Introduction to Time-Frequency Analysis and Wavelet Transforms

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Lecture - 3.3
Discrete time Fourier series

Welcome to lecture 3.3 of the course on time frequency analysis and wavelet transforms. In this module, we are going to learn particularly the discrete time Fourier series. In the previous 2 modules of this unit we have looked at continuous time periodic and aperiodic signals. Now, it is time to move onto the discrete time case so, we are slowly tending to practicality, because in practice we are going to deal with sample data.

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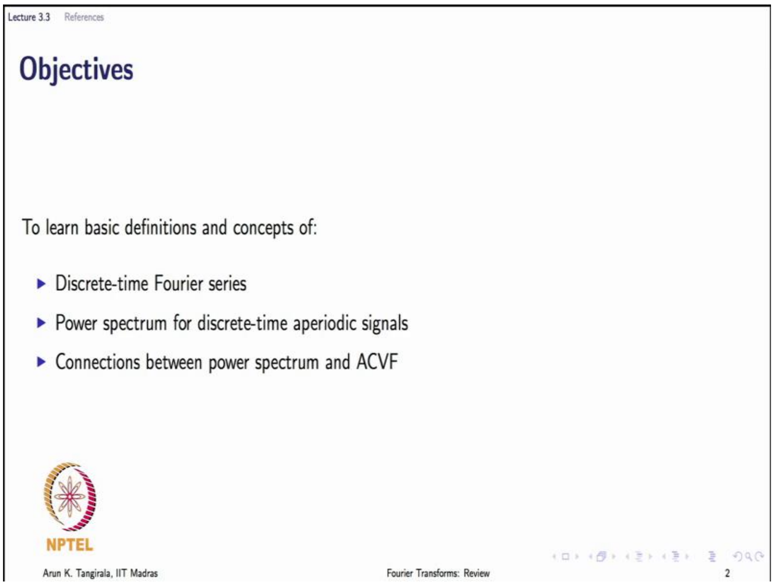

So, the discrete time Fourier series is concerned, discrete time periodic signals as you must have guessed, because we had continuous time Fourier series for continuous time periodic signals. And naturally now, we are going to talk of power spectrum, because recall the analogy or the for the continuous time case. So, the energy spectral density vanishes that we had in lecture 3.2. And now, power spectrum comes back and we will also study a connection between the power spectrum and the auto covariance functions slowly leads us to the properties for the Fourier transforms that will study in the next module.

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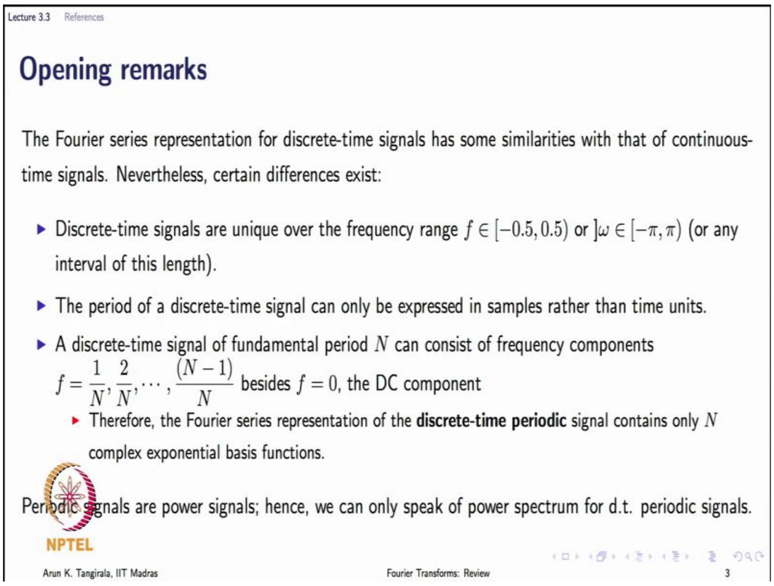

Before we go further, it is very important to understand the consequences of moving from the continuous time case to the discrete time case. And for this recall the concepts that we learned earlier in unit 2 where, we talked about the case of discrete time signals. And we particularly noted that discrete time sinusoids are unique only in fundamental interval which is of width one in cyclic frequency.

This interval could be minus 0.5, 0.5 or 1 0 or if you look at angular frequency it is actually minus pi to pi. And now, the consequence of this property of discrete time sinusoid why are we talking about sinusoid? Because see basis functions are the building blocks in a Fourier analysis or complex sin waves. In the continuous time periodic case, we considered complex sin waves of all frequencies. Now, there is no point in considering complex sinusoid of well, when we say all frequencies, harmonics, but from minus infinity to infinity. Now, the idea in discrete time Fourier serious is going to be the same and I am going to the express the discrete time periodic signal as a weighted combination of discrete time complex exponentials. But with a prime difference being that now I am not going to consider harmonics from minus infinity to infinity.

But I have to, I will restrict myself to this fundamental interval, because outside this interval the basis functions will have eliasis in the fundamental ((refer time 02:58)). So, there is no point in really including them or we can say that they have been already included either way of looking you can look at it. And of course, something to recall that the period of discrete time signal can only be expressed in samples and rather than in time units. Now, the this is the point that I just mentioned the discrete time signal of fundamental period N can consist of how many frequency components? N frequency components, what are those frequencies? Now, these are 1 over N, 2 over N up to N minus 1 over N besides the usual dc component that we have. Notice that we now, switch to lower case for the frequency lower case notation, because now we are dealing with discrete time signals. So, the main message is the Fourier representation of the discrete time periodic signals will not include infinite harmonics, but only a finite number of harmonics, that makes life easier of course, in many ways. And as I mentioned earlier on we are dealing with periodic signals. So, we are going to be going to talk about power and power spectrum, but not power spectral density, because I am only looking at a discrete set of frequencies.

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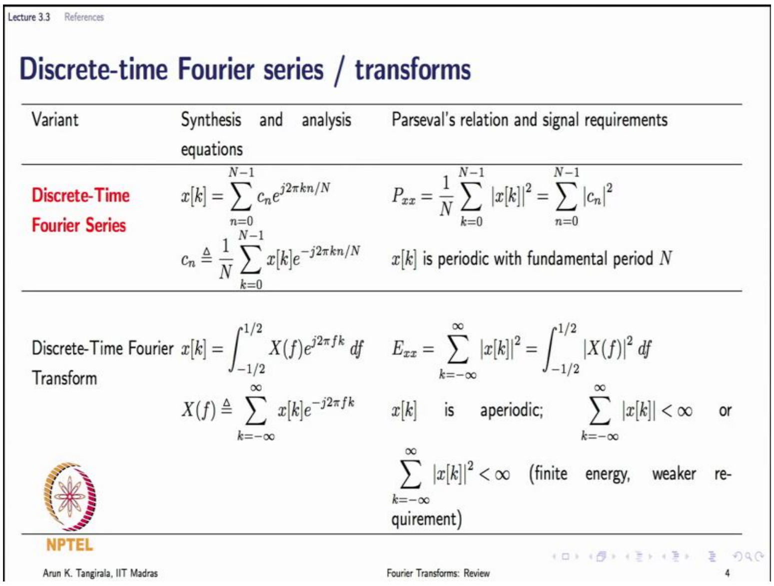

So, just to now, put things in perspective I have the discrete time fourier series this top row which is the focus of the discussion this module. Here, you have the synthesis equations as I mentioned earlier you have a finite summation unlike what we had in the continuous time Fourier series case. The index that keeps track of frequencies this is a small N which runs from 0 to capital N minus 1.

As the index run from 0 to N minus 1, I am actually spanning the entire fundamental frequency range from minus 0.5 to 5.5 or 0 to 1, whichever way you look at it in cyclic frequency. And this is the usual Fourier analysis equations or the expressions for the Fourier coefficient. And on the right hand side you have the power spectral decomposition result, once again due Parseval's which tells me how the power is being decomposed in the frequency domain. And once again I have mod c N square representing the, quantifying the contribution of the nth harmonic which is the discrete time exponential to the overall power of the signal. Of course, the bottom row has the discrete time Fourier transform which is a subject of discussion in the next module.

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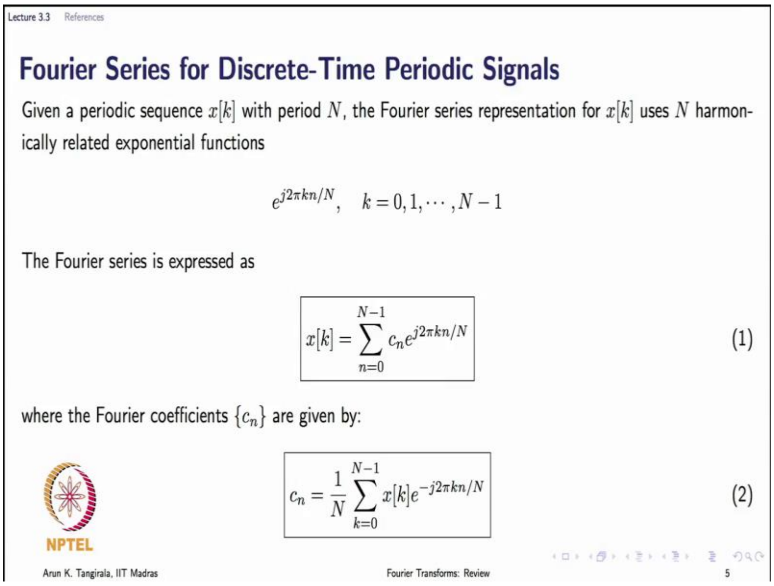

So, we already have had introduction on why we had the summation running only from the fine over the finite length of N and so on. So, let us look at this Fourier analysis equation and synthesis equation bit more in detail. First point again to reiterate, the family of basis functions that I am going to consider is now, finite this is the family of basics functions e to the j 2 pi k N over N. This big N now is a period it is not the number of samples that you have and so on. And here N that is you should not read k here actually N runs from 0 to N minus 1 and the fourier series of course, is express this way while the fourier (refer time: 06:20) are given by this expression as you have seen in the table earlier.

One thing that you can notice is the similarity between the expression for the synthesis and analysis equations. They look very allied except for the dummy indices being different and 2 other differences you have, a factor of 1 over N in front of the summation for C n. And the second difference being e to the j is being replaced by it is complex conjugate. So, that gives me nice advantage when I am computing the Fourier coefficients or recovering the signal. I need to just write one piece of code all I have to do is have this factor 1 over N are in many situations and just switch the sign of j. In many text book you will see in place of 1 over N; you would have 1 over root N in the Fourier series expansion and 1 over root N here. So that the similarity is much more strong on a lighter note, it helps us in memorizing the analysis and synthesis equations in a easy manner.

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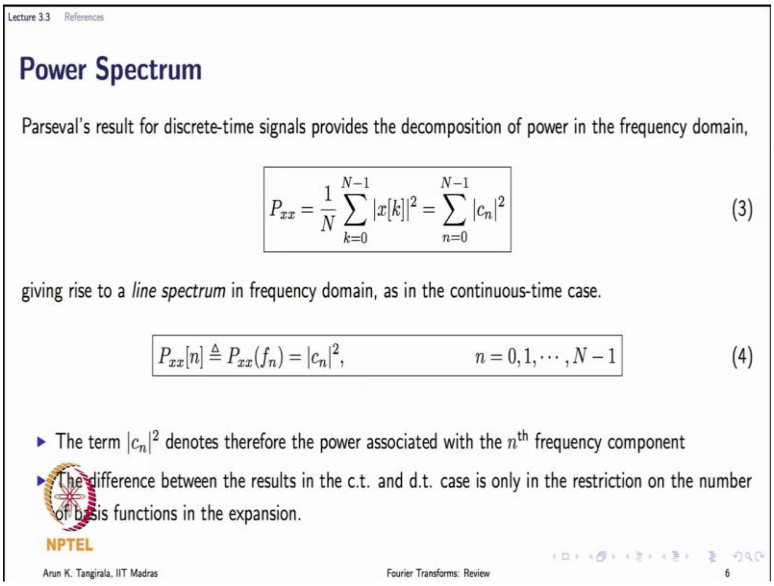

So, moving on to the power spectrum the power spectral decomposition we have already seen this result in the table that I showed you earlier. The main messages here is that mod C n square quantify the contribution of the nth harmonic to the overall power of the signal and N again runs from 0 to capital N minus 1. And the difference again between the results in the continuous time and discrete time case is only in the restriction in the number of frequencies otherwise all the interpretations carry forward. So, this is not much to discuss as far as interpretation of C n is concern. Now, a few interesting properties on this Fourier coefficients t firstly, the Fourier co-efficient enjoy the conjugate symmetry property that is, C subscript n is c star the complex conjugate of subscript big N minus small n minus.

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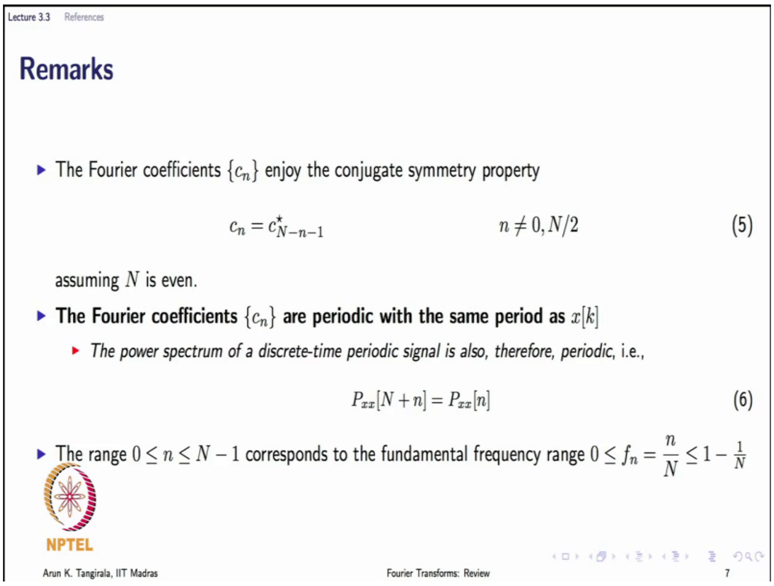

And this is true only when you are looking at indices apart from 0 and N by 2 assuming N is even. And very importantly the Fourier coefficients are periodic with the same period as the signal, which means the power spectrum in fact, repeats itself that again is a consequence of this eliasing property of the discrete time exponentials. So, the power spectrum also enjoys the same periodic property as C signal.

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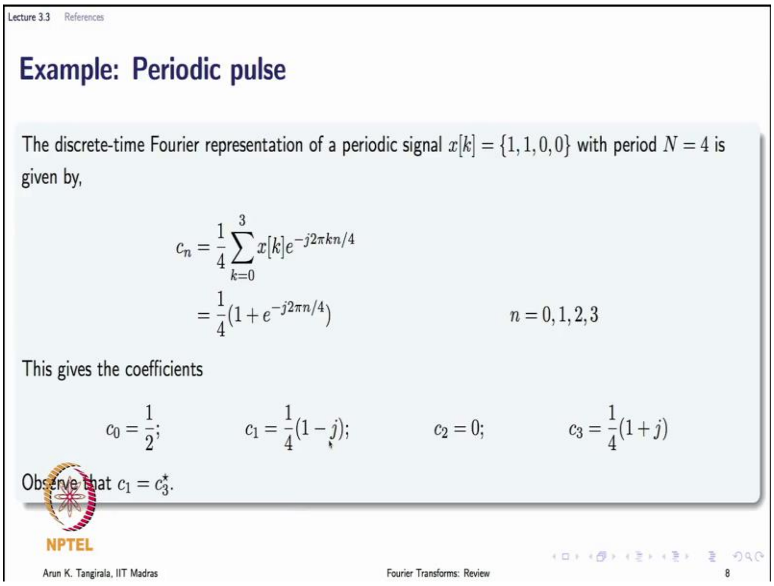

So, let us work through an example to understand the workings of the discrete time Fourier series I just taken a periodic signal and I am giving you the values of this periodic signal over a single period. As you can see the period is 4 and all I do is to compute the fourier coefficients I ignore the expression that I have seen earlier plugging the values of x of k and work through the math it gives me this expression. And if I evaluate the coefficients at all values of n small n 0 1 2 and 3 then I obtain c 0 as 1 over 2 that is theaverage of the signal you should check always the coefficient for frequency zero is the mean of the signal, because of the expression itself. So, plugging N equal 0 here you would obtain 1 over N sigma x k which is nothing but the sample mean or mean of the signal and that is a quick check for your calculations. Then you have c 1 and c 2 and c 3 notices that c 1 and c 3 are complex conjugates of each other by virtue of the property that we discussed here. And this property does not apply to c naught and c 2. So, that is something as a quick verification.

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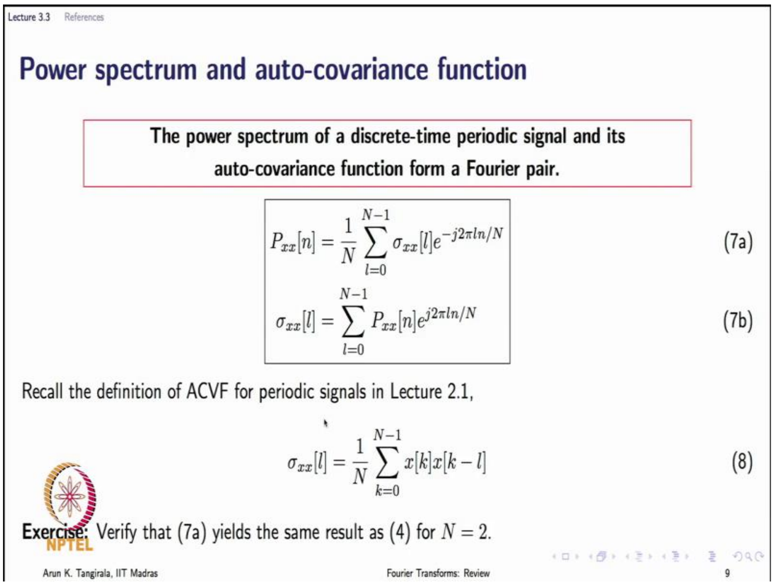

We will conclude this module with a very important result which when extended to the class of random signals leads to what is known as a ((refer time 10:27)) theorem. This result says that the power spectrum of a discrete time periodic signal and its auto covariance function remember we defined the auto covariance function for discrete time periodic signals earlier on in unit 2. They form a fourier pair what it means is the power spectrum is the fourier transform or the discrete time you cannot transform, but the these a fourier coefficients of the fourier series representation of the auto covariance function. So, this is the auto covariance function is the function of the lag l. This is the fourier representation for it that is in 7 b and in 7 a I am calculating the power spectrum as a fourier coefficient of this representation. And this result is fairly easy to prove I will actually leave it to you to verify if this holds.. Of course, you should remember or recall the definition of the auto covariance function for a discrete time periodic signal that we had in the lecture 2.1.

So, plug in this expression for the auto covariance function into equation 7 a and show that it is the same as what we have here in equation four that is p x x of N should get a smart C N square. In fact, as a quick numerical check just set N equals 2 without assuming any values for x and you will able to verify the result. Now, why is this result so important? For two different reasons, one to compute the power spectrum I need not go through the Fourier transform of the signal. I can first compute the auto covariance function and then compute the Fourier transforms of the auto covariance function then you may ask well. So, what right the point is, when I when I move to class of random signals although we do not do that in this course fourier transforms or fourier representation ofr random signals do not exists, but auto covariance functions can be define for random signals.

So, what I do is I define power spectrum not via the Fourier transform of those random signals, but as the Fourier transform of the auto covariance function. Remember when I take the Fourier transform of the signal I have to evaluate the square magnitude in this case it is a periodic signal. So, I am going to compute the Fourier coefficient after computing the Fourier coefficient I have to take this square magnitude. In this case I compute the auto covariance function and the moment I take the Fourier transform of the auto covariance function I get the power spectrum. This result will also be visible to us in the case of discrete time aperiodic signals and is used widely in calculating the auto covariance function. In fact, the x corr x c o r r routine in the mat lab uses it results it actually computes the Fourier transform. And then compute of the given sequence and then constructs the power spectrum and then evaluate the inverse Fourier transform to get the auto covariance function. Because it is computationally more efficient with this note will come to a close this module it is a fairly short one.

In the next module, we are going to discuss discrete time fourier transform which will take us bit more closer to reality, because in reality I am not going to have discrete time periodic signal, I am going to have discrete time aperiodic signals more often. And therefore, I should know how to deal with them eventually end of the discrete Fourier transform which is the most practically relevant version of the Fourier transform to us. So, these are couple of references for your perusal of course, there are plenty of other wonderful references.

Thank you.