Introduction to Time-Frequency Analysis and Wavelet Transforms
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Lecture - 4.3
Instantaneous Frequency and Analytic Signals

Hello friends, welcome to lecture 4.3, this is the third module in the fourth unit, where we are learning fundamental concepts related to time-frequency analysis. And I hope you have had a serious viewing of the previous two modules. The topics are at least one of the topics, which is instantaneous frequency is in continuation of what we learnt in the process of deriving the band-with equation.

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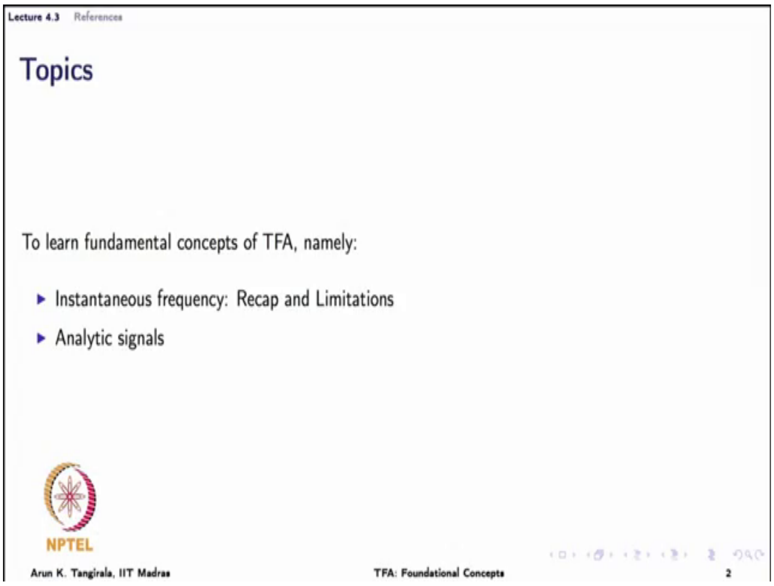

So, we are going to do a recap of the instantaneous frequency and also discuss certain limitations towards the end of this lecture. We shall particularly discuss what are known as analytic signals; we have view, we have encountered this term earlier in the context of looking at mean, frequency, and so on. These analytic signals are essentially complex representations of real-valued signals.

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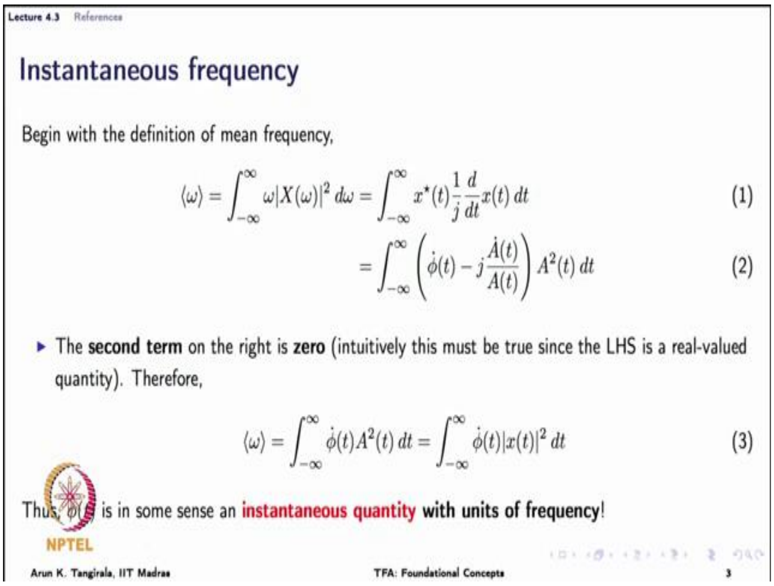

So, let us begin with a recap of instantaneous frequency. Why? This particular term is called instantaneous frequency, whatever definition we have and so on. So, again recalling these equations that we have seen in the previous module, we begin with the expression for the average frequency as you can see in equation 1. As usual this is a definition of the mean frequency. And using the trick for rewriting this expression in frequency domain in terms of the time-domain representation as was given in the previous module. We just rewrite this entire integral in terms of time-domain representations. And then eventually introducing the complex representation that we had for x of t, which is A of t times e to the j phi of t, we have this expression here.

Now, as we argued in the previous module, the left-hand side is a real-valued number; which is a mean frequency, and the right-hand side is an integral of a complex number. Naturally, if you want both sides to match, then the imaginary portion on the right-hand side should go to 0, and it does go to 0, because it is a perfect integrand. So, we are left with this integral here in equation 3 for the calculation of the mean frequency. Now, all I have done is I have rewritten A square of t as mod of x of t square rightfully, because x of t is simply A of t times e to the j phi of t. Now, any average quantity when written in terms of the moments of a density function, it is a first moment of the density function. But here the peculiarity of this integral is the left-hand side is as units of frequency; whereas the right-hand side here is an integral in terms of the time-domain representations.

Whatever may be the case, whether I use mod of x of t squared dt or mod of x of omega square d omega, the energy is preserved due to Parseval's relation. Therefore, this phi dot of t has to have the units of frequency. And it has to be a local quantity, because average quantities are local quantities weighed by some probability and so on if you are talking of random variables; and recalling the analogy that we had between the energy density and the probability density functions, you can think of phi dot of t as a local quantity, which has units of frequency. And because it is a function of time t, we call this as an instantaneous frequency; that is the argument that is presented to derived phi dot of t; it is not straight away that it falls out of some expression; but we are using some qualitative and meaningful arguments to come up with the mathematical expression for instantaneous frequency.

So, let me again reiterate here; instantaneous frequency is a physical concept; it is something that we can think of when look at the changing colors of a leaf or a flower in the morning. You can see that the colors change with time; and because each color is typically associated with a frequency or a frequency band, you can say that, the frequencies are changing with time; and instantaneous frequency will give me an idea of what is the frequency of a given signal at that time. In that perspective, instantaneous frequency is actually a physical concept. What we are trying to arrive here at is a mathematical definition.

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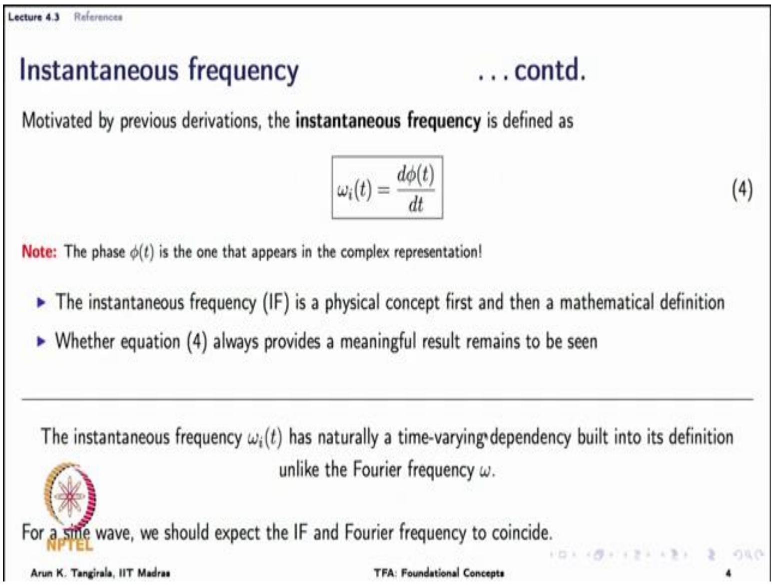

So, the instantaneous frequency or the local frequency – both these terms are used interchangeably in the literature, is the derivative of the phase of the signal. Now, again, this phase comes from the complex representation of the signal that we have seen earlier as well. So, the question that remains to be seen is whether this expression – mathematical expression provides a meaningful result; that means whether the mathematical value that I get by differentiating the phase indeed matches what I have in the signal as a physical property of that signal. And that we will see towards the end of this lecture; we will find that, this definition has certain limitations.

Now, obviously, this instantaneous frequency is a very useful quantity in time-frequency analysis, because that is what; I want to know what frequency is present at what point in time; that, it is a very useful and beautiful quantity to work with. And naturally, for a sine wave, we should expect that, the instantaneous frequency and Fourier frequency coincides. So, now, you have to get used to these two notions of frequency: 1 is a frequency, which is the Fourier frequency, which exists forever, because the Fourier frequency comes from the Fourier-basis functions – sines and cosines, which have this frequency. And it has a same frequency at all times; whereas, instantaneous frequency is changing with time. Only for a sine wave, these two should naturally coincide, because at any point in time, the instantaneous frequency for a sine wave should be fixed. Now, the question is whether I can achieve this identity for the sine waves just by working with the sine waves or do I have to re-represent the sine wave in some other form to establish this; expected coincidence between instantaneous frequency and Fourier frequency.

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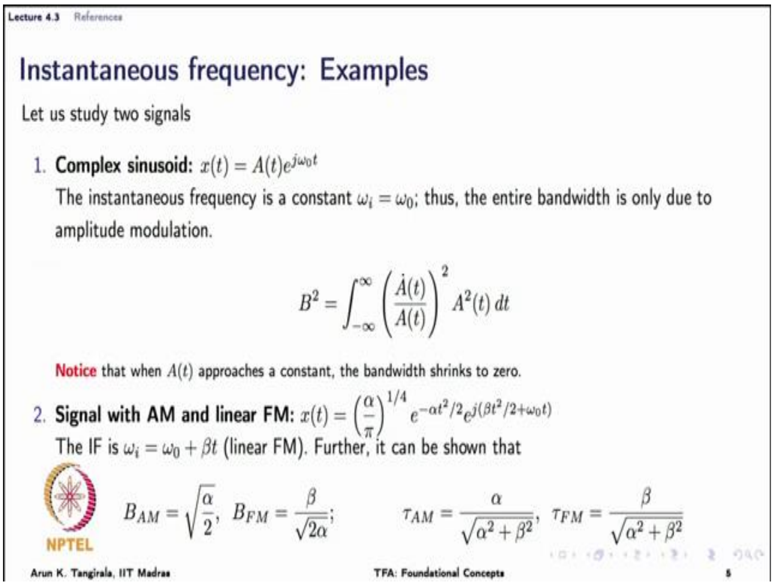

So, let us look at a couple of examples; when I have a complex sine wave, I have this representation; naturally, the signal has complex representations; I do not have to put in much effort; A of t is amplitude. Again, I would like to caution on the term amplitude that we use here. Amplitude in this case is a magnitude of the signal; but in general, if you think of amplitude for a sine wave as we discussed in unit 2, it more or less represents the peak value. But, the nice thing about this complex representation is amplitude and a magnitude of the signal, are one and the same. And that is one of the beauties of working with a complex representation. So, you do not have any confusion between amplitude and magnitude of the signal; otherwise, if you had A of t sine omega t, then the amplitude A of t would be different from the magnitude of the signal, because a magnitude of the signal at any time is amplitude multiplied by sine omega t. So, you should therefore, appreciate the complex representation much better now.

Clearly, the instantaneous frequency is a constant both from a physical view point, because it is a complex sinusoid of fixed frequency. A of t only contributes to the amplitude modulation or changing amplitude. And also, when I use the definition of instantaneous frequency as I have from equation 4, phi of t for this example is omega naught t; its derivative is omega naught. And therefore, the mathematical expression and the physical feature of the signal coincide; and now what happens is because the frequency is fixed; suppose A of t was 1; that means it did not change with time. Then, how would the energy spectral density theoretically look like? It would look like as a peak – exactly a peak at omega naught. So, the bandwidth is 0; there is no spread. But, since A of t is not necessarily a constant and is a function of time, whatever spread that you see in the energy spectral density, that is, the bandwidth that you see for x of t, would be slowly coming from amplitude modulation. And as A of t approaches a constant, you can see that, the bandwidth shrinks to 0. Bandwidth – 0 means that, essentially, that is the single frequency in that signal. So, this is now consistent with all the analyses that have been doing.

But, suppose I have a signal, which has both amplitude modulation and linear frequency modulation; the amplitude modulation part comes from this factor here – alpha by pi raise to 1 over 4 times e to the minus alpha t square by 2; whereas, the frequency modulation comes from the fact that, this phase phi of t here is a quadratic function of time t. Notice with the… Compare with the previous example; the phase was a linear function of time t. And therefore, its derivative is going to be fixed. Therefore, there is no frequency modulation at all. So, whenever the phase is a linear function of time t, there is no frequency modulation. But, when the phase is a function of higher powers of t, then you should expect frequency modulation. Here the phase is a quadratic function of time t.

And naturally, now, the instantaneous frequency is omega naught plus dt; where, we have used this definition in equation 4. So, that is why we call this signal having linear frequency modulation, because the instantaneous frequency is a linear function of time t. Of course, there is an omega naught; and the frequency modulation is therefore, over and top above this omega naught. When beta is 0, again, it will take you back to the no frequency modulation case. In this case, using the expressions that we had derived for the bandwidth, we can easily show that, the amplitude modulation contribution is this. And the frequency modulation contribution is this. As you can see here, the frequency modulation part also has this term alpha. Of course, amplitude modulation is not affected by beta; but whatever may be the case, they both add up, that is, the b am and b fm squares of that add up to give you the total bandwidth.

So, once again, alpha and beta will contribute to the overall bandwidth. But, you will not be able to figure out exactly how much is amplitude modulation, how much is coming from frequency modulation by solely looking at b square. So, given a signal, I would not be able to say; if I only given measurement of x of t, I would not able to say how much contribution is coming from amplitude modulation, how much contribution is coming from frequency modulation; that is the again limitation of using spectral densities alone.

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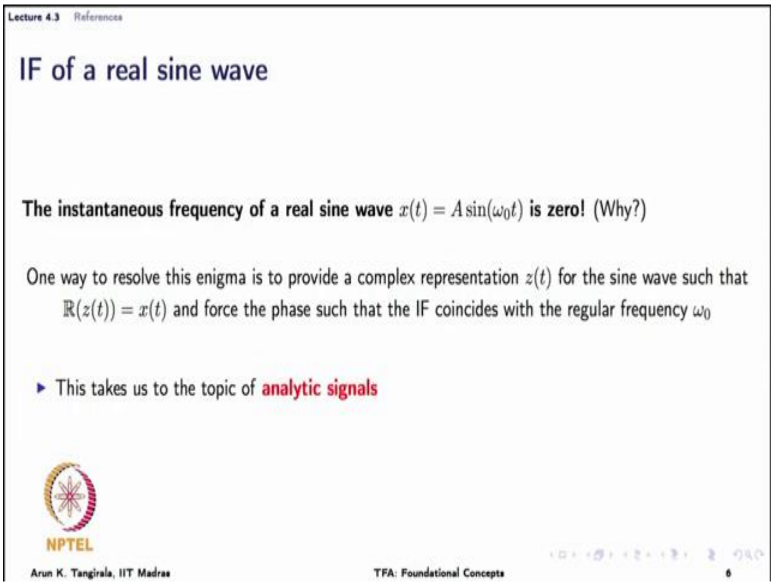

Now, the instantaneous frequency of a real sine wave; so, if I just have A sine omega naught t, it is… If I look at that, it is 0. Why is this? Because if you compare this x of t that I have with the general complex representation that I have for x of t; let me show you this.

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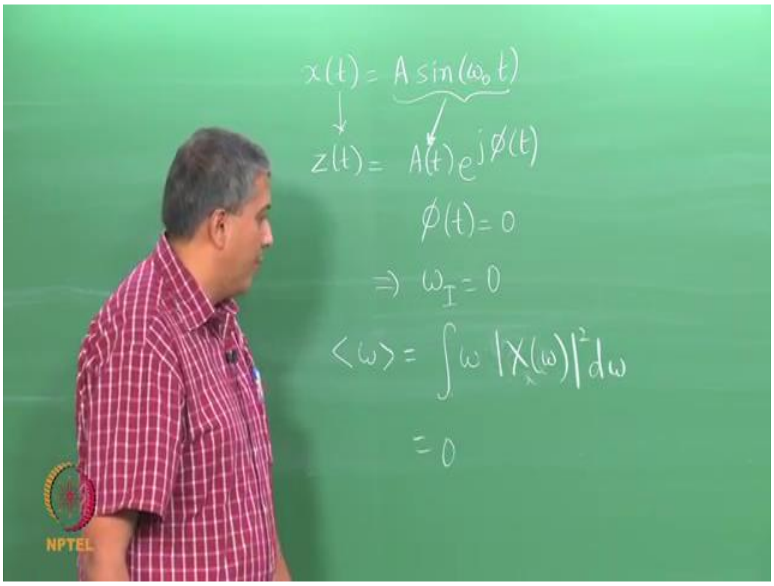

So, I have this signal x of t as A sine omega naught t; whereas, the representation that I use to compute the instantaneous frequency is A of t e to the j phi of t. So, this is the representation that I work with to compute instantaneous frequency. And I have this signal. So, when I cast x of t in the complex form just as is; then clearly, this entire thing goes into A of t, because this is a real-valued signal. And by comparing, it is clear that, there should be no complex portion or imaginary portion at all. As a result of which, I have phi of t being 0. And therefore, the instantaneous frequency is also 0. Now, this is observed, because we know that, the instantaneous frequency of a sine wave physically should be fixed at omega naught. Why is this anomaly occurring? The anomaly is occurring because I am directly working with a real-valued signal; I am working with a real representation itself.

If somehow I find a suitable phi such that I say – so, instead of working with x of t now, I will create a z of t corresponding to x of t; and then say that, the real part of z of t should be x of t and somehow choose phi that everything else make sense. Remember we had this anomaly also in determining the mean frequency. If you recall the example that we had in module 4.1, the mean frequency of a real sine wave; if I just use expression as is, this turns out to be 0, because the spectral density is actually symmetric for this signal. So, we had a similar situation in the computation of mean frequency. There again we pointed out that, this anomaly is occurring because I am working with directly the real-valued representation of x of t. Therefore, we suggested that, a complex representation be used.

So, now, we have two reasons to work with complex representation. One is that, I want the mean frequency to make sense. Two – I want the instantaneous frequency calculation to come out meaningfully. And 3 – of course, which is a good side effect, which is that, the amplitude of the signal, that is, a complex associate – analytic associate as we call, will be the same as the amplitude of the signal itself. So, with this, we will now turn to the discussion on analytic signals. So, the idea now, are the objective now is to come up with the complex representation such that we will force a real part of this complex representation to be the signal itself. But, we are free to choose the imaginary portion. And alternatively, we are saying is that, I am free to choose the phase.

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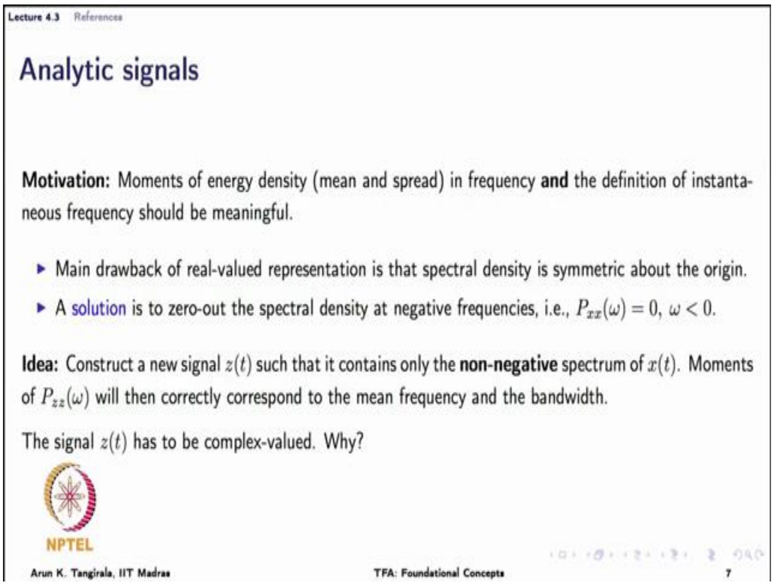

So, let us look at the main motivation as I said main drawback of real-value representation is that, spectral density is symmetric. That is the key point or the key property of the spectral density for real-valued signals, which gives me this observed result that, the mean frequency is 0. Somehow, if I can ignore the negative part of the spectral density and only work with the positive side of the spectral density; then the mean frequency calculation will come out all right. So, again recall the example that we have seen. So, a solution; there are many possible solutions, but a simple solution is to say I am going to construct a new signal out of the given signal such that, the spectral density of the new signal at negative frequency is 0. In fact, this is what you will find as a definition of analytic signal in many texts; all analytic signals will have zero spectral density at negative frequencies. So, the idea is now to construct a new signal z of t such that it only contains non-negative spectrum of x of t. And moreover…

And, what happens is now, the spectral density. I have P here; you should read it as s z z; it is essentially the spectral density – energy spectral density. When I force z of t to be having this property, we will show that, the mean frequency calculations and the instantaneous frequency calculations will come out all right. The question is why should z of t be complex value? The answer is fairly obvious when I have real-valued signals, the spectral density is symmetric whenever I have an asymmetric Fourier transform. Why I have an asymmetric situation here? Because I want the spectral density of z of t, which is the new representation of x of t; I want it to be 0 over negative frequencies. And to be whatever is that of the original signal for the non-negative frequency. So, the energy spectral density now of z of t is asymmetric. Naturally, its Fourier transform is also going to be asymmetric. And whenever I have asymmetric Fourier transforms; that means in time domain, the signal representation is complex or signal is complex valued. So, it is clear that, this new representation that I want has to be complex valued.

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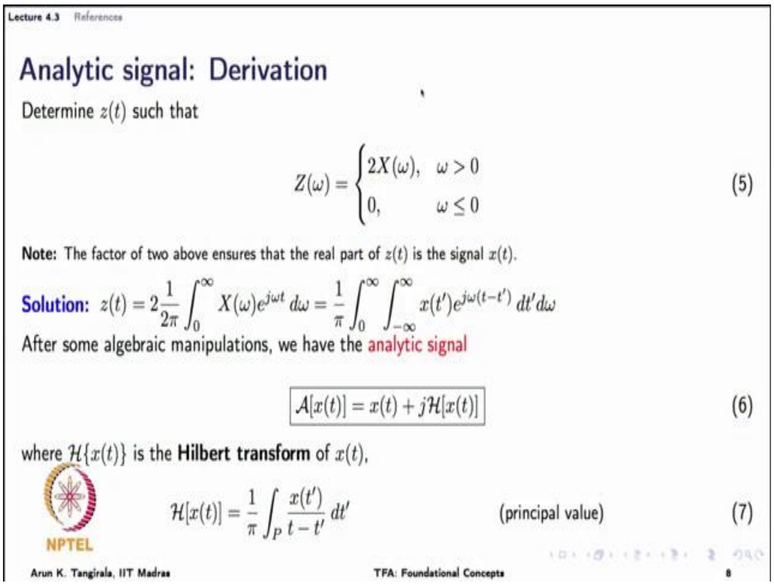

Now, with this, we will now determine the z of t by requiring that, z of omega, that is, its Fourier transform be 2 times x of omega for positive frequencies and 0 for non-negative frequency. I am also forcing the dc component go to zero. There is a reason for that. But, the main question or the key question that you should ask is why do I have factor of 2? Because I only want the spectral density… I want the spectral density of z to match the spectral density of x for non-negative frequency. So, why do I have this factor of 2? You will see that shortly. If I do not have… because I want the real part of z to correspond to the signal itself. And here this point will be clear in the derivation here. So, the solution to this is I begin by taking the inverse Fourier transform of z. So, z of t is an inverse Fourier transform. And I am only integrating from 0 to infinity here. And the reason… because the reason is z of omega is 0 for negative frequencies.

And then I have here 1 over pi 0 to infinity minus infinity to infinity x of t prime e to j omega t minus t prime d t prime d omega. Where did I get this from? I have substituted the expression for x of omega as a Fourier transform itself. This 1 over 2 pi comes from the fact that, we are working with – angular frequencies. If I am working with cyclic frequencies, this 1 over 2 pi would not come into picture. Remember – we are dealing with continuous time signals here. Therefore, the frequencies will run from – ideally from minus infinity to infinity, but z of omega is 0 for negative frequencies or including zero. And therefore, I only restrict the integral from 0 to infinity.

Now, I can actually work through some algebraic manipulations, where we have this; then we arrive at this analytic signal. What is the algebraic manipulation? I use an important property of the integral e to the j omega x. This simple expression is given in Cohen's book; I would suggest that you refer to the Cohen's book. It is an expression for the integral of 0 to infinity e to the j omega x dx. It involves two things: one – it involves the direct delta function; and the other – involves j over t minus t prime. So, let me just right that expression for you.

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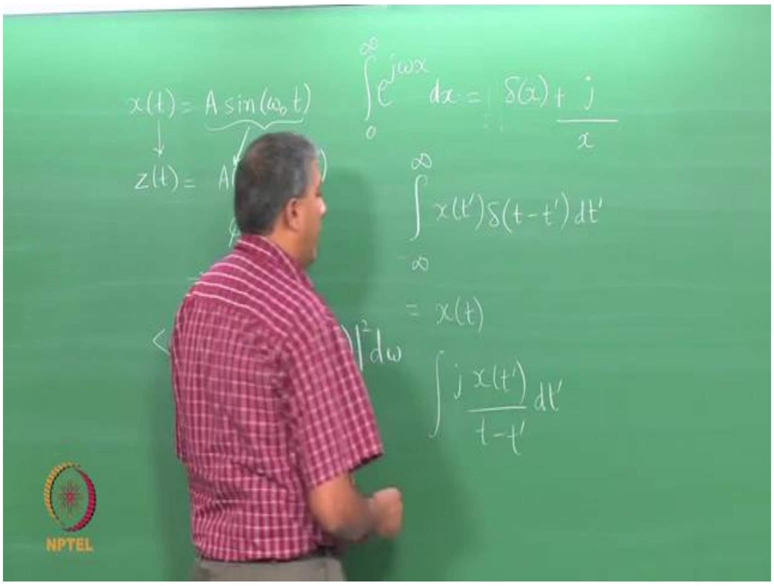

So, the integral 0 to infinity e to the j omega x dx is given by 1 over pi delta x plus j over x. This is the expression that one obtains; there is no 1 over pi here. So, delta x plus j over x. Now, you apply this result to the given integral. What you do is – if you look at the expression here that I have; I have x of t prime times e to the j omega t minus t prime dt prime d omega. First, I am going to evaluate the integral 0 to infinity e to the j omega t minus t prime d omega; and I apply this result. And then I evaluate the inner integral minus infinity to infinity dt prime. When I do that, I have an integral of x of t times delta of t minus t prime. Remember on the board, we have x; but in the integral, we have t minus t prime. Therefore, what I would end up with after using the result is the first term being minus infinity to infinity x of t prime times delta of t minus t prime dt prime. So, let me write that expression for you.

So, the first term would come out to be minus infinity to infinity x of t prime times delta of t minus t prime dt prime. Now, using the sampling prosperity of the direct delta function, whenever I have an integral of this form; by definition, this will yield me the value of x at time t. This is the fundamental property of a direct delta function. Then, I have a second term; where, I have j times x of t prime over t minus t prime dt prime. So, that constitutes the second term that you see on the slide. And that is given here by 1 over pi integral x of t prime by t minus t prime dt prime. The j term being factored out here. This integral here – 1 over pi integral x of t prime by t minus t prime dt prime is known as the Hilbert transform of the signal.

Now, there is a subscript p here; that indicates that, I am evaluating the principal value of this integral. What this means is that, when I evaluate this integral, I will run into a singularity at whenever t prime runs from t; whenever t prime equals t. Remember the limits for t prime is minus infinity to infinity. So, at some point, t prime will hit t; at that point, you will run into singularity. So, to avoid that singularity, there is a special way of evaluating this integral. And that is known as the Cauchy principal value; the resulting integral. So, in evaluating the Hilbert transform of x of t, I evaluate this integral, but using a principal value . So, the analytic representation of a signal is x of t is a complex number, whose real part is the signal itself; and the imaginary part is the Hilbert transform. This is a very beautiful result.

Now, in matlab, you have this command Hilbert; but you have to be careful; it returns not just the Hilbert transform, but the analytic signal representation itself. So, keep that in mind when you use the Hilbert routine in matlab.

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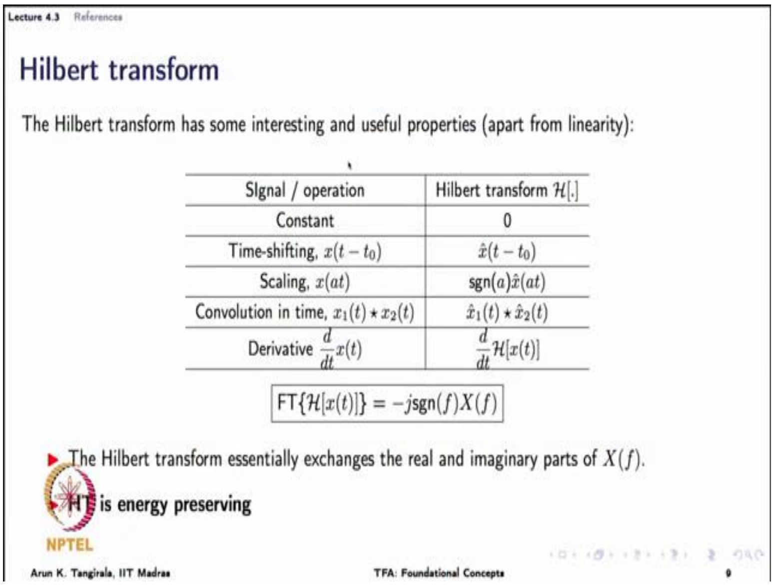

So, these are some useful properties of the Hilbert transform. There are some interesting… These are quite interesting and actually useful. So, for example, the Hilbert transform of a constant is 0. What this means is when I take the inverse Hilbert transform of a signal, there is always going to be an ambiguity of a constant. And that can actually cause problems, which we will talk about in the next module. And then I have this property when I have time shifting in time. The Hilbert transform also shifts in time. Remember – unlike the Fourier transform, Hilbert transform is also function of time t. So, you are not taken to any new domain. So, there are several other properties such as the Hilbert transform of the derivative, is a derivative of the Hilbert transform itself. So, you can use this to derive Hilbert transform of other signals.

And, this particular result that I have shown in the box is useful to understand what Hilbert transform does in the Fourier domain. So, the Fourier transform of the Hilbert transform is nothing but minus j times sine of f times x of f. So, what Hilbert transform is actually doing is if you work out the math here, it essentially exchanges a real and imaginary parts, because you have this multiplication of j in front of x of f. It does not really alter the Fourier transform of the signal. What it does is it exchanges the real and imaginary portions of the Fourier transform and also changes a sign depending on in which frequency you are operating. Most importantly, Hilbert transform is energy preserving. What this means is the energy of the analytic representation of the signal will be twice the energy of the real signal, because the energy of the signal is going to be mod z of t square. That is going to be mod x of t square plus modulus of the Hilbert transform square. But, because Hilbert transform is energy preserving, modulus of Hilbert transform square is going to be the same as the energy of the signal itself. Therefore, the energy of the analytic signal is twice that of the signal itself.

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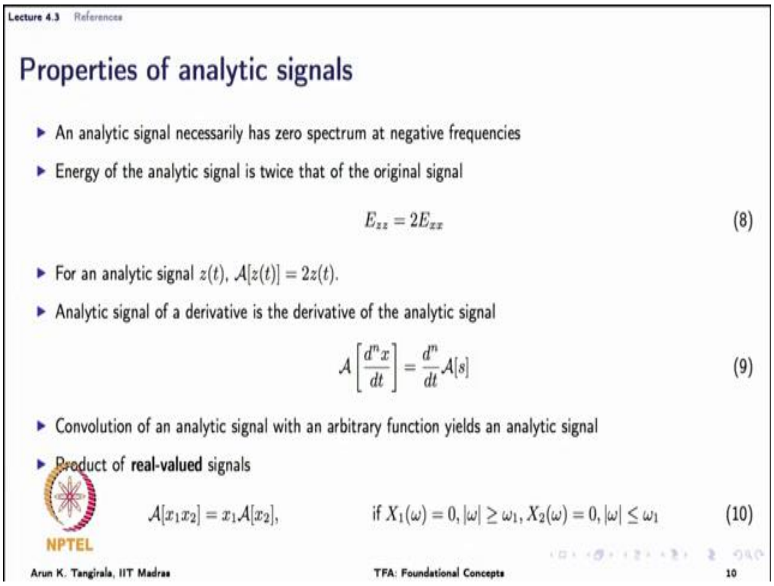

There are some useful properties of analytic signals. The one that you see is the second one is what we have just discussed. And most of these properties follow suit from the property of Hilbert transform. And the last property particularly that I have listed here; the analytic representation of a product of real-valued signals is nothing but x 1 times the analytic representation of x 2 provided x 1 is band limited in this frequency range minus omega 1 to omega 1. And x 2 is band limited in this frequency range. This is useful in evaluating the analytic representations of a product of signals.

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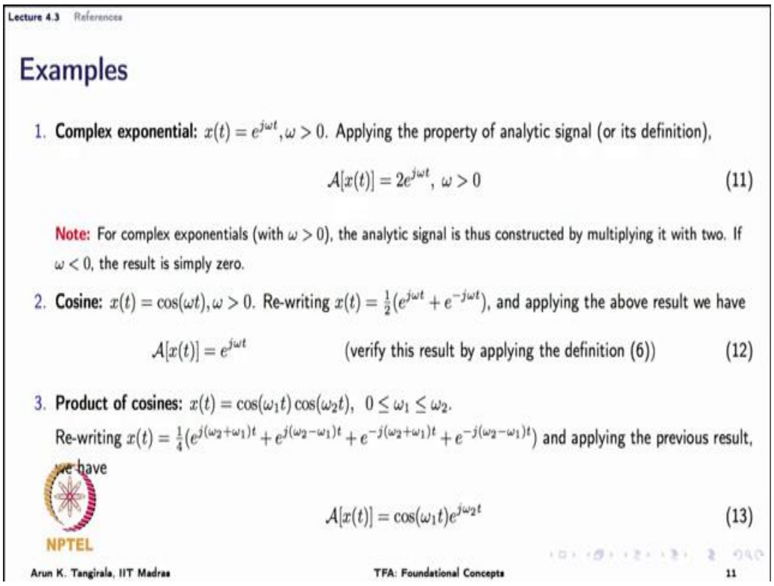

So, I will just want to conclude with a few examples that will show me how to compute the analytic representations and also point out the limitations of instantaneous frequencies. Now, when I have a complex exponential, which says omega greater than 0, it is already… This particular signal is already analytic, because it is not defined for negative frequencies; however, still if I go ahead and choose an analytic representation of this, I get twice of e to the j omega t. And this is a property that we use in deriving the analytic property of the cosine. I am going to leave that derivation of the analytic representation of cosine to you. It is fairly obvious to use this property here. And also you can use this expression, that is, the first result here to derive the analytic expression or the analytic representation of the product of cosines.

Again, we use the same property; we use Euler’s expansion for cosines; and then use analytic expressions here. Something that you should observe; which is a very important point here. The analytic representation of the signal here is cosine omega 1 t times e to the j omega 2 t. That is very interesting. So, I begin with the product of cosines, where omega 1 is less than omega 2. What the analytic representation has done is it has taken the lower frequencies and lumped that into the amplitude. If I compare with the complex representation, then A of t is cosine omega 1 t and c of t is omega 2 t. So, the phase has higher frequency content than the amplitude. This is always the case with analytic representation. This gives us a nice insight into – physical insight into what the analytic signal is doing; it is taking the lower frequencies; putting it into the amplitude and then taking the higher frequencies and putting it into the phase. So, that is something to remember.

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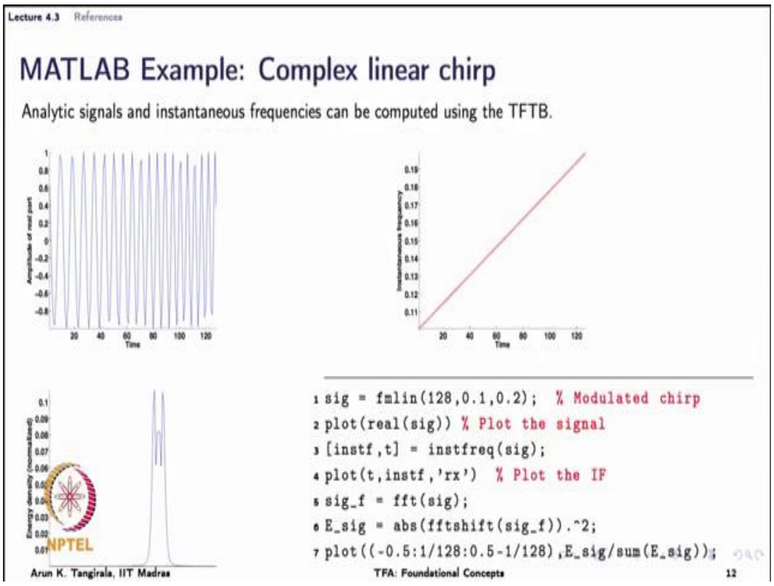

And again, we have now matlab example. So, what I am doing here is I have a complex linear chirp; I am only showing the real value; I use the time-frequency tool box to generate the signal as well as to compute the instantaneous frequencies. The instfreq routine in the time-frequency tool box allows you to compute the instantaneous frequencies. Now, the instantaneous frequency versus time is shown here. Because I say it is a linear chirp, what this means is linear frequency modulation. You will see a linear plot here of instantaneous frequency versus time; it is beautiful. So, I have now directly the time-frequency analysis of this signal; I know how the frequency is changing with time very nicely. So, it is very nicely resolved. So, the instantaneous frequency seems to be a very nice concept to analyze time-frequency analysis.

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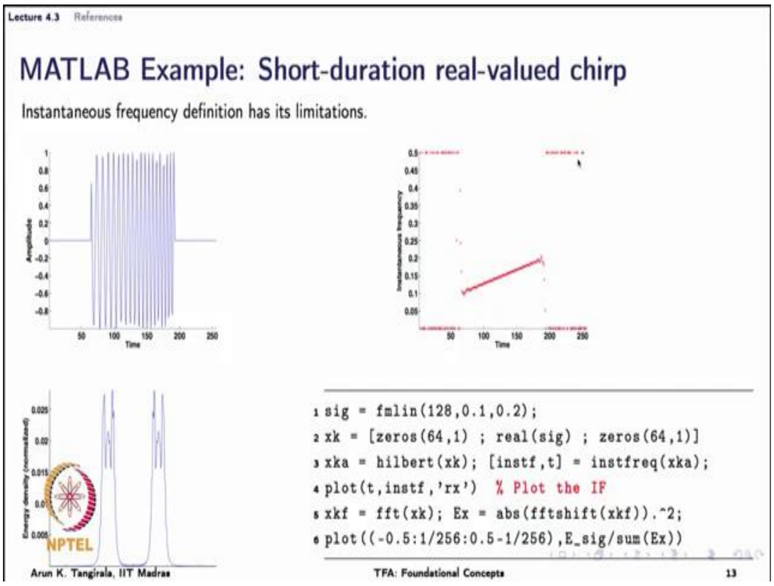

However, it has certain limitations as we will see in the next example. This example is again to show you how useful instantaneous frequencies are. In this case, I am deliberately generating a real-valued signal to show you that, to compute instantaneous frequencies, you have to work with the analytic representations. So, this is the signal of interest to me – 0; this is similar to what we have seen earlier, and then a sinusoid; and then followed by zeroes. So, I generate the analytic representation and compute the instantaneous frequency. You can see the plot here. It says that, in this period, there is no frequency while it is 0 or 0.5, which are more or less the same. And then during this portion, I have a linear frequency modulation. So, I have taken. Remember I am generating a linear chirp, but I am taking the real part and then taking the Hilbert transform. Therefore, it is able to correctly determine the frequency modulation. And after this time, again the instantaneous frequency goes back to 0. So, it has actually done a very good job of detecting the frequency variations in time.

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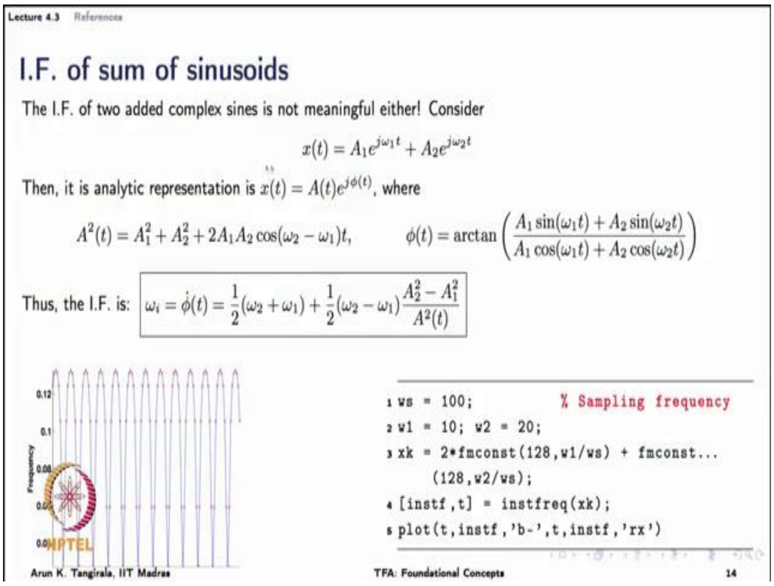

However, when I have a sum of sinusoids like this, then the analytic representation of this sum of sinusoids is fairly complicated; which is given by this expression here – A and phi given here. I think you should take it as an exercise and show that, indeed A of t and phi of t are this for this complex sine wave. Then, to compute the instantaneous frequency, I take the derivative of the phase; and it is fairly complicated. Look at what is happening; the instantaneous frequency does not give me omega 2 and omega 1 separately. It cannot because I have two frequencies at any given time. So, which frequency will it pick? It picks the average and then some product of the difference with this amplitude modulation. So, it is a fairly complicated thing. So, it is not able to tell me that there are two frequencies only; it is actually giving me a function of those two frequencies. This is the fundamental limitation of working with instantaneous frequency as we have defined.

Whenever I have a mono component signal; which means I have a single frequency at any given time; then the instantaneous frequency definition will match with what is happening in the signal. But, when I have more than one frequency at any given time, it is called a multi-component signal. Then, the instantaneous frequency calculations will give me some kind of meaningless results. Now, there is a way out for this; which is given in the form of empirical mode decomposition; which we will talk about it in the next module. But, this is what is the main message of this example.

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So, here is again couple of references for your perusal. And hope you enjoyed the lecture.

Thank you.