Introduction to Time-Frequency Analysis and Wavelet Transforms
Prof. Arun K.Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 3.4
Discrete-Time Fourier Transforms

Welcome to lecture 3.4 of the course on time-frequency analysis and wavelet transforms. In this module, we are going to learn concepts pertaining to discrete-time Fourier transforms. Again as with the previous versions of the Fourier transform, it is largely a review, but mostly the interpretations will matter rather than derivations.

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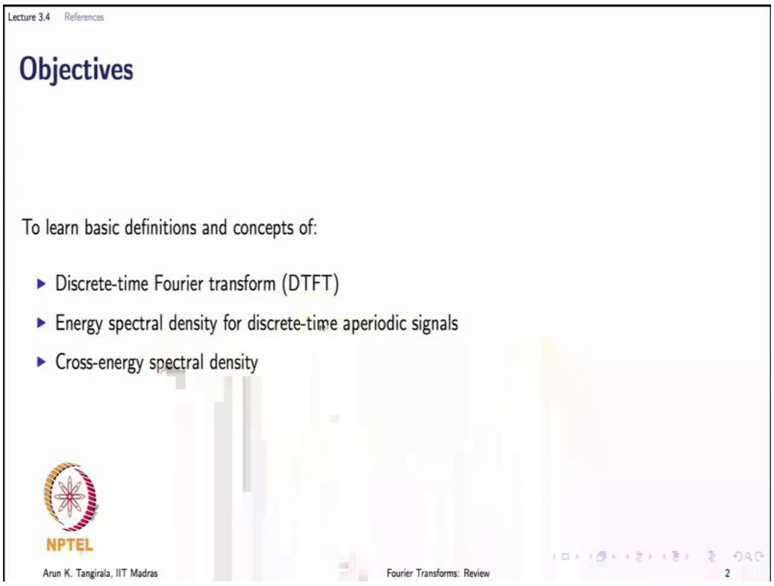

So, in the lecture 3.3, we have discussed discrete-time Fourier series. Now, we move on to discrete-time aperiodic signals like we did in lecture 3.2 for the continuous-time case. As expected we have the discrete-time Fourier transform. And again like the continuous-time case, we have the energy spectral density for discrete-time aperiodic signals. We shall also look at the concept of cross-energy spectral density towards the end.

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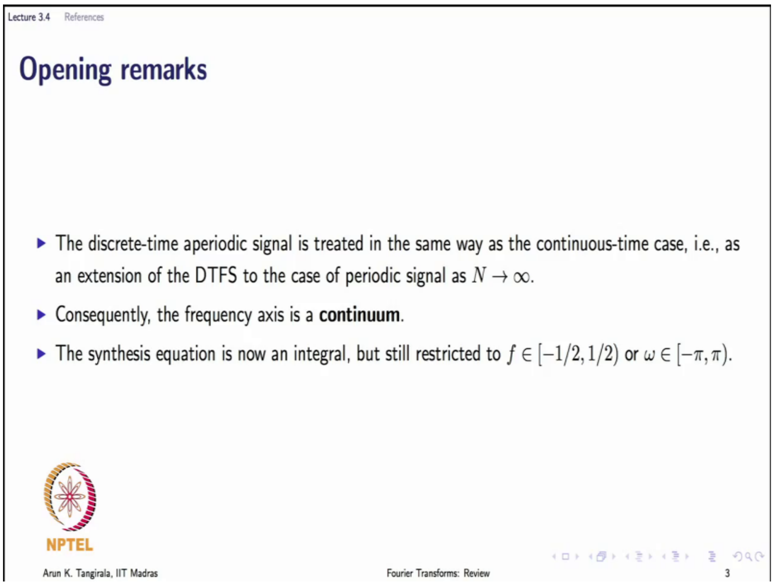

Now, a few opening remarks similar to what we had discussed in lecture 3.3 in the case of discrete-time periodic signals; again the story is more or less similar; the synthesis equation is now an integral like we had in the continuous-time case; but still restricted to this fundamental frequency range, because we are still working with discrete-time sinusoids. The only difference is in the case of periodic signals, we had used harmonics; and now we are going to use the entire set of frequencies, the entire continuum of frequencies in this fundamental interval. Consequently, the frequency axis is a continuum. And once again the discrete-time Fourier transform can be derived starting from the Fourier series equation for the periodic signal by letting the period go to infinity.

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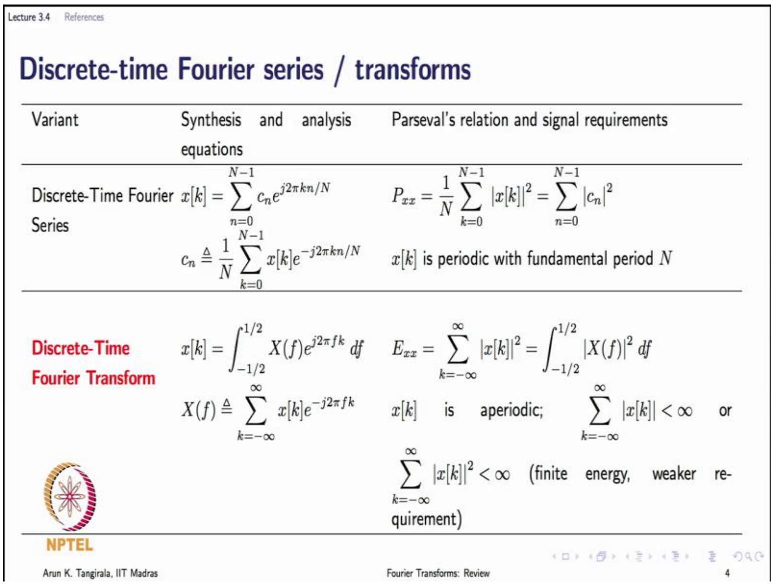

So, recap and a comparison with the discrete-time Fourier series case; this is pretty much similar to what we saw for the continuous-time case. The periodic signal had a summation in the synthesis equation; whereas, now we have an integral in the aperiodic case. And just to reiterate, the integral is now limited to minus half to half; that is because of the behavior of the discrete-time complex exponentials. And as far as the analysis equation is concerned, in place of the Fourier coefficients, we have now the discrete-time Fourier transform – exactly like what we had in the continuous-time case. And once again the summation now runs from k equals minus infinity to infinity; which means we are looking at the entire existence of the signal in contrast to the periodic interval that we considered – the duration that we considered for the discrete-time Fourier series case.

And moving on to the Parseval’s relations, we have the energy spectral decomposition as expected. The area under magnitude square of X of f gives me the total energy or the energy of the signal. And therefore, modulus X of f square acquires the interpretation of an energy spectral density. As far as the condition for the existence of the discrete-time Fourier transform is concerned, the discrete-time signal of course decides being aperiodic should have finite 1-norm, which is a stricter requirement compared to the finite energy, which is a weaker requirement. And these are again similar to what we have discussed in the other three variances.

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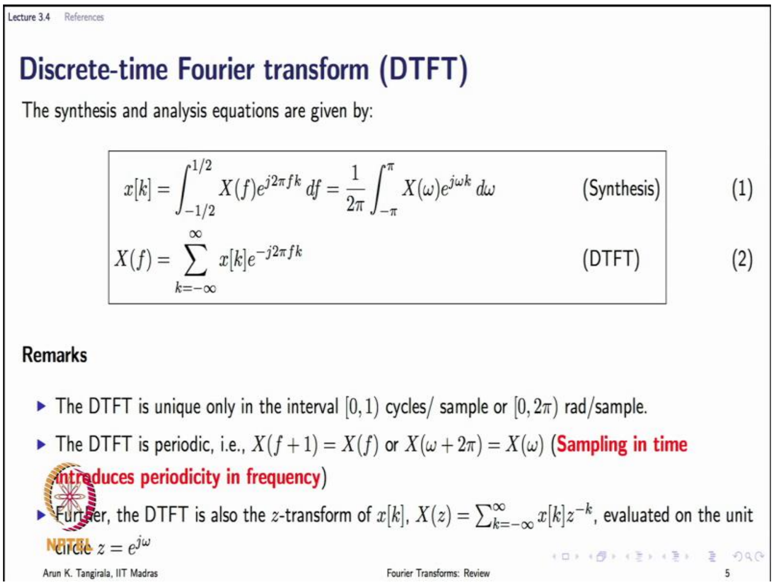

So, now, looking at the synthesis and the analysis equations in detail, we have the expression for the synthesis equation here. I have given you both versions in terms of the angular frequency as well as in terms of the cyclic frequency. Note this factor 1 over 2 pi that appears in front of the integral when you use angular frequencies; that is fairly straightforward to see, because f is omega over 2 pi. So, in moving from this integral to the angular frequency integral, you have to accommodate this 1 over 2 pi. And the discrete-time Fourier transform equation is the same whether you evaluate in terms of the cyclic frequency or the angular frequency. But, there are a few interesting remarks that I would like to make here. First of all, the discrete-time Fourier transform as we had mentioned earlier is unique only in the interval 0, 1 in terms of cyclic frequency or minus 0.5 to 0.5. Or in terms of angular frequency 0 to 2 pi.

Very important ((Refer Time: 05:20)). This is quite important to note, because when we move to discrete-Fourier transform, we will make a dual observation of this remark; which is that, the discrete-time Fourier transform is periodic as you can see from the equation 2 here. And we say now that, sampling in time introduces periodicity in frequency. This is a very important observation, because when we move to discrete-Fourier transforms, we would be sampling the frequency domain and there we will observe that, sampling in frequency introduces periodicity in time domains. So, the time and frequency domains are very close duals of each other. When we discuss the properties of Fourier transform in the next module, we will also observe this strong duality between the time and the Fourier frequency domain. So, this is something to remember.

And finally, the discrete-time Fourier transform is also the z transform of this discrete-time signal evaluated on the unit circle. On the z-transform is a generalized version of the Fourier transform; where, z is a complex number unlike the pure imaginary number that, it becomes when you evaluate it on the unit circle. So, the z transforms of signal can accommodate a larger class of signals – even those signals that Fourier transforms cannot accommodate. For example, they can accommodate even signals that do not have a finite 1-norm. So, that is the beauty of using the z transform. So, we have used extensively in the analysis of linear-time invariance systems and so on. But we will not use z transforms here.

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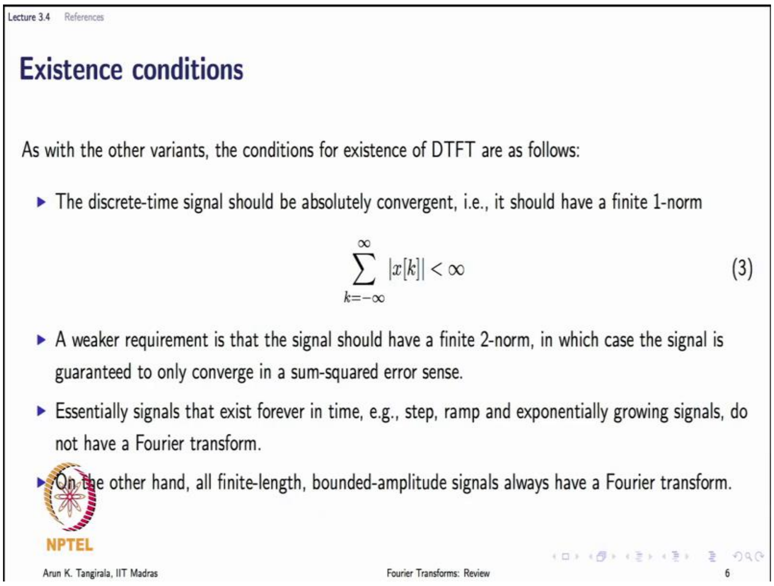

Existence conditions – I have already remarked the discrete-time signal should be absolutely convergent; that means it should have a finite 1-norm and a weaker requirement that, the signal should have a finite 2-norm; which means finite energy. Now, what this means is that, signals that exists forever in time and are aperiodic such as step and ramp and so on. And they do not have a Fourier transform; whereas, all finite length bounded amplitude signals will have a Fourier transform. And this was also the case with the continuous-time signal.

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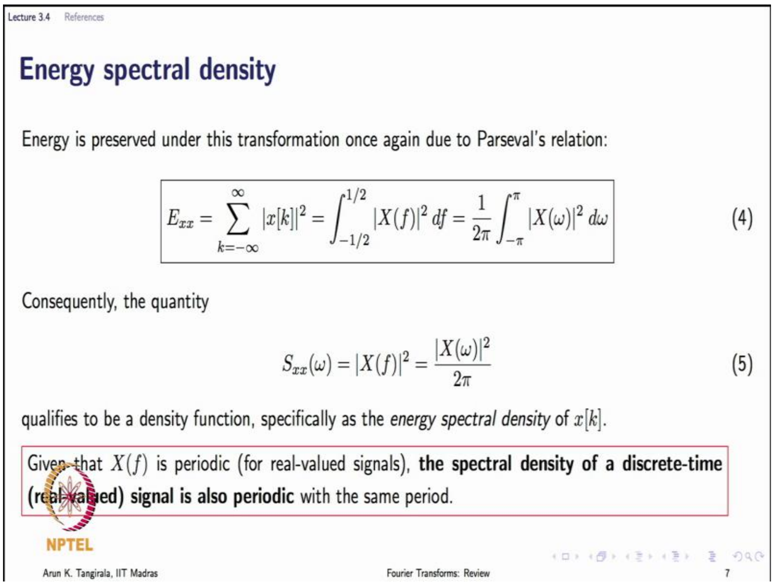

As expected now and as also discussed earlier, now, we have this energy spectral density, which is a very practically useful measure. The area under magnitude square of X of f gives me the energy due to Parseval's relation. And again, notice that, I have given expressions in terms of cyclic as well as angular frequency. And the moment you use an angular frequency here – the angular frequency, you have 1 over 2 pi appearing in front of the integral. Ignoring this 1 over 2 pi would give you wrong value of the spectral density – the energy spectral density. So, as we have argued before with other – with the continuous-time case, the mod X of f square or mod X of omega square by 2 pi gives me the energy spectral density of the signal or the energy density in the frequency domain.

Now, given that, X of f is periodic as we have noticed earlier for real-valued signals, the spectral density is also periodic. Remember we are going to evaluate the spectral density only in the fundamental frequency range; it does not mean the spectral density does not exists outside this interval; it is just that it is periodic. So, there is no need to plot it; but if you want to plot this spectral density for the entire frequency axis, you plot over the fundamental frequency range and repeat it; that is the main idea or the main consequence of this observation that the spectral density of a discrete-time signal is periodic. That is the fundamental difference between the continuous-time aperiodic case and the discrete-time aperiodic case when it comes to spectral density. There the spectral densities are not periodic.

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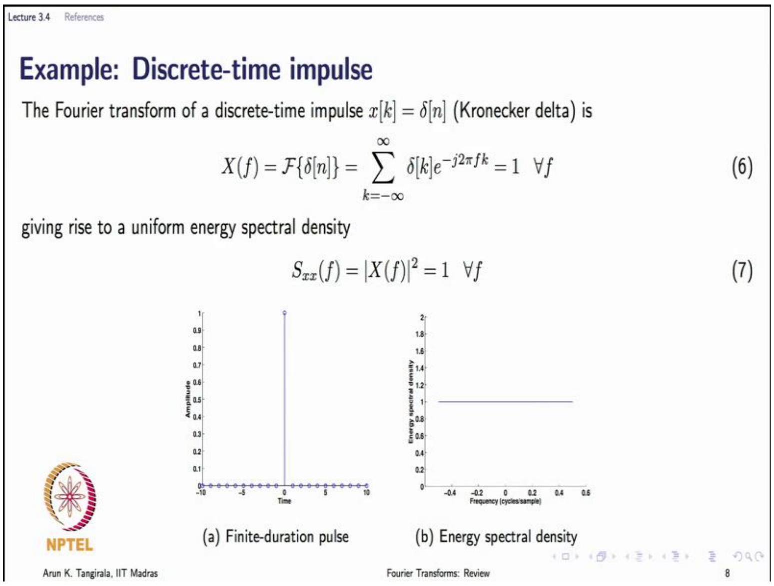

So, let us walk through a couple of examples to understand how this discrete-time Fourier transform works and get a better feel of it. So, here we take a very simple example which is a discrete-time impulse; which is also known as a Kronecker delta. Now, unlike the continuous-time impulse, this Kronecker delta is physically realizable. And just plugging in this expression for the signal into the Fourier transform expression clearly brings me this very interesting result that, the Fourier transform is unity – assuming unit impulse at all frequencies. What this tells me is that, when I have a signal that is highly localized in time; now, we are showing you the signals – the signal here on the left and the energy spectral density on the right. The signal plot is also the same as the energy spectral density plot in time.

Again, the same observation that we made yesterday – when we have a highly localized energy density in time or very finely localized, we have a very broad spread of the energy density in frequency domain; in fact, it spans all the frequencies uniformly; this is again a consequence of the duration bandwidth principle that we have seen. So, the duration here is the ideal, that is, zero-duration you can say, so just one sample. And the bandwidth here is the complete fundamental frequency interval. Of course, we are talking of duration and bandwidth in a very qualitative sense; but as I explained in the previous lecture – the lecture 3.2, the duration and bandwidth are actually the central moments of the energy densities in time and frequency respectively. So, this is something that we should repeatedly observe, so as to reinforce a concept of duration bandwidth principle in our minds.

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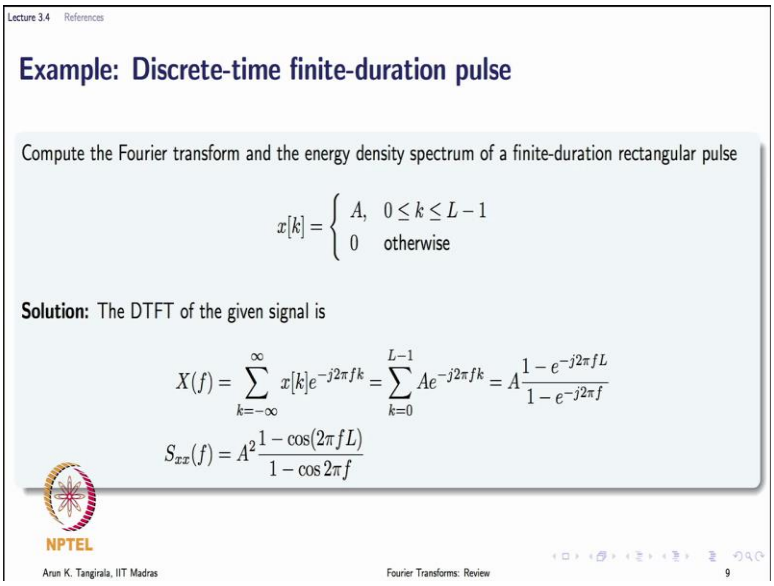

Now, when we evaluate the discrete-time Fourier transform of a finite-duration pulse, we have this expression here. Again this is fairly easy to derive; just walk through the math and you will get this answer. And the expression for the energy spectral density is given by evaluating the squared magnitude of X of f. L is the duration of the pulse that we are talking about.

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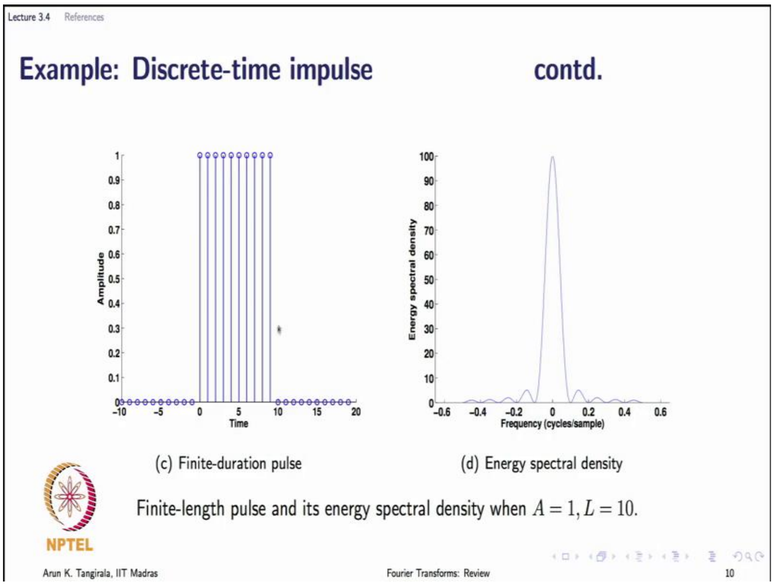

To give you an idea of how things look like, I have evaluated this for L equals 10 and A equals 1, that is amplitude being 1. Once again you see the same story that we had seen earlier. Now, the signal has a larger duration than the case of the impulse. The duration definitely is finite unlike the case of the impulse. Naturally, now, what has happened is the bandwidth or the spread of the energy spectral density has shrunk from being full from uniform – from being a very large value for the impulse case. In fact, here the energy density spread over all frequencies; whereas for the pulse case, the energy density – spectral density is localized over this interval. So, there is significant energy here in this range, but insignificant once outside this interval. So, the bandwidth is definitely smaller than the case of impulse. Again I mean the same story – the duration bandwidth principle is playing the game here.

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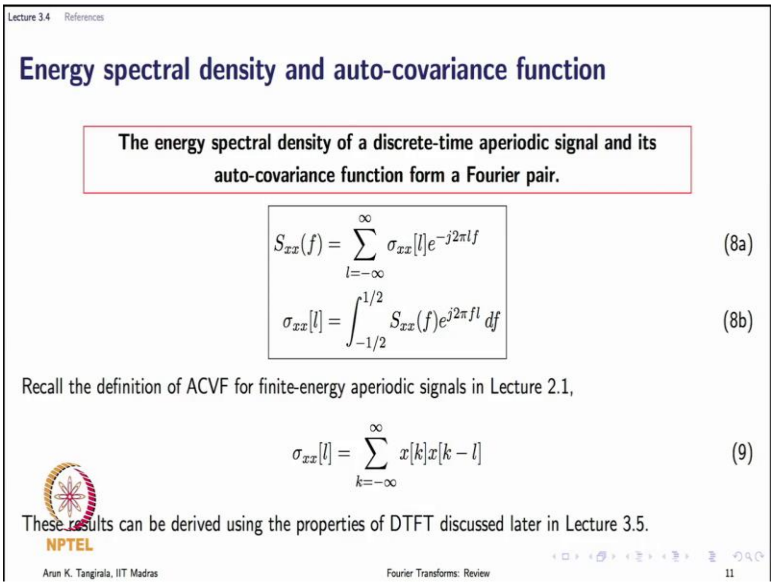

So, when we talked about the discrete-time periodic signal, we noticed these connections between energy spectral density and auto-covariance function. We have a similar result here; the energy spectral density of a discrete-time periodic signal and its auto-covariance function form of Fourier pair as given by equations 8a and 8b here. I have this energy spectral density as a Fourier transform of the auto-covariance function. And likewise, the auto-covariance function being the inverse transform – an inverse Fourier transform of the energy spectral density. Notice the limits here; the limits for the auto-covariance function run from minus infinity to infinity; whereas, the limits for frequency run from minus half to half. If you were to use this spectral density in terms of angular frequency, then you necessarily need to have a 1 over 2 pi in front of this integral to get the correct values of the auto-covariance function. Now, in order to prove this, you need to use this definition that we had given in lecture 2.1 for the auto-covariance function of a discrete-time finite energy aperiodic signal.

Now, it turns out that, you can also derive this result using the properties of discrete-time Fourier transform, which we will discuss in the next module. Again, all of these lead to what is known as the Wiener-Khinchin theorem as I had mentioned in the previous lecture, which is extensively used in computing and defining the spectral density of random signals.

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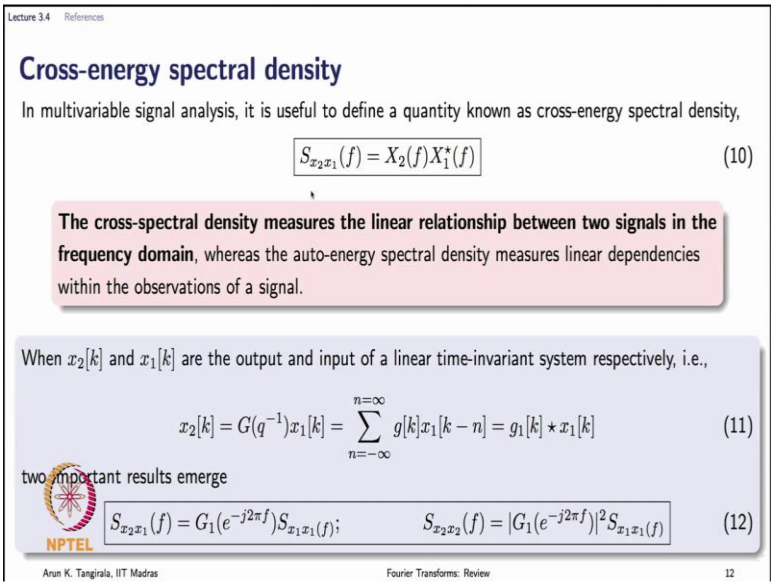

Now, it is useful to know the concept of cross-energy spectral density. The cross-energy spectral density essentially is an extension of the auto-energy spectral density, that is, a regular energy spectral density that we have been looking at for a single signal. Now, we are looking at two signals. We would like to measure the linear dependence between two signals in frequency domain. Remember – we said the cross-covariance function is actually a measure of the linear dependence between two signals that are shifted in time by lag L. In fact, similar to the relation that we have here, that is, the auto-covariance function and the auto-energy spectral density being the Fourier pair; here we have the cross-energy spectral density and the cross-covariance function also forming a Fourier pair although I do not sight that in this slide here

The most important point to keep in mind is that, the cross-spectral density or the cross-energy spectral density measures the linear relationship between two signals in the frequency domain. You can also define what is known as a cross-power spectral density. Again, you will find a similar relation between the cross-power spectral density and the cross-covariance function for periodic signals. Why is this result useful and where is it used? It is used extensively in the analysis of linear-time invariant systems. If you think of x 2 being the output and x 1 being the input to a linear-time invariant system; then we know from linear systems theory that, the output and input of the discrete-time linear invariant system are related through this familiar discrete-time convolution equation; the star here denotes convolution. And one can start from this convolution equation and use the properties of discrete-time Fourier transform to derive these two central results in the analysis of LTI systems in the frequency domain. The first one is relating the cross spectral – it is expressing the cross-spectral density in terms of the auto-spectral density of the input. So, here this is the cross-spectral density between the output and input. And that is related through what is known as the frequency response function. So, this big G of e to the minus j2pif is nothing but the discrete-time Fourier transform of the impulse response coefficients that we have; which are denoted by the small g.

And here we have the expression for the output spectral density in terms of the input energy spectral density. But, now, again the frequency response function comes into picture, but the magnitude square now plays a role. So, you can see from these two results that, the cross-energy spectral density is a complex valued quantity unlike the auto-energy spectral density. And that is to be expected, because the cross-energy spectral density is the Fourier transform of the cross-covariance function. And because the cross-covariance function is in general asymmetry, you would obtain a complex valued number.

Whereas the auto-covariance function is a symmetric function; you are guaranteed that, the auto-energy spectral density is going to be real valued quantity. Of course, we are talking of real-valued signals. Therefore, this cross-energy spectral density has a magnitude and phase; and the phase of the cross-spectral densities are used in delay estimation and so on; just like we said cross-covariance functions can be used in delay estimation. These frequency domain results are very useful in for example, quantifying linearity, testing linearity in time-invariant systems; defining what is known as coherency; which tells me at each frequency, what is the extent of linear relationship between the input and output of a system and so on.

And of course, an important application of these results is in the computation of the frequency response functions in system identification; where, I am given input-output data; and therefore, I can compute the left-hand side – the cross spectral density and this second factor on the right-hand side; knowing these two, I can obtain an estimate of the frequency response function. This is called the spectral analysis method of estimating the frequency response function. Whereas, the second equation here is used in estimating power spectral densities of outputs given the system impulse response description or the frequency response description, and the input sequence. So, there are several applications of this result.

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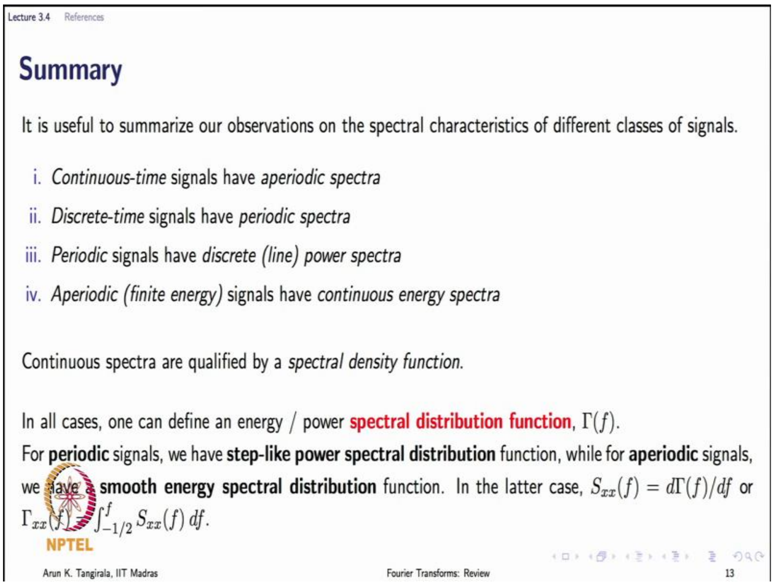

So, just to summarize now, all the four different transforms that we have looked at; the first observation is the continuous-time signals have aperiodic spectra as I had mentioned earlier on in this module. Whereas, discrete-time signals have periodic spectra; regardless of whether this discrete-time signal is periodic or aperiodic, they have periodic spectra. And likewise, for the continuous-time case, regardless of whether this continuous-time signal is periodic or not, the spectrum is aperiodic. And periodic signals have discrete power spectra. So, these are very nice results to remember. What we are saying here is regardless of whether the signal is discrete or continuous in time, periodic signals will always have line spectra. So, there is no notion of spectral density. Whereas, aperiodic finite energy signals have continuous energy spectra again regardless of whether this is continuous-time or discrete-time. And therefore, I can define a spectral density. So, as I mentioned here, continuous spectra are qualified by a spectral density function.

Now, finally we should also know that, in all cases, you can always think of a spectral distribution function like the probability distribution function or the mass distribution function in mechanics. For periodic signals, this spectral distribution has a step-like shape. Essentially, the spectral distribution function is going to give me the amount of power contained in all the frequencies from minus infinity up to that frequency. So, let me just explain that to you.

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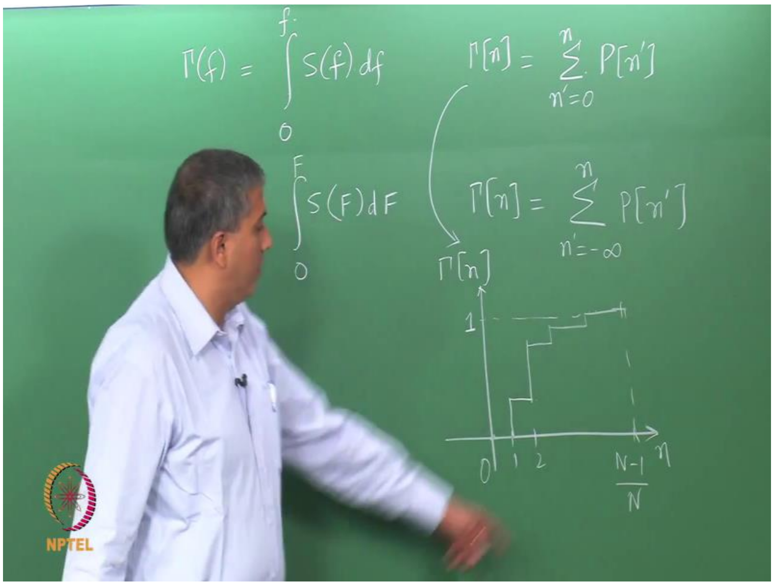

So, let us consider the case of aperiodic signals; in which case, I have gamma as a continuous function; I have dropped the subscripts here. Then, it is the area under the energy spectral density up to the frequency f. So, this is the case of discrete-time aperiodic signal that have finite energy. And likewise, for the continuous-time aperiodic case, I have gamma in terms of the big F. Once again, the interpretation is the same. Now, when I look at the continuous-time or discrete-time periodic; let us take the discrete time periodic case; I have gamma now – a function of the n-th harmonic. So, I have here the n prime running from 0 to the n-th frequency; here I have P of n prime. So, I am going to add up the power spectral density cumulatively up to the n-th frequency. And likewise, one can derive here the expression for the continuous-time periodic case as well. The only difference being here – now, that n prime runs from minus infinity to n; once again I have P of n prime. So, what I meant by saying that, the spectral distribution function having a step-like behavior for the periodic case is you can recall that, these power spectra – now, these are power-spectral distribution function; these are energy-spectral distribution functions. The power-spectral distribution now will have a step-like behavior, because each of this here is only defined at those respective n prime. So, if I only show you for the discrete-time periodic case, it should suffice. And I plot from n equals 0 up to N minus 1 by N. So, this is n on the x-axis.

Assume that, the signal has no power at zero frequency; which means its DC component is 0; then this is n equal to 1. At this point, I have a jump and I have 2 here and so on. So, the jumps of course, need not be identical at every n and so on. Depending on the nature of the frequency distribution, this magnitude will change; but overall, you will see a step-like shape for the spectral distribution; and it saturates here. So, the value here at n equals big N minus 1 by big N is nothing but the total power in the signal.

Now, if I normalize the discrete-time periodic signal to have unit power, then the value at this n will reach a value of unity when we say that, signal has been already normalized. And this is called the normalized spectral distribution. So, that hopefully explains to you how the spectral distributions look like for the aperiodic case, where I have energy spectral distributions; and for the periodic case, I have the power-spectral distribution. Regardless of whether the signal is periodic or aperiodic, I can always define a spectral distribution; but I can only define a spectral density for aperiodic finite energy signals.

So, with this, we come to a close of this module. What we are going to learn in the next module is the properties of discrete-time Fourier transform. We are not going to discuss all the properties; we are going to discuss those that are relevant to us in the context of time-frequency analysis and also look at the discrete Fourier transform in the subsequent modules. So, thank you and hope you have a enjoyable session.

Thanks.