation here for x of t and I get t A dot times A plus j times t times phi dot times A square, which I have intentionally rewritten in this form. All this derivation is borrowed from Cohen's book. So, if you have any confusion, you can refer to the text book by Cohen. So, now, if you look at this first term, it is a perfect integrand. Therefore, the integral of this term is going to turn out to 0 when you integrate from minus infinity to infinity. And, this second one here evaluates to minus half assuming that, we have normalized the signal to have unit energy. So, the second term is minus half with that normalization assumption. And, the third term is nothing but j times the covariance. So, you can see when I integrate j times t phi dot of t by definition integral t times phi dot of t assuming that, average time and average frequency as 0, is nothing but the covariance itself.

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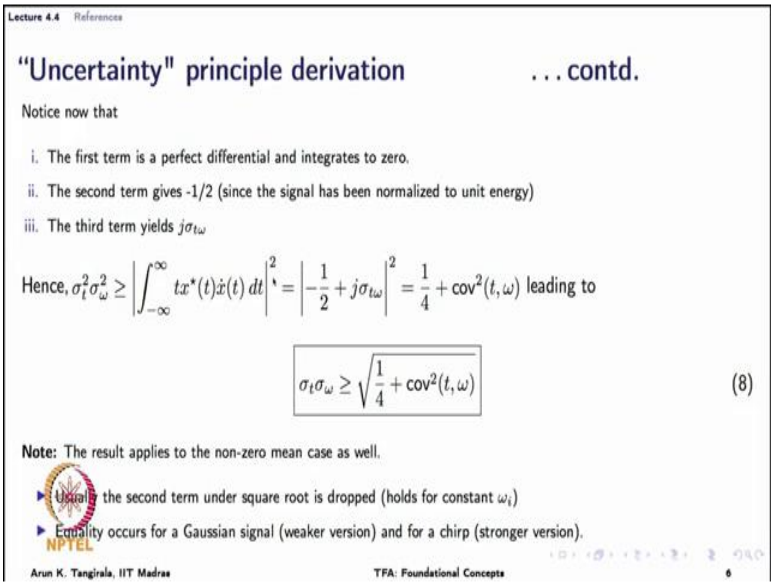

Therefore, I have this result here. Sigma square t time sigma square omega is greater than or equal to minus half plus j sigma t omega mod square, that is, a magnitude square of this. The inner term here is a complex number. And, the moment I take the magnitude square, I get 1 over 4 plus covariance square t omega leading to this result that we have stated earlier. So, this is a more rigorous statement. This is the complete statement of the duration bandwidth principle. Normally, this covariance square of t comma omega is omitted from the general statement of the result. Is that right? It is right whenever you are looking at signals that have constant frequencies, because when signals have constant frequencies like a sine wave and so on; then, the covariance is 0. We have seen that in the example.

However, the issue is not whether covariance is 0 or not, the main point is that, the product of duration and bandwidth is bounded below by a finite number. Consequently, whenever sigma t increases, sigma omega false down; and, whenever sigma t decreases, sigma omega increases. So, that is the most important point that you have to remember rather than really worrying about whether this 1 over… It is the right-hand side is half or square root of 1 over 4 plus covariance square t comma omega. Now, when does the equality occur? The equality occurs for a Gaussian’s signal; you can see the worked out example in Cohen’s text and also for a chirp; that is, the weaker version equality occurs for a Gaussian signal; the product of sigma t sigma omega evaluates to half for a Gaussian signal and for a chirp when you are looking at the stronger version. Both these examples are worked out in Cohen's text. And, I strongly recommend you look up the workings of that example.

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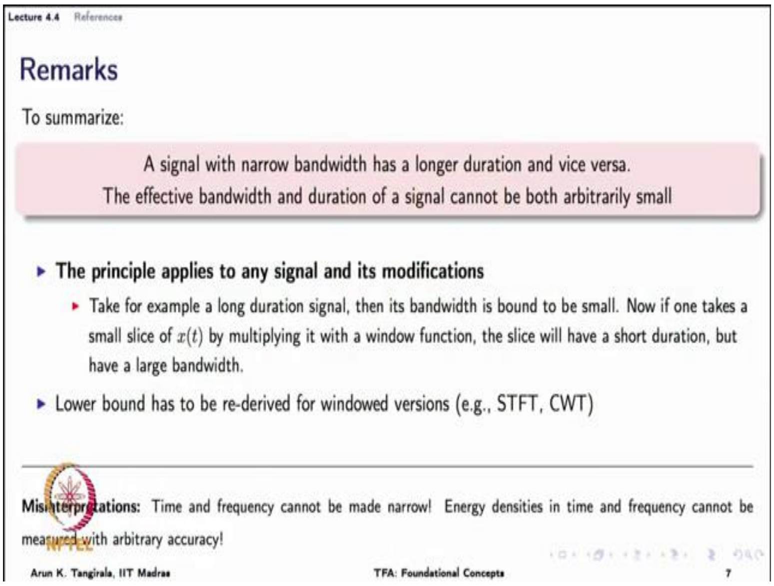

So, that brings us to the close of this module. I just want to make a few remarks again with respect to the duration-bandwidth principle. What it says is a signal with narrow bandwidth has a longer duration and vice versa. And, or, that the effective bandwidth and duration of a signal cannot be both arbitrarily small. Now, this principle applies to any signal and its modification. What we mean by modification is in short-time Fourier transform for instance, I window the signal. So, the duration-bandwidth principle again applies to this windowed signal as well. There you are looking at the duration of the windowed signal and the bandwidth of the windowed signal. So, you have to be careful in asking – to which signal does this duration bandwidth principle apply. And, we will derive the lower bound for the local quantities in time-frequency analysis. That is what we mean by localize in joint time-frequency analysis; I will be looking at conditional bandwidth and conditional duration and I will have to re-derive the bounds for the product of the conditional duration and conditional bandwidth. We will understand this better when we talk of short-time Fourier transforms and CWT transform and so on.

Now, there are two common misinterpretations or wrong statements given for duration-bandwidth principle. One – that time and frequency cannot be made narrow. That is a very weird statement; there is no sense in that. Or, that energy densities in time and frequency cannot be measured with arbitrary accuracy. There is absolutely no statement here with respect to measurements here. It has got nothing to do with your ability to measure; it is just got to do with the spread of the energy densities in time and frequency. How you measure is not dictated by the duration-bandwidth principle, nor does it place any limitations on it. There is also sometimes this statement called delta t times delta omega is greater than or equal to half; where, delta t and delta omega are the resolutions in time and frequency. That is also wrong. The sigma t and sigma omega are not the resolutions in time and frequency; delta t is the sampling interval that is dictated by the sampling rate; and, delta omega is in turn dictated by the sampling rate as well, because we know in DFT, the frequency resolution is limited by the number of observations as well. So, delta omega is dictated both by the sampling rate and the number of observations. That has got nothing to do with the duration and bandwidth. So, if you see statements like delta t and delta omega greater than or equal to half; where, delta t and delta omega are being referred to as resolutions in time and frequency, that statement is incorrect; the correct statement is sigma t times sigma omega is greater than or equal to half.

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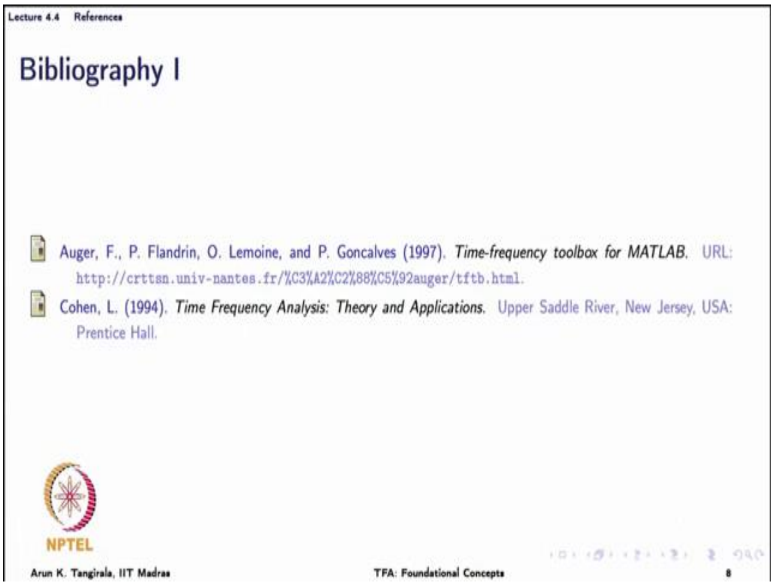

So, with those remarks, we will close this module. And, once again, I welcome you to refer to Cohen's book and also work out a few examples in the time-frequency toolbox frame work for you to understand how this duration-bandwidth principle works.

We will meet again in the next lecture.

Thank you.