Introduction to Time-Frequency Analysis and Wavelet Transforms
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture No - 4.1.
Duration and Bandwidth

Hello friends, welcome to lecture 4.1. This is the first module in the unit on time frequency analysis that is on the foundational concepts of time frequency analysis this is where our formal study of time frequency analysis begins. In this particular lecture, we are going to talk about the concepts of duration and band width in a formal manner. We have seen these concepts before at least in a qualitative sense we have some feel for them,, but today we learn the formal definitions.

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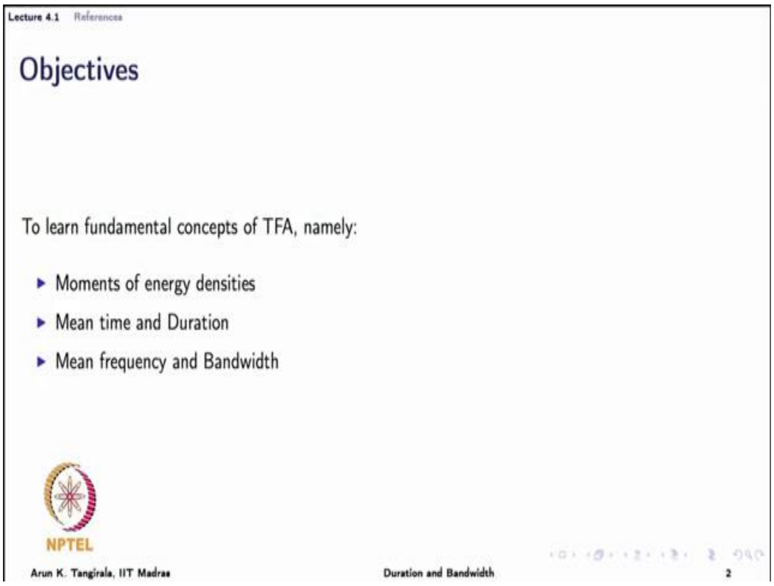

So, the objectives of this module is, to learn the fundamental concepts of time frequency analysis not all of them,, but this is the first module in that journey. Specifically we look at the moments of energy densities namely mean time and duration and mean frequency and band width. A lot of this these definitions are easy to follow again if you draw analogies with probability density functions as I had mentioned in the previous modules as well. Now, one of the prime purposes of time frequency analysis is to know what frequency content is present in a signal if not at a particular time instant,, but over a small interval of time. So, the until now we have seen at least a couple of examples in the previous module all like unit particularly, that the Fourier transform or particular is the spectral analysis is unable to give us that piece of information.

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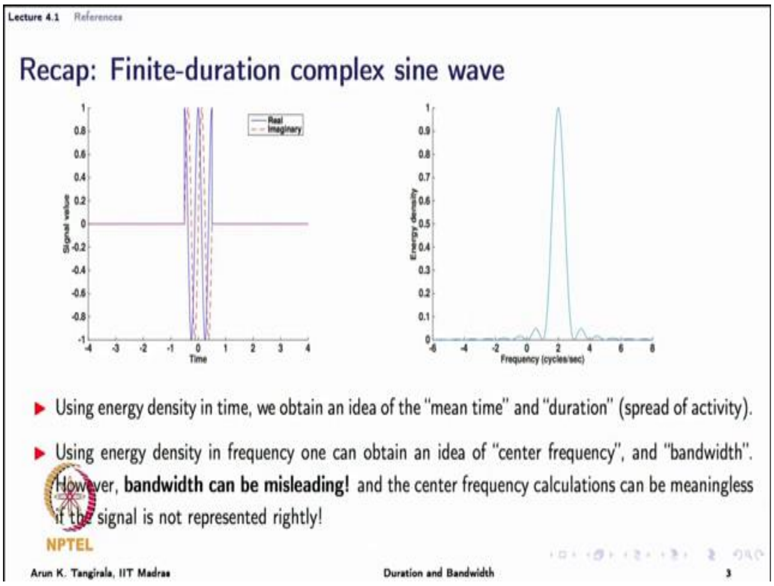

So, let us recap this example in that context. So, this is an example that we have seen before in the context of continues time Fourier transforms where, we have signal that has no activity in the beginning and the end. And in the middle somewhere in the center we have a couple of cycles of a sine wave of a given frequency. Now, when we say duration or when we are looking at moments of energy density in time, we are interested in knowing what is happening in this sigma, what is the center of activity of the signal and how far the activity is spread. So, that is a vital piece of information, because let us say I am looking at I am searching for fault in a signal. Then, I would like to know these two pieces of information at least around what time the fault occurred or this particular feature of interest was present in the signal and how long was this feature present. That is a these are the two at least vital statistics . So, to speak of the signal that I am interested in. I am also interested in knowing what are the frequencies contained in the signal whenever it was active and for that I turn to the energy spectral density as we have learnt earlier also.

But what we have learnt earlier also that this band width can be misleading. So, if you can if you look at these 2 plots here, we know already that the signal wherever it is active has only a single frequency over this region here. But the energy spectral density is not reflective of this reality. If I just look at the energy spectral density without paying attention to the signal, then I am compelled to believe that there are many frequencies presents in the signal whenever they were present. So, there are 2 problems here; one that I do not know when this frequencies are present. And two; it is not just showing the frequency that is contain in the signal. Of course, the center frequencies still coincides with the actual frequency content of the signal whenever it was active. But then I have this spread of frequencies which is now giving me some misleading information’s. I am compelled to believe that there are frequencies apart from this 2 cycles per second.

So, these are the 2 problems of using the energy spectral density to determine the frequency content and the particularly the location in time of that particular frequencies of those particular frequencies at the energy spectral density is indicating. Of course, what I can do is I can keep the energy density in time and spectral density by side by side and then examine these 2 plots. But common sense tells us that it is not going to be a useful exercise in practice, because in reality signals may have more than one frequency component. In which case and also the frequencies can come and go this is a very simple example therefore, it is easy for me to determine what is happening by a visual inspection. And in practice it is going to be very difficult and that is the main motivation for turning to time frequency analysis, which will tell me what frequencies where present over a given interval of time.

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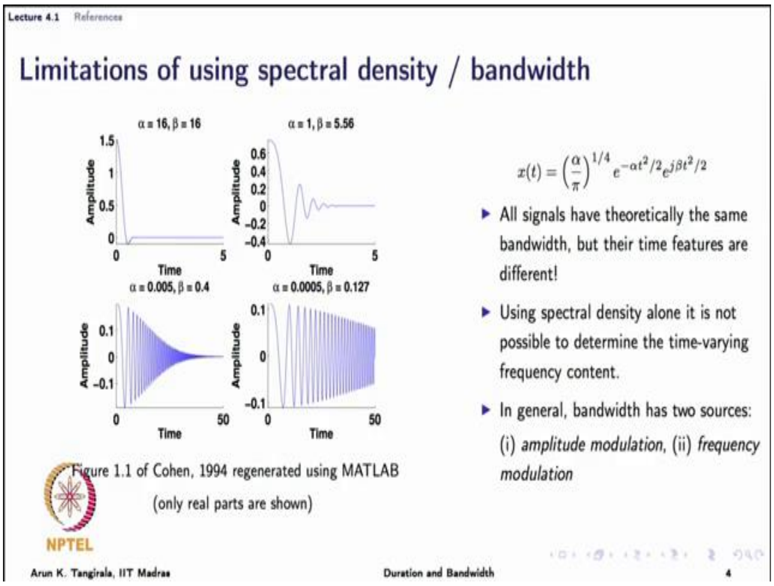

Now, there we will look at one more motivating example which tells us a limitations of using spectral density or band width. So, before I again go over this example let me take you back to this example itself. So, that I can introduce to you at least the concepts of amplitude modulation and frequency modulation. So, if you look at this energy density here in time it is representative of what is happening in the signal in time. Wherever the energy density is 0, the signal has zero activity wherever it is non zero it has non zero activity or there is something happening in the signal. Now, if you look at the beginning and the concluding portions of this signals there is no activity. I can look upon these 2 portions in 2 different waves, I can say that in these regions the frequency of the signal is 0. So, regardless of the amplitude the signal will have what we say is zero activity. Now, of course,, we can also say now that the signal had some frequency we do not know what frequency it had,, but then the amplitude of that signal over that period of time whether it is here or towards end it is 0. Remember, the signal represent in the signal representation for the periodic signal I have a cosine omega t or a slightly more generalize representations a of t cosine of theta of t.

So, I could have a of t being zero or I could have the frequency going to zero and a being non zero. So, I could actually represent the signal either as a amplitude, whenever I have change in frequency or change in frequency content I can in principle represents the signal either as a amplitude modulation or frequency modulation. The main point in discussing the limitation of spectral density is, the band width is unable to tell me in fact, this spectral energy spectral density is unable to tell me whether there was a frequency modulation or amplitude modulation. And time frequency analysis will help us discover that. So, this is an example in that direction I have 4 signals all of which have the same band width the expression for this signal is given on the right hand side at the top. All these 4 signals has we will also see in the next module will derive formal I will give you the expression for the band width for these signals the value of this band width all these 4 signals have the same band width. However, now you can see that their time domain profiles are completely different. This is an example that I have taken from Cohen’s book in fact, most of the discussion in this module and probably in this unit itself is going to closely follow the introductory chapters of Cohen’s book.

So, this quickly tells me,, but that just by looking at the band width or the spread of the energy spectral density alone I will not to be in a position to comment on what is happening in the signal. So, you can see these two this plot here is an example of an amplitude modulation and here also I have an amplitude modulation. In some of the signals I have both amplitude and frequency modulations and. So, on. We will see this later on once we derive the equation for the band width we will revisit this example. So, the message is spectral density, energy spectral density or even for that matter power spectral density will not give me the freak time varying nature of the frequency content of the signal. And as a preview to the band width equation let me tell you that the band width has contributions from both amplitude modulation and frequency modulation. And you could adjust one for the other to get the same band width, that is the basic limitation of band width.

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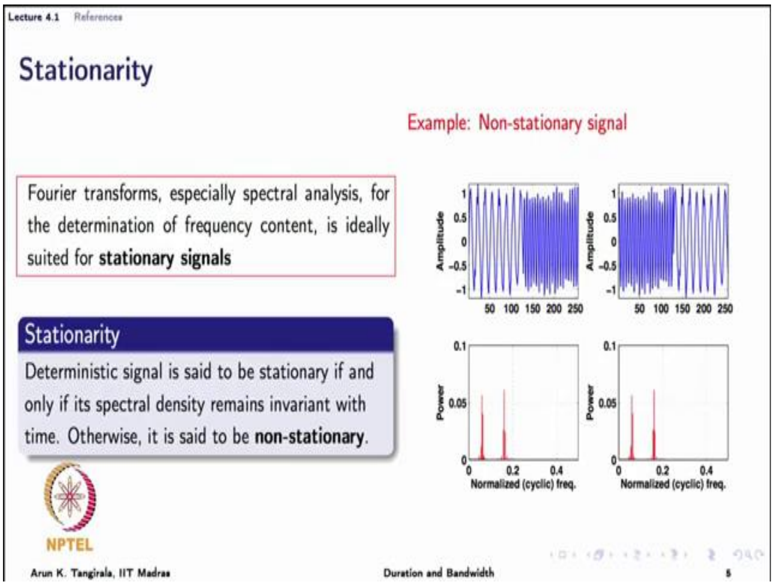

So, having discussed this time varying frequency content feature of signals it is time to define what is known as stationarity for deterministic signals. Now, the Fourier transforms especially the spectral analysis for the determination of frequency content is ideally suited only for stationary signals. What is this stationarity? The stationarity is a property of this signal which requires that the spectral density remains invariant with time. That means the frequency content should not change with time that is a very important and very strict requirement for the suitability of Fourier transforms. In fact, if you think of it more signals in reality may not even satisfy this stationarity requirement. So, let us look at this example from our introductory slides where I have the signal with 2 frequencies in succession..

In the first example I have low followed by high, and the second one I have high followed by low. Given beneath are the spectral densities you can see that the spectral densities look alike. So, it is unable to distinguish this switch of frequencies are the change infrequency content and we say these 2 signals are non-stationary in a deterministic sense. You can you also encounter this definition of stationarity in the context of random signals, where again you can define you can have a similar definition. And you can say that the random signal is stationary in particular this is second order stationary if it is spectral density remains in variant with time. So, similar definitions therefore, exist for both deterministic and stochastic signals. So, the main motivation for turning to time frequency analysis is to know how to analyze or to analyze the class of non stationary signals that is something to keep at the back of domain.

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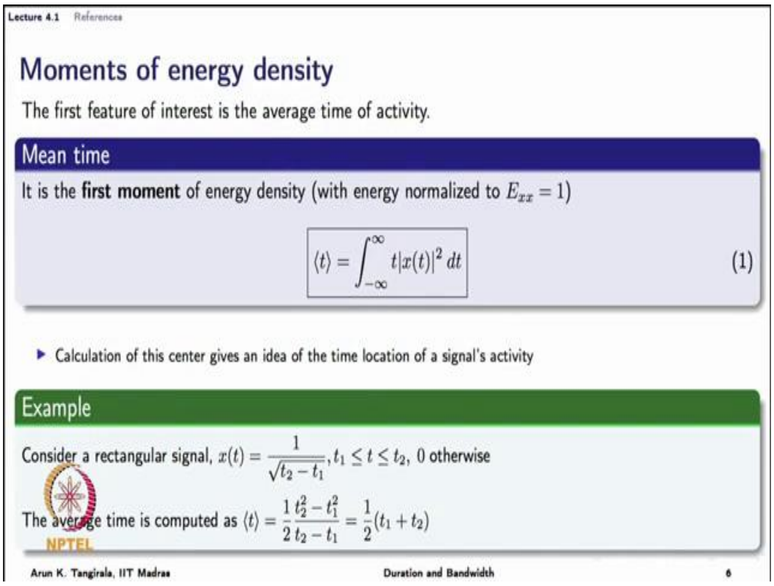

So, we had early on mention that the way I use the energy density plots or the spectral density plots and. So, on is I search for a particular feature in these density plots. In particular I am interested in knowing when a particular feature existed in the signal and the energy density is a reflection of what is happening in the signal. So, I want to know when a particular feature existed and how long this particular feature existed. Now, we formally introduce terms that will help us quantify what we are searching for, both in time and frequency. The first a quantity of interest is the mean time which is a measure of the center of activity of the signal. Now, again here as I mentioned earlier these definitions become easy to follow if you are able to recall the analogy of probability density function for a random variable where we talk about mean variants of a random variable based on the probability density function. So, let me actually show you the similarity of this expression for mean time with the mean of a random variable.

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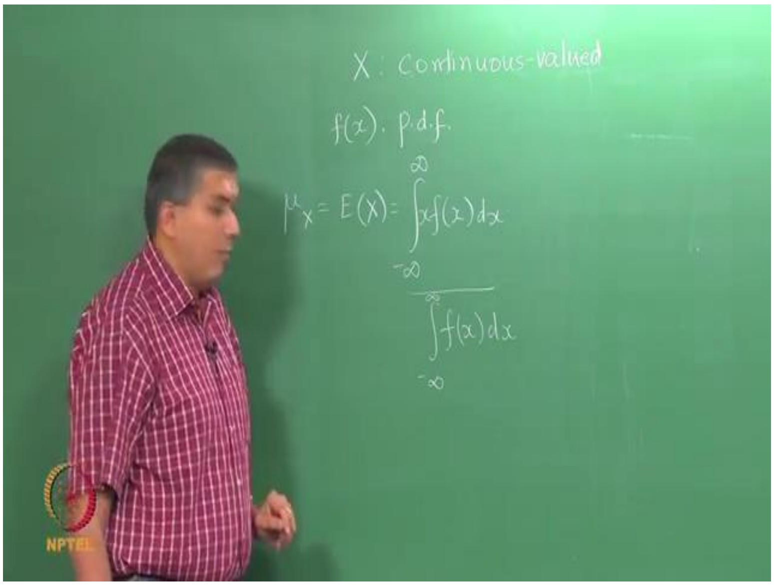

So, here I have for a random variable X they do not confuse this x with the signal x that we are using. If this random let us assume that this random variable is continuous value. So, that I can think of a pdf, normally by convention pdfs are denoted by lower case and we use a lower case x here. So, this is the pdf of this random variable when I want to know the expected outcome of the x or the center of the all possible outcomes of x. Then I turn to this measure called mean or expected value or average value which is defined as the first moment of the pdf. In this, definition we again integrate over the entire place. So, minus infinity to infinity here in the in this case refers to the extreme outcome so, you looking at the entire outcome space.

Now, strictly speaking there should be an integral f of x dx in the denominator. However, normally all pdf s are normalized to have unity area. So, that the probability of the occurrence of any occurrence of this random variable is normalized unity. Therefore, in many text books you do not see this, the denominator appearing in the definition of mean. The same applies to the definition of mean time as well. So, here we have mean time now, you can see this similarity.

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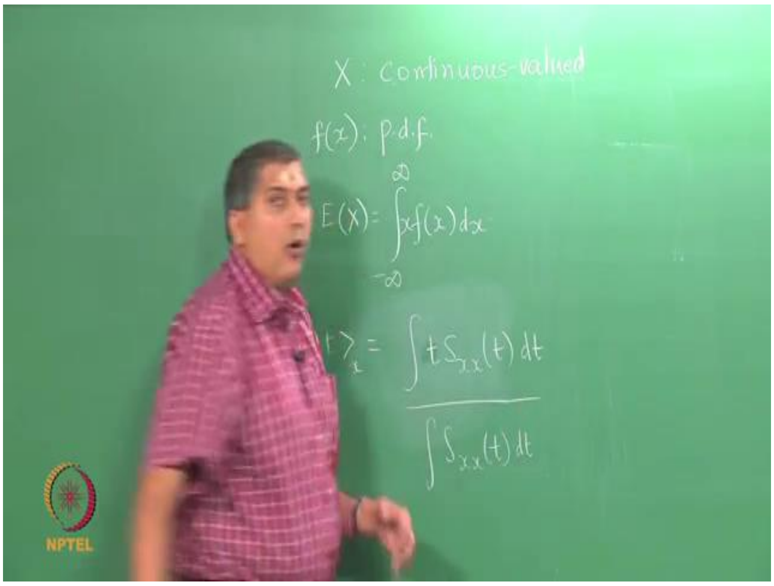

Now, here x now instead of x we have time t it does not mean the time t is are random variable here. Please notice that I am only giving this examples just as an analogy,, but not they are not necessarily identical. So, now, I am interested in the mean time of a signal x that is governed or that is given by the first moment of this energy density in time. Again ideally I should have an S x x of t dt. But if you have normalized signal to have unit energy then you can switch off this denominator and which is what I have assume in the definition as well. Therefore, we will omit this,, but you should keep at the back of your mind that there is a denominator and you should certainly make sure that the signal has been normalized to have unit energy.

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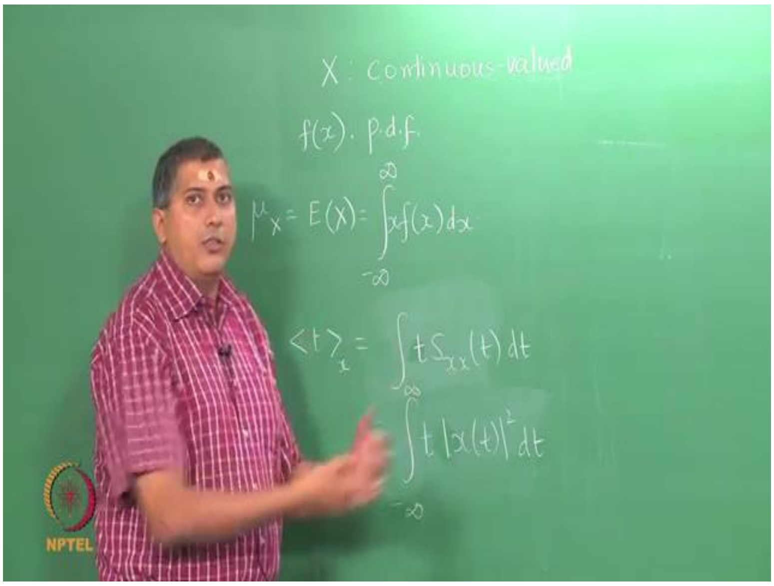

Now, we know the definition of the energy spectral density that is it. So, this is what I have from minus infinity to infinity over the entire existence of this signal. So, what does this give me? This gives me an idea of the center of activity of the signal this is the mean time. So, I hope now, that is clear that is the analogy of mean time with the mean of a random variable is clear and likewise we will you can extend these analogy to the second central moments that we shall discuss shortly. So, there is an example here in the bottom for a rectangular signal. This rectangular signal extends over this period t 1 to t 2 z. So, that is the duration over which a rectangular signal is active outside this interval it is 0. And I have here 1 over root t 2 minus t 1 you should verify that the signal given by this expression has a unit energy spectral density it is fairly obvious. Therefore, I do not need to divide by this denominator in calculating the mean time which is shown here.

The mean time here turns out to be half of t 1 plus t 2 which agrees with our intuitive understanding of this center of time when I have a rectangular signal with in this interval t 1 to 2; obviously, the center of activity has to be at t 1 plus t 2 by 2. Therefore, this expression really gives me the correct answer,, but for more complicated signals is a very simple signal. These expressions are going to give me the correct answer,, but my problem will be to arrive at the intuitive answer, because rectangular signal is a very symmetric it is a very flat signal and so, on. But in practice signals are going to have complicated shapes therefore, I need to have a feel for the answer and that is going to make difficult. The best way is therefore, to calculate the center of time using this expression. You will have plenty of opportunities in the assignments to evaluate the center of time for other signals as well.

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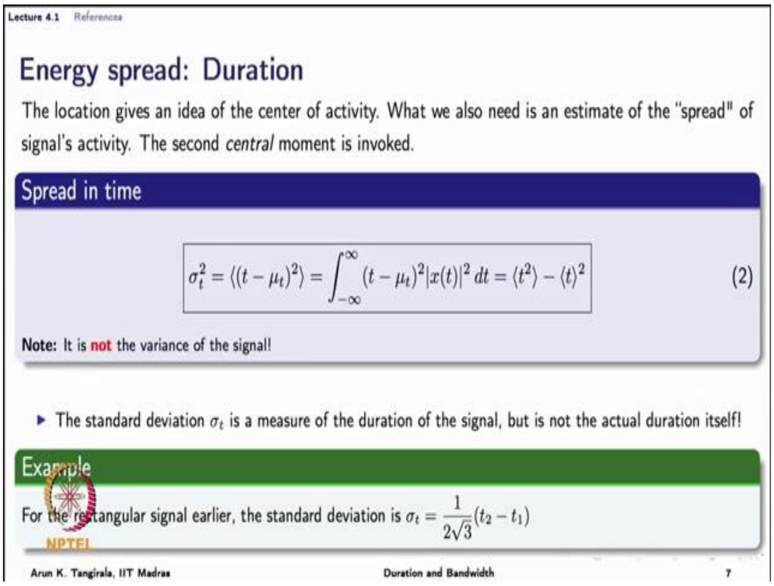

So, we turn now to the second central moment, because just knowing the center of activity of the signal is not sufficient. I need to know how long this particular feature or activity was present. And naturally I turn to what is known as the spread or the second central moment again in analogy with the variance that we have for a random variable. Variance of a random variable is also the second central moment evaluated using the probability density function. So, is denoted by sigma square t this definition is a hopefully straight forward here, t minus mu t square the average the brackets notation sorry I have not mentioned earlier. The left hand right angle brackets here denote average averaging operation, similar to the expectation operation that we conduct for random variables. So, central moment it is not simply average of t square. Again, I use the energy density to compute this second central moment, it is fairly easy to show likely show even in the case of random variables that, this spread can be written as average of t square plus the square of the average. So, that is a very standard result to show.

Here, we have assumed that the energy density has been normalized to have unit area alright. Now, a big word of caution this is not the variance of the signal. Just, because I see a symbol sigma square t here this is not a variance of signal. In fact, the subscript t should tell you that, I am evaluating the variance here in the sense of time not for the signal and again time is not a random variable here. So, let us not draw some absurd conclusions from this expression. Now, this positive square root of this variance is called a standard deviation and it is a measure of the duration of the signal. Again, for the rectangular signal the standard deviation is given by 1 over 2 root 3 t 2 minus t 1. The way you look interpret the standard deviation is the same way that you do for random variables. The standard deviation will give me a measure of the spread, it is not the entire spread itself. So, for example if I have a Gaussian distributed random variable; that means, with my pdf looks like as a bell shape then approximately 95 percent of the outcome space are is actually covered by 2 standard deviations. And 3 standard deviations will cover 99 percent and. So, on..

The same applies here if the energy density here is a bell shape 1 then, 2 stand 1 standard deviation will give me idea of about 66 percent of the signals overall actual duration in time and. So, on. One thing that we can definitely say if sigma t is large then the signal duration is also going to be large. However, the actual interpretation of sigma t with respect to duration depends on the shape of the energy density. Like in the case of random variables, iit has a () square distribution random variable then the sigma t does not necessarily straight away give me complete idea of the duration. But it still representative of the overall existence of the signal or the duration of the signal. So, henceforth we will say here due by duration will always mean sigma square t. So, it does not duration does not now, refer to how long the signal existed, but exactly the sigma square t and mean time being the 1 that we have defined earlier.

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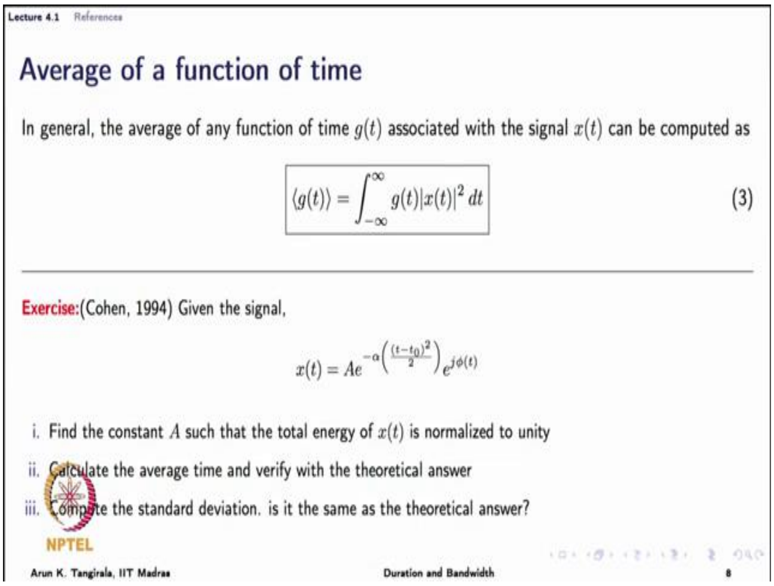

Now, before we move on, here, I would like to give an expression for the average of a function of time that can be useful in many situations. This is again the same expression that we encounter in the context of random variables where I find expressions for expectations of functions of random variable. There you would have g of x and instead of this time averaging I would have averaging across on some less space and I would have in the integral g of x times f of x dx. This is useful in certain calculations that we may find later on. Now, here is an excise that I give from Cohen's book, given the signal I would I invite you to calculate first the constant a. So, that the energy of the signal is total energy is normalized unity and then compute the average time and verify with the theoretical answer. By looking at this expression here, you should be able to say what is the center time? And also compute the standard deviation again by looking at this expression you should be able to say, what is the spread of the signal, in time. So, with this we move on this is a simple exercise, home work exercise for you.

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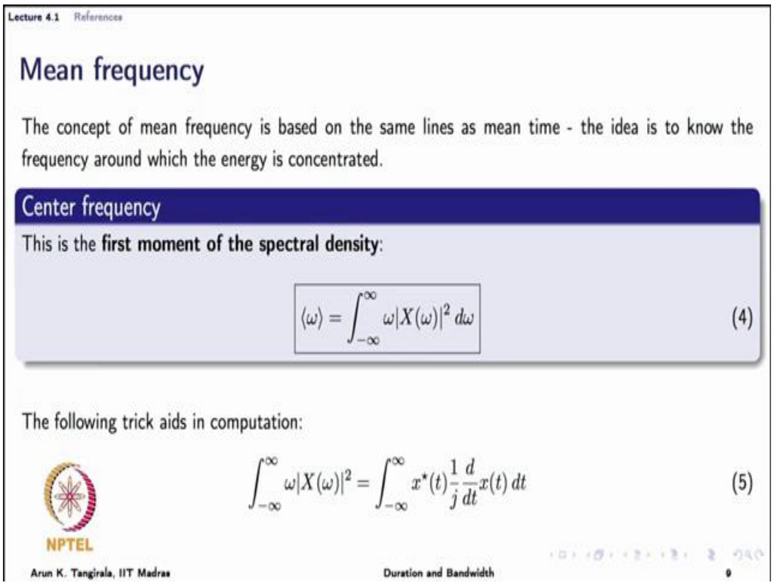

Now, we turn to the frequency domain and we ask what is the center frequency, which again we have use this term center frequency before when we talked about complex morlet wavelets when we are discussing the scaling property of Fourier transforms. So, here is a definition of the center frequency it is a first moment of the spectral density exactly along the lines of the mean time that we are defined. And once again we assume that the energy spectral density has been normalized to have unit area. The rest of the interpretation is straight forward again omega is not a random variable here we are just calculating integral, sorry average of omega with respect to x. And we are integrating the same way as we do for the mean time except that, I replace x of t square mod x of t square with mod x of omega square and so, on. Now, this following trick that again I give from Cohen's book typically aids in computation of this integral that why this is a nice expression, because in order to evaluate the mean frequency I do not need to evaluate x of omega first. If I am given the signal expression I can actually straight away compute the mean frequency without having to go through the step of computing the Fourier transform. We shall use this later on in deriving the equations for the bandwidth.

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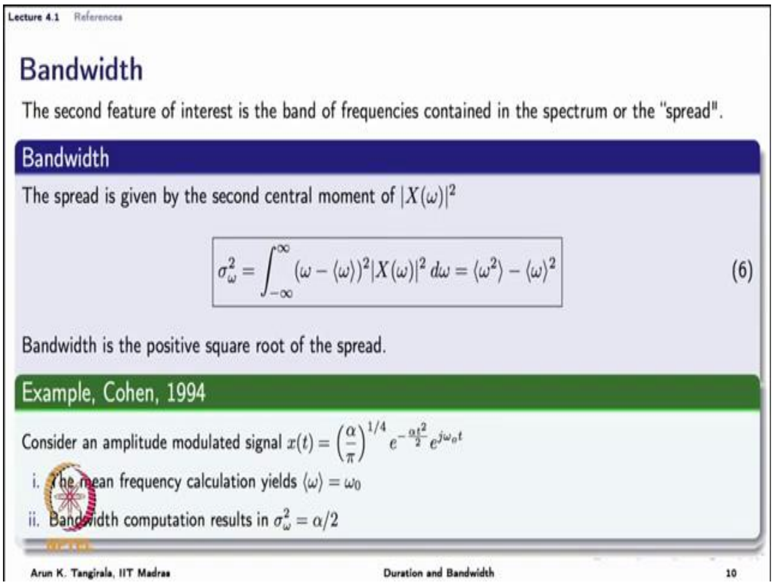

Of course, when I say equation of the band width it is a later module here I am giving you the definition of band width. So, band width again is a second central moment of the energy spectral density and again along the same lines as we have defined duration. Earlier we had said duration is sigma t now, we are going to say that the band width is sigma omega or sigma square omega whichever way you want to call it duration also you can refer to a sigma t normally that is way it is referred,, but sigma square t is also legitimate representative of the duration. So, sigma square omega once again is based on is defined as is the second central moment of a energy spectral density which again you can show, is the difference between the average of omega square less the square of the average. And band width is the positive square root of the spread.

But when we talk of bandwidth equation we will give an equation for sigma square omega, because they are easier that is an easier quantity to work. Now, if you consider this amplitude modulated signal as an example again from Cohen’s book the mean frequency calculation if you plug in the signal into the expression for the mean frequency. You can use this equation 5 to compute the mean frequency it turns out that you get omega naught and it is fairly obvious here this is what is happening here is this is a complex exponential here e to the j omega t and then I have an amplitude modulations The rest of the factor in x of t is amplitude modulation. So, the actual frequency present in the signal is omega naught only. This is what we had seen in the first example, of the opening example of this module where I had complex sine wave for a certain duration. And then front and the end there was no activities you can say there was an amplitude modulation.

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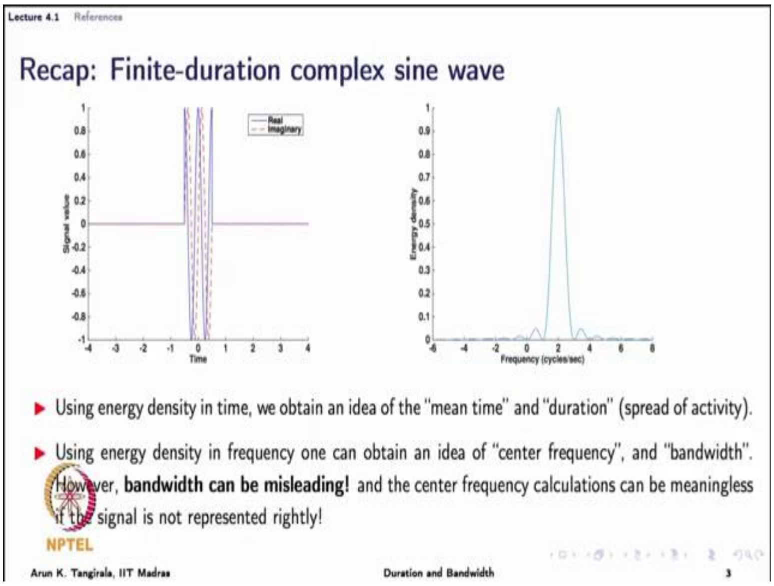

If you recall the energy spectral density that I had shown, the center frequency of this is still (). So, it turns out alright. The only issue with respect to the bandwidth ideally I should have expected a nice spike there,, but I would not, because I am going to have these components here contributing as well. One of the ways of looking at it is of course,, as we are explained earlier the band width has contributions from both amplitude modulation and frequency modulation.

The other way of looking at it is Fourier transform imagines signal to be made up of several weighted sinusoids at any point in time. And what happens is when the signal has zero activity that is where it has an issue it represent, it imagines this zero activity signal to be as if there are sinusoids canceling out each other. So, it represents the zero part of the signal as an addition and cancellation, subtraction of sinusoids which is not necessarily true. As a simple example, if I am standing outside the room I do not hear any sound from the room. I can think of two possibilities at least that there is no one in the room or nothing in the room makes any noise, or there are at least 2 people in the room or 2 sound sources in the room which are actually canceling out each other.

Now, what is actual reality you have to going to the room and check. What happens is

Fourier transform always imagines this zero activity or this no sound region to be made up of several sound sources canceling each other. That is why it tries to bring in other frequencies to explain the zero activities, that is the main limitation of Fourier transform. Let us return to the example here, I have the mean frequency calculation already discussed. Now, the bandwidth equation gives me alpha by 2. Now, you see the band alpha belongs to the amplitude modulation factor you can now see that the bandwidth has contribution from alpha by 2. So, the moment that I have amplitude modulation I will see a spread. Ideally I am not supposed to see this spread but that is not incorporated into the Fourier transform. Both amplitude modulation and if there was a frequency modulation into the signal that would have also appeared in this expression for the bandwidth which is what we are going to see in the bandwidth equation.

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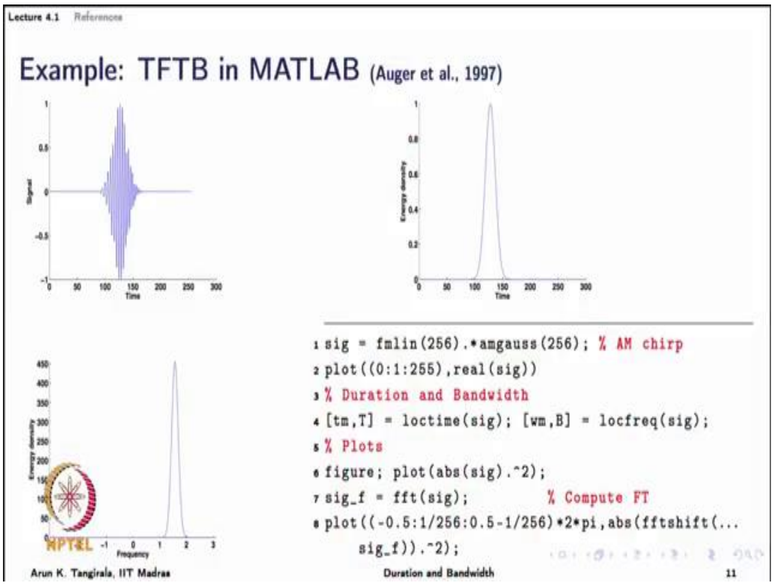

So, let us conclude this module with a couple of examples in mat lab and which will give us the motivation also for looking at analytic signals later on. So, this is an example illustrated in mat lab using the time frequency tool box I given the reference to this at the end of the lecture have generated amplitude modulated signal here. As you can see here I have fm lin will give me simple sinusoid and modulating that with an amplitude mode sorry fm lin will give me a linear frequency modulation linear chart and that is been modulated by this Gaussian amplitude modulation. And I am plotting the real part of the signal, by default you will obtain a complex valued signal in the time frequency tool box and the, execute this commands. And there are 2 routines in time frequency tool boxes or tool box for computing the duration mean time and bandwidth center frequency. These are loc time and loc frequency again I welcome to type this commands by yourself and if you make mistakes, it is a good learning experience,, but if you do not also it is good.

But it is please go through these commands one by one and what I show you here are the energy densities in time and here at the bottom energy spectral density. You can clearly see that the duration well the center frequency comes out alright here that as I have used. In fact, you should get these values and verify that is run these routines loc time and loc frequency and verify that the answers you get for t and d coincide with the give you fairly correct idea of the duration of the signal and the bandwidth. And that the tm which is the center of activity here, you should upfront determine manually what is the center of the activity here that would be the average of these two time instance. Because it is a symmetric modulation that we have introduced and the average frequency being here you can read off and you check that wm tallies with the center frequency given out by this energy spectral density it is a symmetric one. Therefore, whatever you visually observe should more or less coincide with what you get from the routine.

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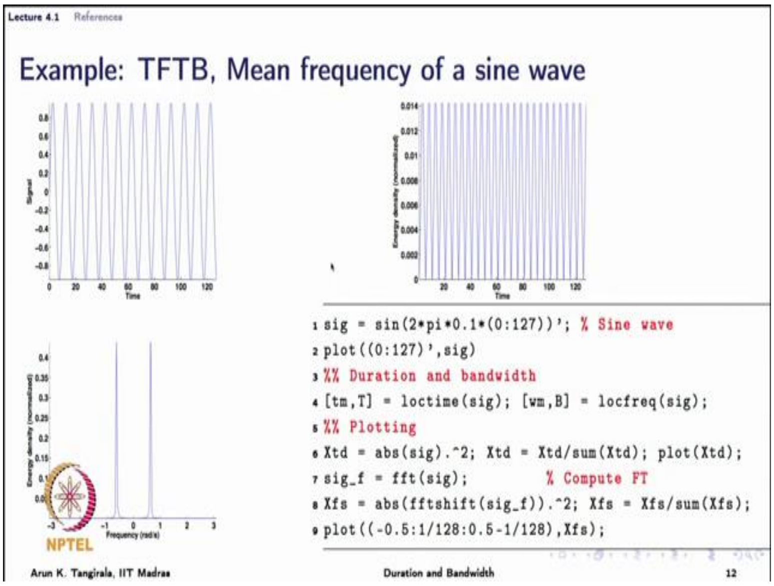

Now, let us turn to an interesting example which is a concluding example in this lecture I am going to take a sine wave right of this long. So, it is present over the entire existence single frequency no amplitude modulation. The center time should be this somewhere here that is a mean time,, but the point that I want to draw your attention to is the center frequency. So, if I look at the energy spectral density that I am showing you here. Of course, you may think of a power spectral density,, but you are looking at a finite duration sine wave will not worry about what is happening outside you can assume it is zero.

Therefore, let us stop of energy spectral density I showed the energy spectral density here you can see that there are 2 peaks symmetric peaks around zero. So, if you look at the center frequency manually it should turn out to be zero which is right. In fact, it will tally with the value given out by your loc frequency. It is actually going to return a value of zero for wm check it out for yourself. Now, that does not make sense how can the mean frequency of a sine wave be zero when I know very well that the sine wave has a non-zero frequency so, what is the issue here? The issue here is, the way I have represented the signal in computing the center frequency. This is where if I turn to a complex representation.

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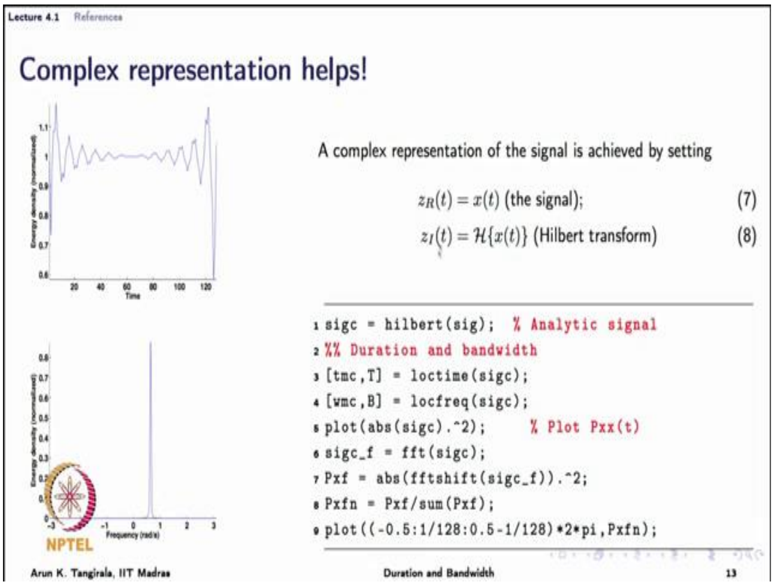

So, x of t is given signal I am going to construct a complex representation we call the one of the lectures in unit 2 where we talk of signal representations. I am going to construct a complex representation of the signal whose real part is signal itself. Imaginary part is the Hilbert transform of the signal which put together this complex signal is known as the analytic associate or analytic representation of the signal when I work with z of t instead of x of t. Then, I have this energy spectral density for z of t and it correctly gives me the center frequency not only manually that is by the visual inspection. But also when I use the loc frequency you can check that wmc which is center frequency for this z of t exactly coincides with the frequency that I used in x of t. This motivates us to look at complex representations which we call as in particular, this complex representation which is known as the analytic associate of the signal x of t.

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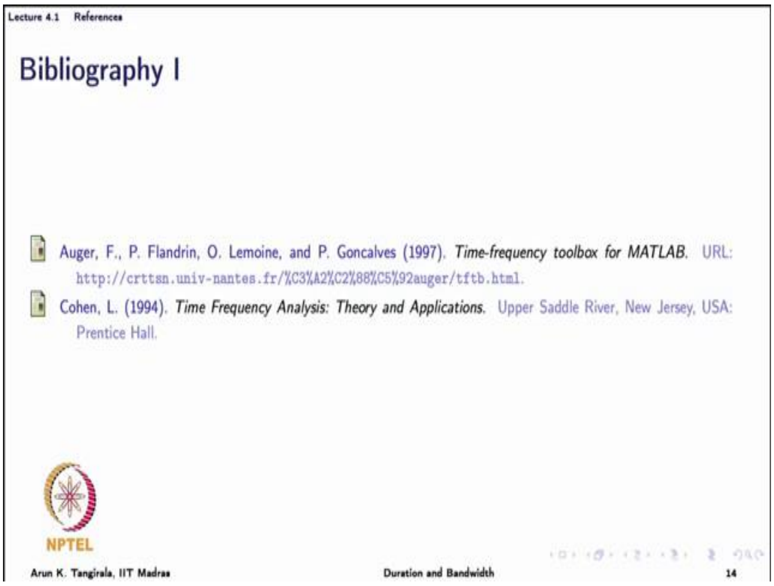

So, that brings us to the close of this lecture. And in the next lecture which is 4.2, we will drive the bandwidth equation and also studied the concept of instantaneous frequency. So, these are some other references for yourself you should go through this. The time frequency tool box reference is given even in the introductory slide or you can look up the next module as well.

Thank you.