Introduction to Time-Frequency Analysis and Wavelet Transforms

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Lecture - 3.5
Properties of Fourier transforms

Hello friends, welcome to lecture 3.5 where we are going to review certain useful properties of Fourier transforms. Now, in this I do not mention discrete time or continuous time, because the properties that we are going to discuss are equally applicable to both domains. But since I use the term Fourier transform, obviously we are referring to the class of aperiodic signals. There exists similar properties for the Fourier series case, but we are not so particularly interested in them right now.

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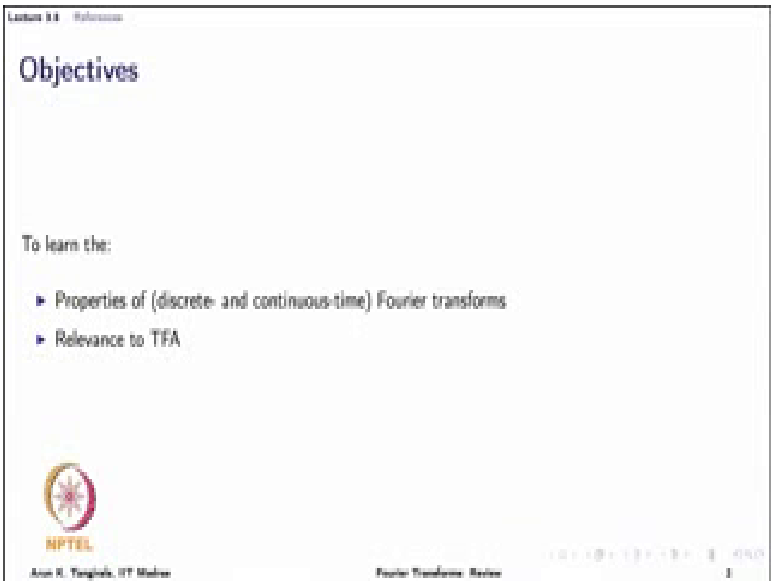

So, the objectives of this module is, I have to discuss the properties of the discrete time and continuous time Fourier transforms. And in particular those that are relevant to time frequency analysis. I only highlight the utility of these properties. We will actually apply them when we study the particular tools, such as short time Fourier transforms, wavelet transforms and Wigner Ville distributions.

Now, just to recap we have in the last 4 modules studied the 4 different types of theoretical transforms - the continuous time periodic and aperiodic, and the discrete time periodic and aperiodic signals cases, where we have the Fourier series and Fourier transform cases. In all this we have really avoided the derivations, and rather stated the results; occasionally we have shown the derivation of some results, but by and large, we have kept away with derivations. Primarily, because this is a unit where we are reviewing the concepts rather than learning them fresh. In fact, these are as I have said earlier also prerequisites, but we are just revisiting them in the context of time frequency analysis. We will adopt a similar approach here as well. We will learn the properties rather than trying to derive them, but a lot of these properties can be actually derived by hand without referring to any text. In fact that would be a good exercise for you to go through.

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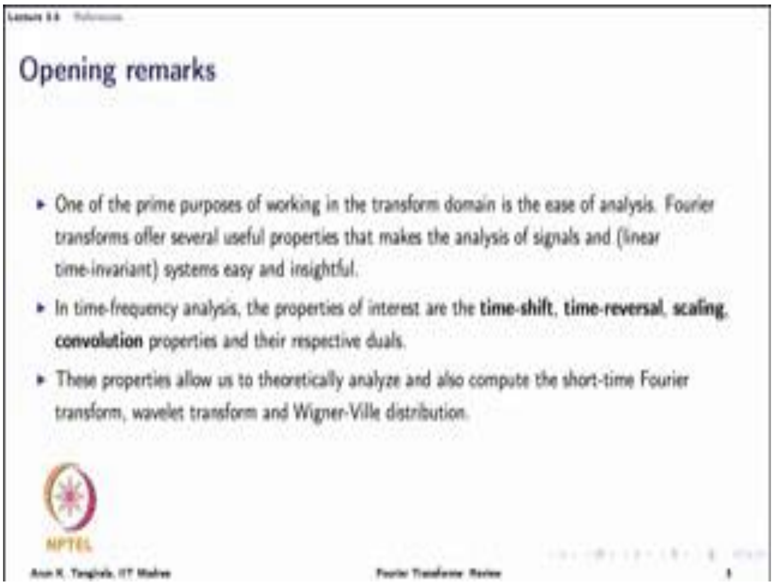

So, just a few opening remarks as usual. To recall one of the prime purposes of working with transforms is that they provide ease of analysis. So, for example, if you take Fourier transforms, very useful in the analysis of linear time invariant systems and signals emanating from emerging from these systems. In a number of different ways, of course we are not going to really study all the useful applications of these properties, but we will restrict ourselves to time frequency analysis. And in particular among the many properties, we will focus on the time shift, time reversal, scaling, convolution, and as a consequence correlation properties of the Fourier transforms. We will also study the dual properties. I am mentioned this earlier. We have seen this in duration and bandwidth as well. The time and Fourier frequency domains have a very nice duality, and so do these properties as well, as we will shortly go through.

The main point that I would like to keep you in mind is, these transforms are being reviewed with the purpose of analyzing and computing the other transforms that we are going to study in this course, including the Wigner Ville distribution. Wigner Ville distribution is not ideally a transform, but it can be viewed as a transform with an adoptive basis.

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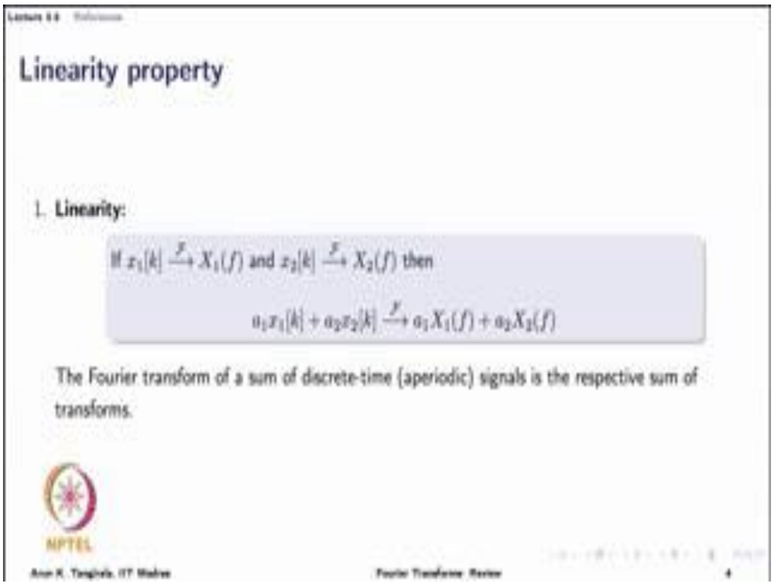

So, let us start looking at the linearity property which is fairly obvious result to derive. So, if I have 2 signals with Fourier transforms, x 1 and x 2, then the Fourier transform of sum of these 2 signals, is the respective sum of the Fourier transforms which is a fairly easy one to derive. And you should expect this because these are essentially, the Fourier transforms are essentially summations, and they are linear combinations. So, this property should not be surprising. Of course, this is, this works behind the scenes in all the Fourier analysis, and even the analysis of other transforms and so on.

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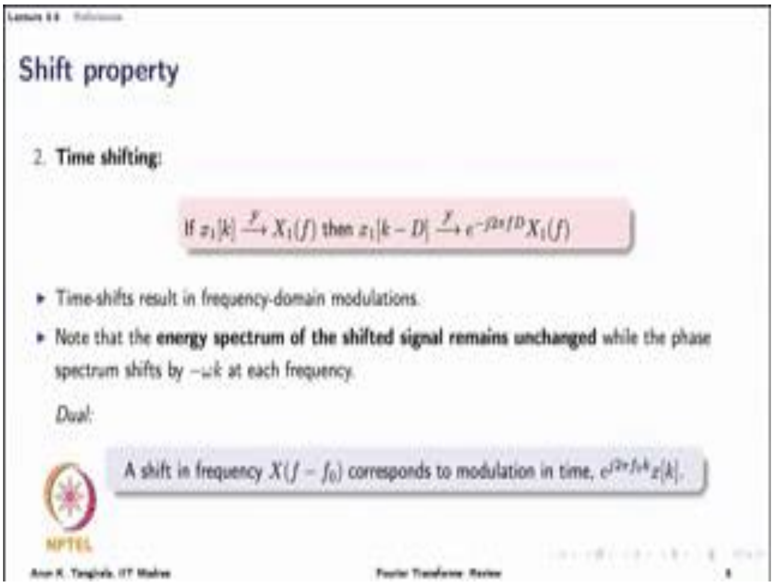

So, the first, it is second, but really the first property that we are going to use in the analysis of other transforms, is the time shifting property. Now, this time shifting property states that if I have a signal whose Fourier transform is, let us say, signal is x 1 k, and its Fourier transform is x 1 of f, then when the signal is shifted in time by a certain among d then the Fourier transform is multiplied with e to the minus j 2 pi f d. This result is, as it is stated is given for discrete time signals. But, as I mentioned earlier, it is equally applicable to the continuous time case as well.

So, the way to remember this is the time shift result in frequency domain modulations. Now, this is a very useful result in delay estimation widely used in signal processing and so on. We shall use this property to study the transforms such as wavelet transforms and short time Fourier transforms, particularly the wavelet transforms. For example, if I translate the wavelet function, then how does its Fourier transform modify, and so on. So, we will learn the particular application of this in the respective time frequency analysis tools later on.

An important point to remember is time shifts do not affect the energy spectrum. This is also true for power spectrum as well. So, the energy spectrum of the shifted signal remains unchanged; obviously, because if you look at the Fourier transform of the shifted signal, the magnitude of the Fourier transform of the shifted signal and the original signal are the same, because the magnitude of e to the minus j 2 pi f is the unit, regardless of the value of d.

Therefore, the energy spectrum will remain the same; however, the phase is affected by this delay. And it is this property that is used in delay estimation. There are number of classical delay estimation methods formulated in the frequency domain based on this property. So, you exploit the phase to estimate the delay. In fact the spectrum shifts by minus omega d at e frequency or you can say, minus 2 pi f d, the phase shifts by minus omega d or minus 2 pi f d at e frequency.

Now, the duality of this result is that a shift in frequency. So, this is just a reverse case. Now, shift in frequency corresponds to modulation in time, that is the nice duality that this properties share. Now, once again I leave this to you to prove this relation; that is fairly straight forward. So, you start with the definition of Fourier transform, and then evaluate the Fourier transform of the shifted signal, and make some necessary adjustments; in about 1 or 2 steps you should be able to derive the final result. This is some, so both these properties are going to be useful to us later on.

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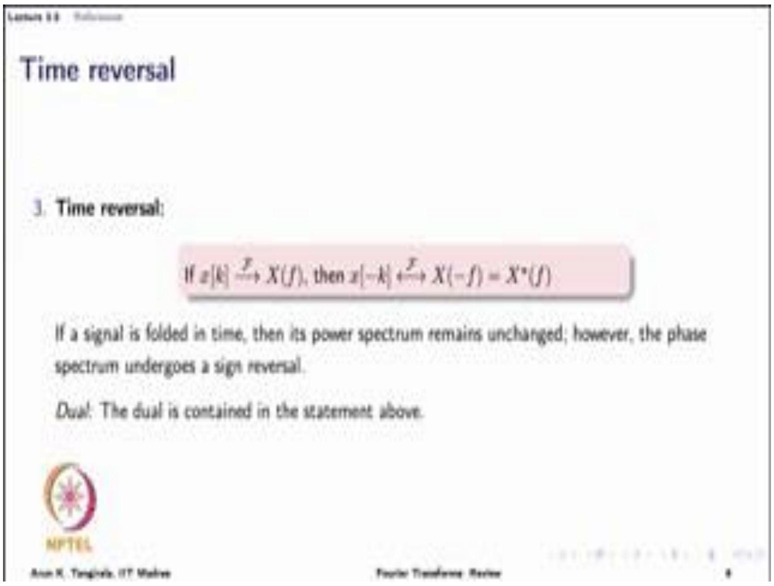

The next property of interest is the time reversal. By time reversal we mean reflection in time. So, if I have a signal whose Fourier transform is x of f, then the Fourier transform of the reflected one is the complex conjugate of the Fourier transform of the original signal. So, reflection results in a complex conjugate behavior. Remember, complex conjugate is obtained by replacing j, the imaginary part of the number with its negative. So, you are really transferring the negative sign in time to the imaginary portion.

And the duality also, is also contained in the statement. So, if I am asking, if I reflecting frequency, what is the consequence in time? The consequence in time is that I am reflecting in time, right. So, that is because if I reflect the frequency it is equal to taking the complex conjugate. So, the dual is contained in this statement. So, this is useful later on in writing an expression for the computation of wavelet transform. In fact all wavelet transforms are actually rewritten in frequency domain using this property, and another property as well.

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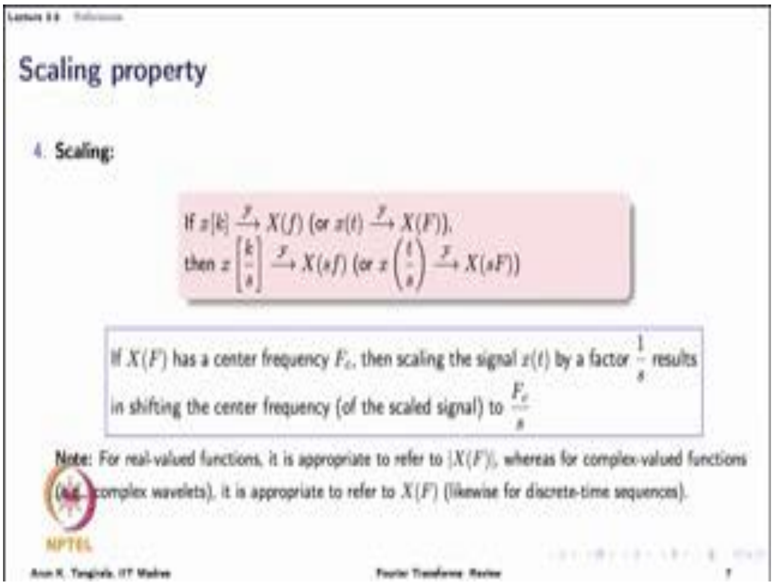

Now, coming to one of the most interesting properties to us, particularly in the context of wavelet transforms, is the scaling property. This is a property that needs to be understood well. So, we shall spend a few more minutes than we spent on the other couple of properties. So, what this result says is if I have a signal whose Fourier transform is x of f, then the Fourier transform of the scaled signal; what we mean by scaling is if I have x of k then the scale version is x of k by s, as we can see here. Then the Fourier transform of this scale signal is x of s f.

It is a fairly easy result to prove. Again I am avoiding the derivation here, but the simple, the starting point is the expression for the Fourier transform. So, what you want do to is you want to evaluate the Fourier transform of the scale signal, write the expression for the scale signal, and then you should be able to make a change of variable, just one change of variable will get you this result. So, such a fairly straight forward result to prove.

Rather than worrying ourselves about the proof at this moment we shall ask what is a consequence of this property in time frequency analysis and in filtering and so on? So, if x of capital F, so now I am referring to a continuous time signal here. If the Fourier transform of a continuous time signal has as a center frequency f c, then scaling the signal that is a continuous time signal by a factor 1 over s results in shifting the center frequency to f c over s. So, let me explain this to you briefly on the board, and there is also an illustration of this point in the next slide.

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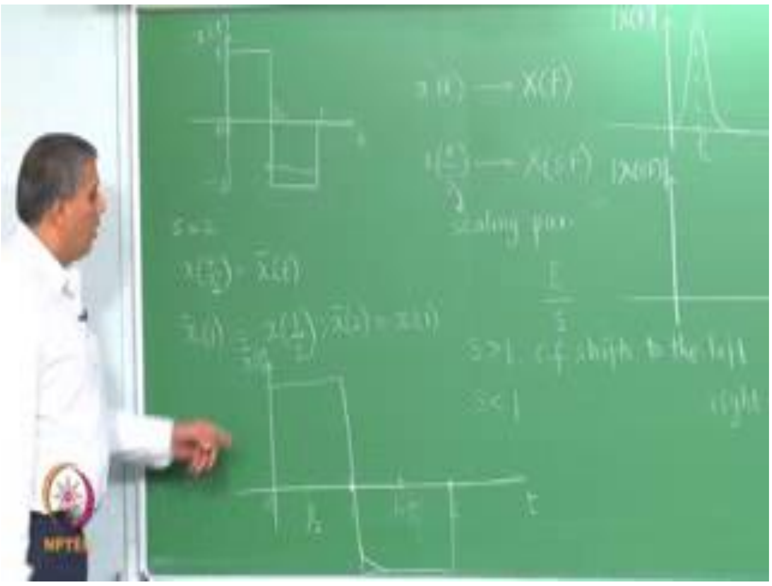

So, what this result says is, if a signal has the Fourier transform x (f), then x (t) over s match to x (s times f) in the Fourier domain, alright; where s is the scaling parameter. Now, the statement there says, if x (f) has a center frequency f c, what do we mean by center frequency? For no,. imagine that x (f) is a real valued quantity; if it is not, then you take the magnitude. When is x (f) real valued? When x (t) is complex, right. You can have complex functions; for example, when you are dealing with complex wavelet such as Morlet wavelet, then this function here or signal here, is a complex one. If it is symmetric, then you should get a real value Fourier transform.

Either way, whether it is complex or real I can look at the magnitude. And let us say that the magnitude of x (f) has this kind of a shape, just for the sake of f. So, here I have the center frequency f c; this is what we mean by center frequency. We can think of x (t) or even x (k) as the impulse response of band pass filter. Later on we will make this observation.

So, scaling this signal by a factor of 1 over s results in x, the Fourier transform in x of s of s times f. Obviously, now what happen is, that the center frequency of this here, of this x (x f), I were to plot, then now this, what I would see is a scaling of, the shift of this center frequency by a factor of 1 over s. Now, to understand this I have the illustration later on. Whether it shifts to the right or to the left, depends on the value of s.

If s is greater than 1, then the center frequency shifts to the left because f c over s is going to be less than f c. If s is less than, so this is center frequency shifts to the left, and that is the reason I have not drawn it here, how it looks like. And then when s is less than 1, the same thing here shifts to the right. Now, this left and right are with respect to the f c, that is the center frequency of the original signal, alright. I have not drawn this, but I have drawn this for you in the slide. This is the basic point that I am trying to make there.

Now, before we go to the next slide, it is important to understand what is the consequence of scaling this signal? That is, how does the signal look like when I scale it? So, to understand this, let us take the simplest functions. Although I am not going to use this; in the next level I am going to use different function. But, to understand the consequence of scaling, let us say that the signal of interest has the shape like that of a Haar wavelet. And let us say, this is 0, half, 1.

We will not particularly worried about the magnitude, but we can say here this is 1 and minus 1, just for the purpose of the discussion, that is really irrelevant. Now, when I look at x of t over s, suppose s is 2, alright, suppose s is 2, then what happens is I have a dilated signal which means I am really taking the signal and stretching it like I stretch an elastic band. Why does it get stretched? To see why it get stretched, define this x of t of s as some x bar of t, just to understand what is happening. So, this is a new signal, x bar of t. We have just introduced a notation which is related to this x of t through this relation x of t over s.

So, let us say I am asking what is the value of x at 1, at time 1, x bar at time 1, t equals 1? Now, because s is 2, I am going to replace s with 2 here, as a consequence I have x bar of 1 as x of half; which means what is happening here? The value at half, so if I were to draw the x bond as a function of time t, what happens is, the value at 1 for x bar is actually the value at half, alright.

So, let me actually draw a longer time axis because it is going to be a dilated signal. So, here I have 0, half, 1 and 1.5 and 2. So, the value of 1 for x bar is a value of x at half which is actually 0. And the value at 2, likewise is going to be, of x bar at 2 is going to be the value of x at time 1. So, which means, at this point I am going to have this here. I am just showing you the cross over points here and symbolically it will utilize. So, you can see clearly that I have an elongated or a dilated signal.

And now, I leave it as a simple exercise to you, to see that values of s less than 1 will result in compression, we really compressing the signal. Obviously, what is happening is that the area under this is going to be more than the area under this. So, one needs some kind of normalization and so on. We will talk about this when we discuss wavelets. So, the point to remember is when I scale a signal this way, and values of s greater than 1, will result in dilation or elongation. And values of s less than 1, is going to result in compression.

And in frequency domain, the same story is happening, why? Because, there is one more thing that I will show you; we have discussed here that the center frequency shifts to the left for values s greater than 1. So, what is happening here is, the moment I elongate this, think of very high values of s, then what would happen is, this becomes a really elongated pulse. So, it has much more lower frequency content than this one. So, this is changing fast, this is changing slowly.

Whichever changes slowly, we say it has low frequency; whichever changes fast has relatively high frequency. It is all relative to this original signal, nothing is obsolete here, right. So, that is why now the center frequency shifting to the left; predominantly, this is a low frequency signal relative to this signal here, and that is why I have a center frequency shifting to the left when s is greater than 1. So, this is the new, this is f c prime; this is the center frequency of the scale signal.

Also, you should observe that I have the spread of this scale, of the Fourier transform of the scale signal narrower than the one that I have for the original signal. Now, this is again a consequence of the duration bandwidth principle, right. What is happening here? When I am elongating the signal, I am extending the duration of the signal, as a consequence I would be really narrowing the bandwidth, alright.

So, what is the scaling factor here? The duration is being modified by this factor s, right. So, the duration is actually increasing. So, if sigma t is the duration of x of t, then x of t of s has a duration of s times sigma t. And the bandwidth here, if suppose x of f has a bandwidth which is a measure of the spread of this; it is not this exactly, but it is the measure of the spread, then the bandwidth of this is going to be sigma b over s.

The product of these 2 is still sigma t time sigma b; that is the beauty of this scaling. So, now, again you can see, when s is greater than 1 I have a longer duration signal, but that results in a narrower bandwidth; and when s is less than 1 I have a shorter duration because I am going to compress this, but then at the cost of increased bandwidth. So, hopefully, this gives you an idea of what we are going to see in wavelet transforms.

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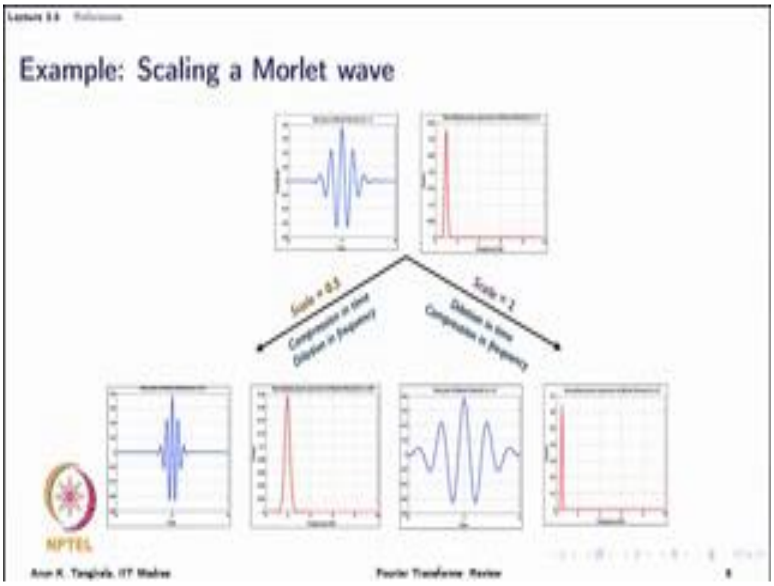

So, let us see this with an illustration on the Morlet wave. I have shown you this slide in lecture 1.2 when we were giving a, going through an overview of wavelets. So, what I have on the top is a real portion of the complex Morlet wave. I could plot the imaginary one as well. And what I have on the right is mod x of f square, it reads power. So, it still, curvature of this is the same as modulus of x (f). So, this is now the situation that I have for x (t); this is x (t), this is real part of x (t), and I have here mod x (f).

When I scale it by a factor of 0.5, as I mentioned earlier, I have compression here. But then what do I have? I have the center frequency shifting to the right. So, what is happening here? Relative to the original filter or the original energy spectral density that I have, this now has the filtering characteristics in the higher frequency region because the center frequency has shifted to the right, but the spread has increased. And when I scale it by a factor of 2 which is greater than 1, then I have a dilated wave or wavelet, because now this is the mother wave and these 2 are the wavelets, I have the dilated wavelet.

Now, the center frequency has shifted to left, and the spread has also decreased, because of the reasons that I have explained, alright. So, this is something that should be understood. There can be some confusion initially in understanding this, but again going through the exercise repeatedly, doing this by yourself will help a lot. And this is very important to understand how wavelet transforms really work. If you have understood this, you are kind of understood 50 percent of how wavelet transforms work, right, what is the basic idea.

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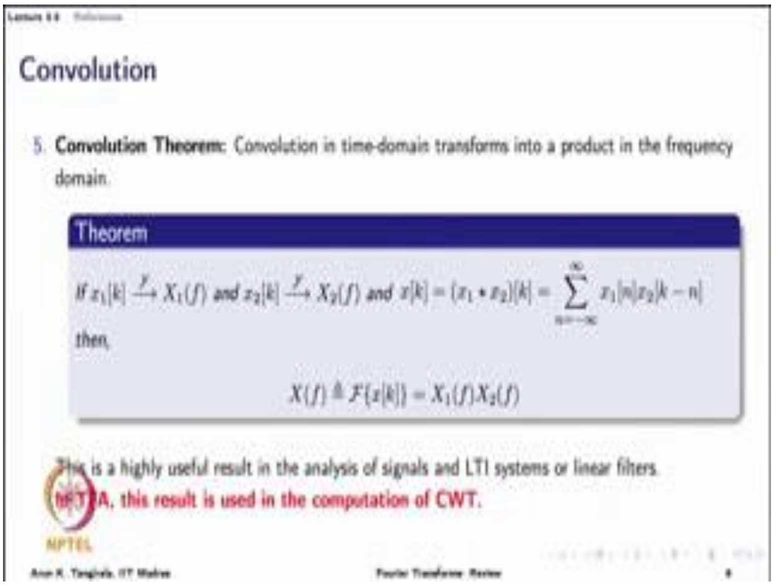

So, one of the penultimate properties that we are interested in is a convolution theorem which is a very famous and celebrated result in linear systems analysis. Convolution in time domain transforms into a product in the frequency domain. So, if I have 2 signals and I am looking at a third signal which is a convolution of these 2 signals, then the Fourier transform of the convolution of this is a product of the Fourier, respective Fourier transform.

So, it is a beautiful result because it paves way for a number of things for a frequency domain analysis of linear time invariant systems for computation of convolution itself because there exists efficient algorithms for computation of Fourier transforms by a way of ffts, fast Fourier transforms. I can first compute the, in order to compute the convolution, this convolution here, I can first compute the respective Fourier transforms, multiply them and then take the inverse Fourier transform, I will recover the problem. So, that is the beauty of this result, but that is in the computation side. In the theoretical side, this leads to the analysis of linear time invariant systems in frequency domain, we have what are known as frequency response functions and so on.

We had preliminary discussion of this in the when we were discussing the discrete time Fourier transforms and so on. As for as time frequency analysis is concerned, this is very useful in the computation of continuous wavelet transform. We can write the continuous wavelet transform as a convolution and then evaluate the continuous wavelet transform in the frequency domain.

At this moment it may sound strange because wavelet transforms help you do a time frequency analysis, but then we are saying we shift to the Fourier domain and come back, but they are all interrelated, alright. So, we will see later on how this is used in the computation of CWT, the details of them.

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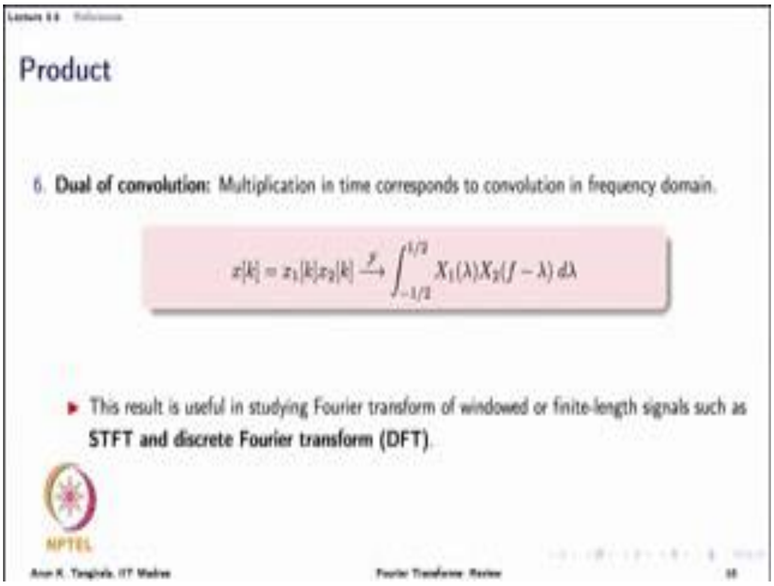

Now, the duality of this convolution result is a product. When I have a product of the signals, and I am looking at the Fourier transform of that signal, then in the frequency domain, I have the convolution of the respective signals, right. Now, how is this useful? Well, product is a relatively easy thing to evaluate. So, we are not really going to use in computation. But, we are going to use this result in studying Fourier transforms of windowed signal. What we mean by windowed is, segmenting, as we do in short time

Fourier transforms and so on.

Windowing not only occurs in short time Fourier transform but also in the regular Fourier transforms of finite length signals. All the Fourier transforms that we apply in practice are for finite, to finite length signals. And you can always view the finite length signal as a segmented version of the infinitely long signal that we are unable to obtain observe.

Therefore, this result is useful in asking what happens to the true Fourier transforms that is that of the infinitely long signal, when I take only a part of it which is what I can do in practice, and evaluate the Fourier transforms.

So, how does the Fourier transform of the observed signal or observed series relay to the Fourier transform of the theoretical one, is the question, and that is nicely answered by this. And this is called windowing, and they can show this result in spectral leakage and so on. We shall talk about this in the next module when we talk of discrete Fourier transform. It is also useful, therefore, in understanding how short time Fourier transform works.

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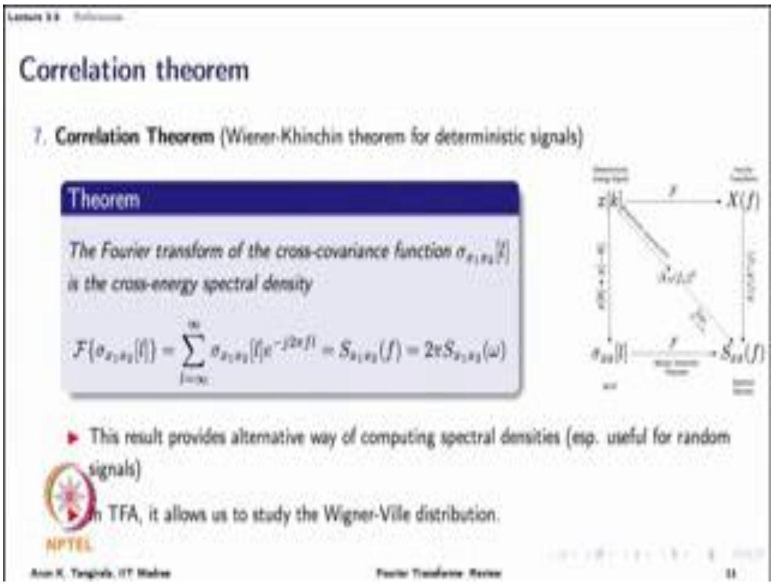

So, finally, we have the correlation theorem, we say it also the Wiener Khinchin theorem version for deterministic signals. Although, Wiener and Khinchin gave the result for the stochastic signals, just slide adaptation of those terminologies here. So, the Fourier transform of the cross covariance function is the cross energy spectral density. We have seen this in the context of discrete time Fourier transform earlier, and also in the case of discrete time Fourier series.

So, the expression here says, that the Fourier transform, discrete time Fourier transform of the cross covariance function gives me the energy spectral density. Again, I have given you relations both in terms of cyclic and angular frequency. The figure on the right essentially tells you now 3 different ways of computing the energy spectral density. We have discussed this earlier as well; how this theorem or this result gives me a way of computing the energy spectral density.

So, one way is a classic way, I take the Fourier transform, take the magnitude square; the other way is through the Wiener Khinchin theorem. Now, the way to prove this result is to actually take the, to express the auto covariance function or the cross covariance function as a convolution between the signal and the reflected version of itself, if you are looking at the auto covariance function; or, the reflected version of the other signal if you are looking at the cross covariance function.

That is why I have a convolution of x 1, and the reflected version of the other signal. So, that is of the signal itself. Then, I have the auto cross covariance function, and I take the Fourier transform, I can get the energy spectral density.

The third rule is to compute what is known as the discrete Fourier transform which I will discuss in the next module, and compute the magnitude square. The discrete Fourier transform is concerned with the Fourier transform of a finite length signal. Now, when I increase this number of observations to infinity, then I can also recover the energy spectral density.

So, these are 3 different ways of computing the energy density. In time frequency analysis, this result is very useful in understanding the Wigner Ville distribution. So, with this we come to a close of this module on the properties of Fourier transforms. As I have mentioned earlier on, although we have stated the results for discrete time signals aperiodic signals, these are also useful for continuous time aperiodic signals as well.

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And I suggest that you derive some of these properties by yourself. They are fairly easy to derive by hand without actually referring to a text. Of course, our teaching assistants and myself, we are always available to help you with the derivations or the answering your question on the forum. Good luck and see you in the next module.

Thanks.