Introduction to Time-Frequency Analysis and Wavelet Transforms
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Lecture - 4.2
Bandwidth equation and Instantaneous frequency

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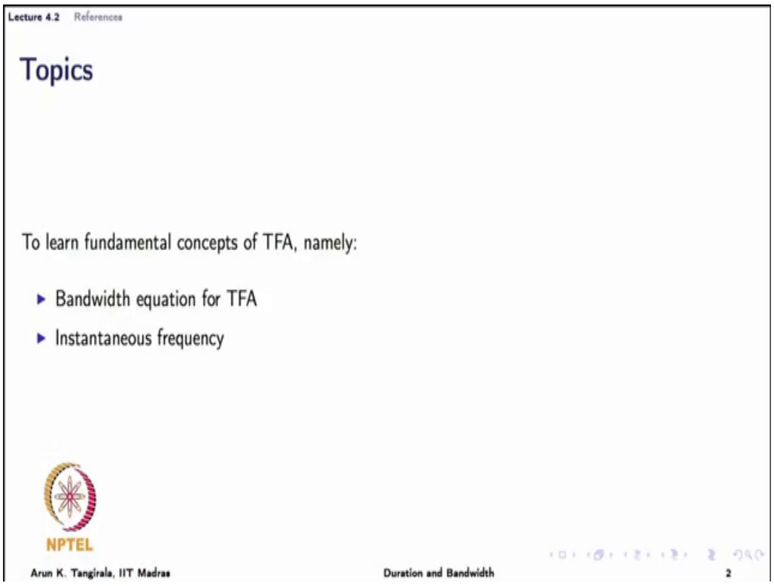

Hello friends, welcome to lecture 4.2 on bandwidth equation primarily, and we will also introduce the concept of instantaneous frequency. So, this is a continuation of what we have discussed in lecture 4.1. The objective again is to learn these specific concepts in time frequency analysis, which is the bandwidth equation for TFA that is in the context of time frequency analysis, and then obtain an introduction to the concept of instantaneous frequency. We will continue the discuss on instantaneous frequency in the next lecture.

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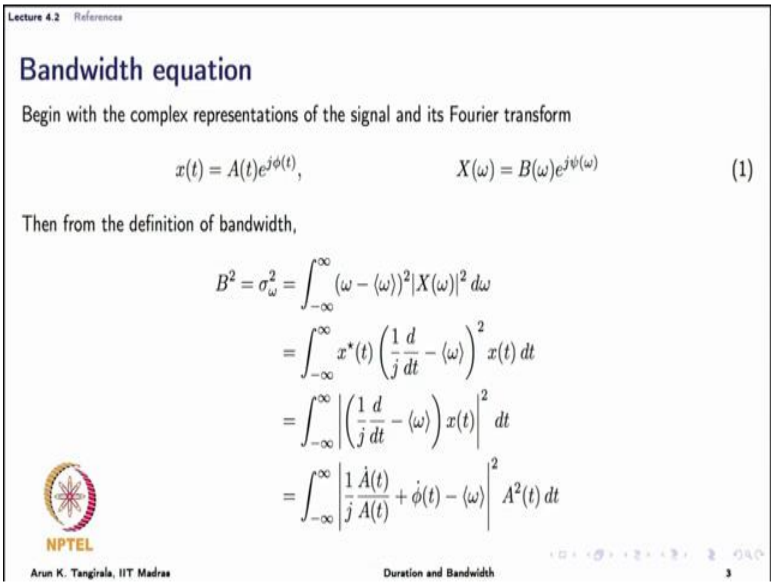

So, let us begin with these signal representations, complex representations, we concluded the last lecture with an example on why, and how complex representations can help us in rightly determining the center frequency of even a simple sine wave. So, therefore we will assume henceforth that I would like to work with complex representations. So, this is a complex representation for x of t, and here I have the complex representation for x of omega.

What are this phi of t and phi of omega is not really, so much of a concern at this moment, our objective is to derive the expression for the bandwidth. So, these representations and the definition of bandwidth, and also the simple trick that we went over in the definition of bandwidth you should go and refer to that is slide. We can rewrite this integral here in terms of the signal in time itself, where this d over d t is now as an operator operating on x of t.

Now, we could rewrite this product of x star of t and this square of 1 over j d over d t minus average of omega times x of t, simply has the modulus square, that is a fairly simple algebra to go from this integral to this integral. We should recognized that essentially the integrand here is a modulus square of this quantity 1 over j d over d t minus average of omega times x of t. Now what I do here is I bringing the expression for x of t which is a of t times e to the j phi of t, and the derivative of x of t would have two terms applying the chain rule. The first one would be a dot of t e to the j phi of t, and the second term would be a of t times phi dot of t times e to the j phi of t. So, what I have done here is, I have essentially taken a of t out of this the factor within the modulus and just rewritten it. So, that is see only difference here otherwise the algebra is fairly straight forward.

Now, this is a complex number phi dot of t is a real number, average of omega is a real number. So, phi dot of t minus average of omega is a real part, and 1 over j a dot of t over a of t is a imaginary part. So, you are looking at magnitude square of a complex number.

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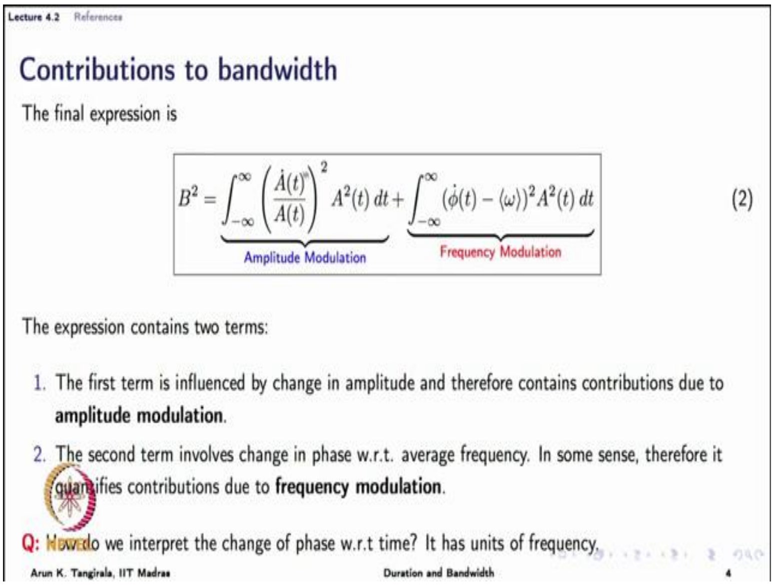

So, this essentially if I take this square magnitude, I will have two terms; one being magnitude of a dot of t over a t of square times a square of t here. So, I have here a dot of t over a of t square times a square of t d t, of course I could cancel out, but there is just for simplicity say got for a point that we want to illustrate we have retain the this way, and a second expression would be phi dot of t minus new omega, average of omega square times a square of t d t. This is a fundamental equation known as a bandwidth equation, and what reminds is the interpretation of these two terms. Because this first term goes to zero whenever a of t is constant that is whenever a dot of t is zero; that means, whenever there is no amplitude modulation, this a dot is nothing but the derivative of a of t. Whenever there is no amplitude modulation this term would vanish, and whenever a of t is non zero, this term will exist.

And therefore, this term solely accounts for the amplitude modulation contributions to the bandwidth. And here I have phi dot of t minus average of omega square a square of t d t. Now look at this way a square of t d t is nothing but my energy density in time, and average of omega is the mean frequency here, and I also have here phi dot of t. At this moment although I say frequency modulation, it is not immediately clear why this should be frequency modulation unless phi dot of t really only has some units of frequency or can be given some interpretation of the frequency. So, how do interpret this change of phase with respect to time. Remember phi of t is called the phase as we have even defined earlier in the basic definitions module. It has units of frequency obviously, because you are subtracting the average frequency.

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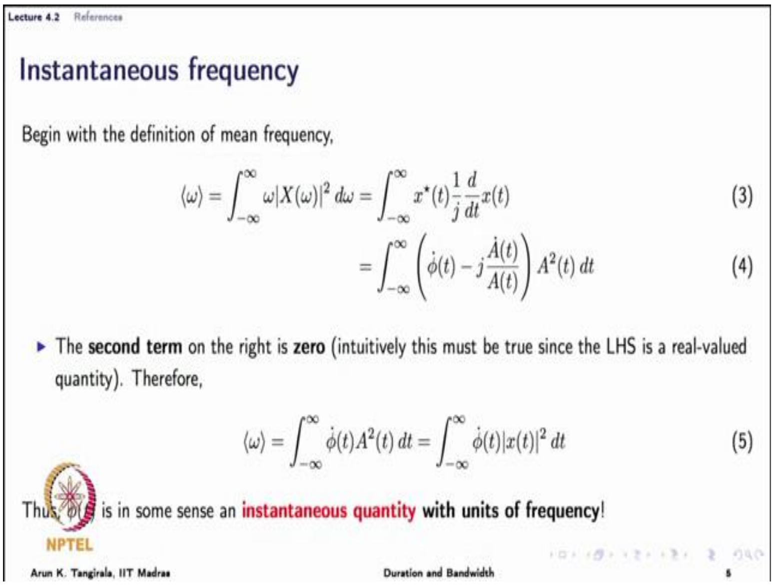

So, to understand phi dot of t we go back to the definition of mean frequency again invoke the definition, rewrite the same definition in terms of x of t rather x star of t and x of t both, and what we do here is we substitute again for x of t, the signal representation that I have here and obtain this expression. Now, if I look at this expression the left hand side is a real number, average of omega is real number, whereas on the right hand side I have two terms; the first term is a purely real number - it is integral phi dot of t times a square of t d t, and a second term is a real imaginary number. I have a real number on the left hand side, and I have right complex number on the right hand side. So, if things have to agree, I know this definition is I have started from the definition so everything is consistent.

Therefore, the imaginary portion has to go to zero, that is an intuitive way of zeroing out the second term, if you are not convinced then you can evaluate the integral for the second term, and you will quickly see that is a perfect integrand, and therefore it goes to perfect integrand, therefore it goes to zero.

So, what we are left with is this beautiful result which says that the mean frequency although define this way turns out to be phi dot of t integral of phi dot of t a square of t d t. Remember a square of t d t is the energy density in time, and on the left hand side I have an average quantity, therefore phi dot of t has to have some interpretation of a instantaneous frequency, and that is very important. So, what we are saying is, on the left hand side I have average. How do I obtain averages? The way obtain averages is I take the instantaneous value, and roughly way it with the energy density times d t, and take the integral, that is how we have even derived the mean time.

Now that is a same story here, I have average of omega being phi dot of t times a square of t d t, and therefore phi dot of t has to have some interpretation of an instantaneous are a local frequency and that is where the concept of well, mathematical definition of instantaneous frequency is one. The concept of instantaneous frequency is the does not require all this mathematical formulation, it is a fairly easy think to imagine, it is a physical concept it is essentially the frequency of the signal at a given point in time. But this phase way for the mathematical definition of instantaneous frequency, it is essentially the derivative of the phase which phase when I this phase that appears in the complex representation of the signal.

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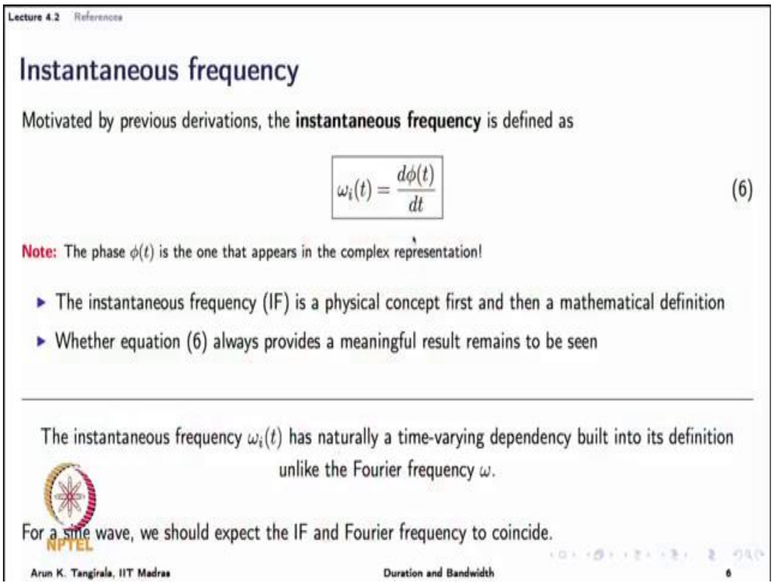

Now, I define this instantaneous frequency as the derivative of the phase, again this phase is not the regular phase that we talk of in sine of omega t plus phi and so on. This phase is the phase that appears in the complex representation of the signal, in particular we will say that the analytical representation of the signal. Now, of course what remains to be seen is whether this mathematical definition, tally is and in reality gives me the correct value of the instantaneous frequency of a signal and so on. That is something that will discuss later on, but clearly because phi dot of t is in general a function of time this omega is also function of time sub denoted by the subscript i.

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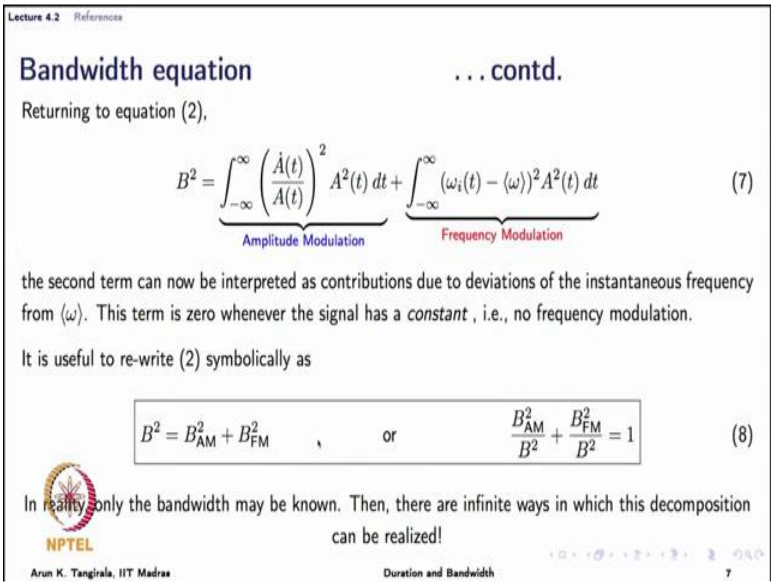

And now we written to the bandwidth equation in place of phi dot of t I substitute omega i of t. So, this term now is nice is ready for interpretation, I can straight away interpret this as the contribution of the frequency modulation, because whenever there is a frequency modulation frequency is going to change with time. In which case omega i is not going to be equal to the average frequency, but when I have a sine wave of a fix frequency then the instantaneous frequency is going to be fix in time exactly equal to the average frequency itself, therefore this term will go to zero.

In other words when there is no frequency modulation, there is this term will vanish and whenever there is a frequency modulation this in general with exist, and therefore I can say this term contributes, the represents the frequency modulation contributions to the bandwidth. Now I have this beautiful equation here in equation 8, the bandwidth is a some of two terms; one due to amplitude modulation, other due to frequency modulation. This is a normalized version of this expression. This goes on to show that the bandwidth by itself cannot tell me whether the signal was being amplitude modulated or frequency modulated. In fact I can think of the signal that is given to me as another signal with a different amplitude modulation and frequency modulation.

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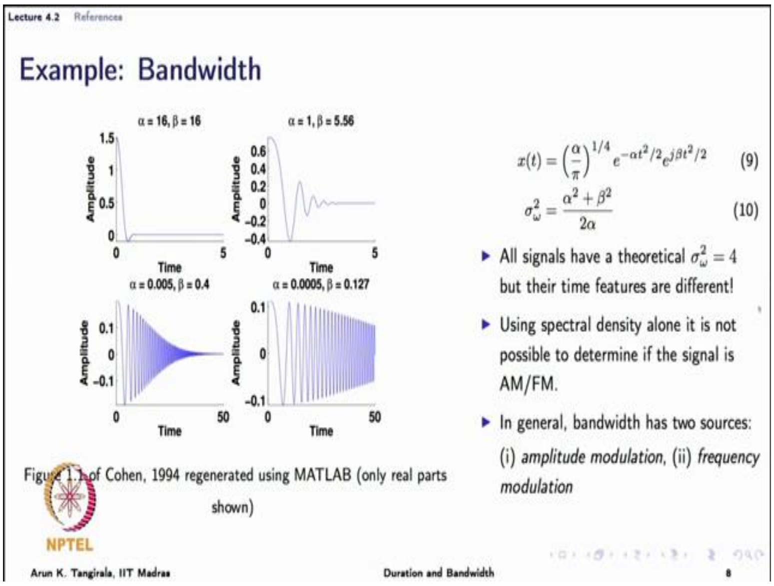

So, that the overall bandwidth the same for both, that is exactly the example that we had discussed in the previous lecture and written to that, I have here this same signal expression that I have given earlier. Now look at this. So, this part here represents a amplitude modulation alpha by pi raise to 1 over 4 times e to the minus alpha t square by 2 and e to the j beta t square by 2 represents the frequency contributions, and frequency modulation. If you plug in this expression into the expression for the bandwidth, you can derive sigma square omega as alpha square plus beta square by two alpha. When alpha is 0, you can say that essentially the signal is zero of course, but when alpha goes to very small value, then the amplitude modulation comes down. And when beta is zero essentially, there is no frequency modulation ((Refer Time: 11:36)) a pure amplitude ((Refer Time: 11:38)) signal, and these four signals here use different values of alpha and beta to show you how the signal time profiles are affected by different choices of alpha and beta, I have here very high value of alpha and very high value of beta. And then again I come here to the extreme case, where very low values of alpha and beta being relatively greater than alpha.

And what I want to show you here, it is that I have different values of alpha and beta all giving you the same bandwidth but completely different time profiles, because as I change alpha and beta, I am changing the extent of amplitude and frequency modulations, but preserving the bandwidth. So, looking at the bandwidth alone I will not be able to determine which of these signals is the actual time profiles. So, looking at a spectral density alone will not tell me, give me an idea of what is happening in time, so that is the main point that I want to illustrate.

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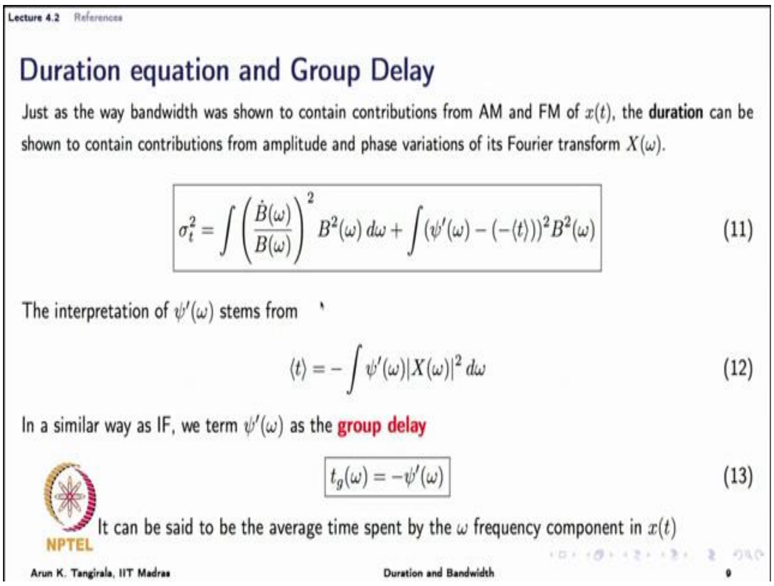

So, will conclude this lecture with this concept of group delay that we may or may not really a pursue extensively in this course, this group delay is an analog of the instantaneous frequency, its of instantaneous frequency is a frequency that is a function of time, and it tells me at this instant what is the frequency? The group delay will tell me the average time spend by a particular frequency component.

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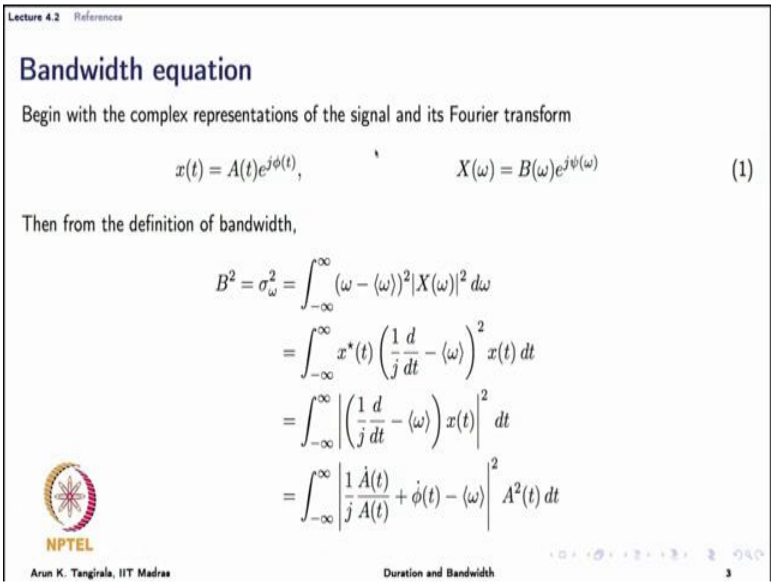

The definition of group delay is given here. It is essentially remember our instantaneous frequency was based on phi of t here, the phase in the complex representation of x of t. So, naturally you should expect the group delay to be the dependent on the phase, but of the complex representation of its fourier transform, that is exactly what has occurred here. So, I have group delay as a negative derivative of psi prime of omega. Again in order to deriving instantaneous frequency, we started with the definition of mean frequency, in order to derive the group delay, we start with the definition of average time. And then we realize that the average time can be represented as psi prime of omega times mod x of omega square d omega.

Once again the same interpretations and arguments can be given here. Psi prime of omega should represents some local function of time, a function omega which has units of time. So, that it is averaged by this spectral density to give me average time. And therefore, because of the negative sign here, the group delay is the negative derivative of psi of omega. As I said group delay has interpretation that it is average times spend by this frequency component of frequency omega in the signal. That is a nice piece of information as well I would like to know basically may be there is some frequencies, which has spent longer time in the signal some frequencies which has spent shorter times that will help me in feature extraction and analysis of the signal.

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So, is a couple of references again for your reading. In the next lecture we are going to talk about instantaneous frequencies and analytical signals.

Thank you.