Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 08
Reynolds Transport Theorem and the Equation of Continuity

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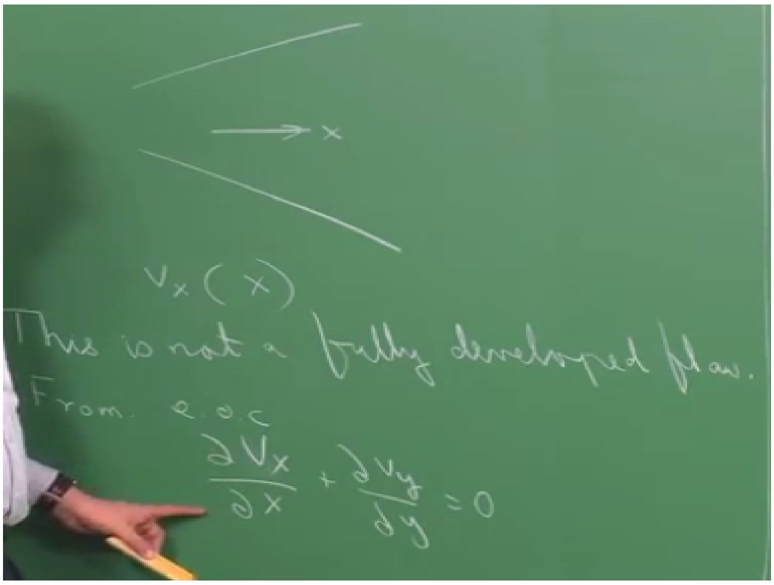

Okay, so welcome to lecture 8 of the course, end of last lecture, Shubham who is not here asked a question regarding the fact that I was actually neglecting the y velocity component in the diverging channel, so let us just go back and take a look at this channel and if you recall one of the things that happens is that the x; this is the x direction velocity component and we realized that vx, the x velocity component is going to be a function of x.

Because it changes in the direction of the flow okay, so what this means is that the flow is not fully developed okay, so developing flow and because vx is a function of x, it implies that you will have a y velocity component and that you can see by looking at the equation of continuity. So, point is that this is not a fully developed flow okay and from the equation of continuity, which we are going to derive later on today is; we know that dvx/dx + dvy/dy equals 0.

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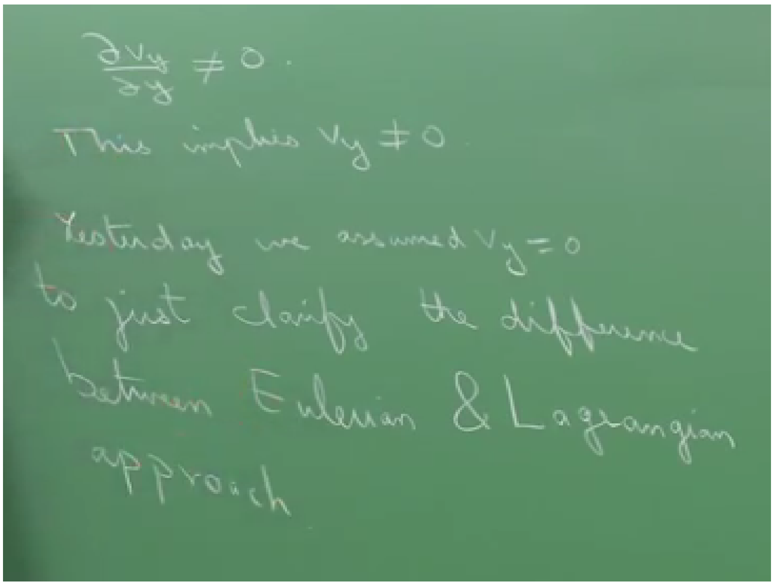

And since this is non zero because velocity is changing with x, we have dvy/dy is != 0, which implies that there is going to be some kind of a variation of the y velocity component in the y direction okay and this implies vy is != 0, so in the strict sense, what I should have done is taken into account the fact that we have both the components vx and vy but I just wanted to illustrate the concept of the Eulerian approach and the Lagrangian approach.

And so, I did not really worry about this vy, okay I just wanted to talk about the fact that the Eulerian approach, you are looking at a fixed point and you just talk about variations at that point and in the Lagrangian approach, you move with the particle, I just want to illustrate the idea. So, in some sense, I made a simplification, I just wanted to clarify that before we move on.

So, yesterday we assumed vy equals 0 to just clarify that the difference or illustrate the difference between the Eulerian and the Lagrangian approach okay. Now, come back to what we wanted to do today, which is the Reynolds transport theorem. How many of you have seen this before? Okay, quite a few and quite pretty much everybody okay, that is good but then I did not realize this, so we are going to do that anyway today and then we will move on.

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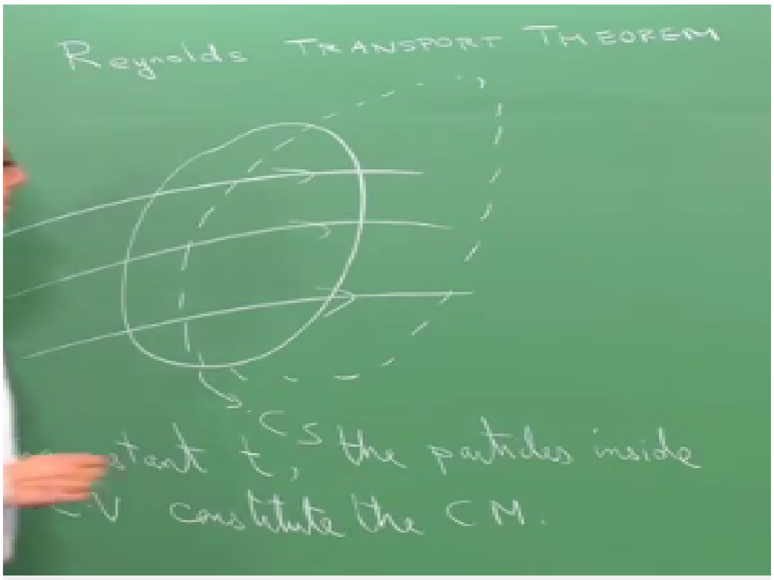

So, what we did yesterday was the derivation of the Euler's acceleration formula okay, today we are just going to extend this to a finite volume region. So, let us consider a flow field, liquid is moving in this direction, I am going to look at this region, which is a fixed region in space and that is my control volume okay. So, just to keep life simple, we just doing everything 2 dimensional okay but if you want, you can make it to a 3 dimensional thing.

Since I can draw things 3 dimensional on this 2 dimensional board, so we just doing 2 dimensional flow fields, so now you have this ellipsoidal oval shape thing which is my control volume and this boundary of this control volume is my control surface okay. What I am going to do is; and remember what we are interested in this is; trying to relate changes in control volume to changes in control mass okay.

So, I am going to look at a particular instant of time t and I am going to look at all the matter, all the molecules, all the particles which are inside this control volume and I am going to call that my control mass. So, at time instant t, the control mass is occupying my control volume okay. So, this is my control surface and this is the region inside is my control volume and at time instant t, the particles inside the control volume constitute the control mass, okay.

So, instant of time whatever is inside is my control mass and that is occupying a particular region that I am identifying as my control volume. What is going to happen is at a later instant of time, a delta t later because of the flow, the particles which were inside are actually going to move out okay because there is a flow and this; the control mass is going to occupy another region in space, the control volume is fixed okay.

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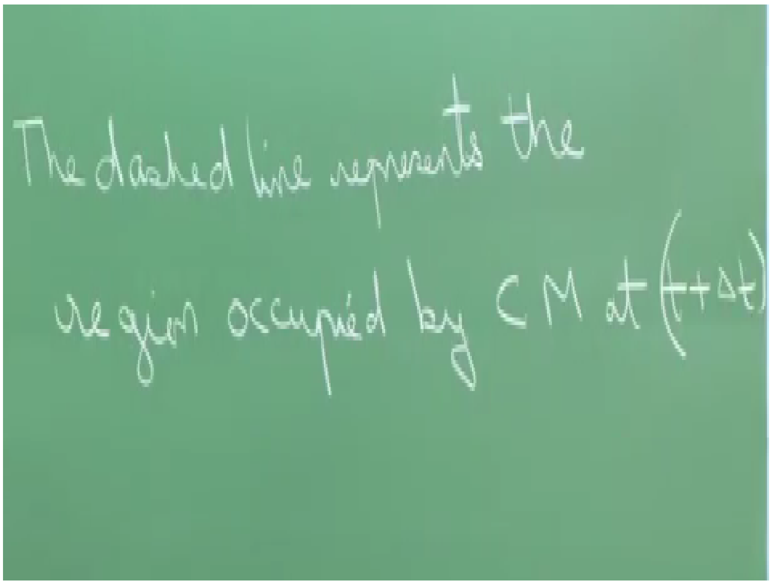

This is my solid surface, solid curve that I have, this dashed line represents the boundary of the control mass at a later instant of time t + delta t okay. So, the dashed line represents the region occupied by my control mass at t + delta t okay and clearly the mass can move right because of the flow and so it is occupying a different region in space and that has given by this dashed line and time t, the control mass is colliding; coinciding with my control volume.

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What I want to do is; I want to; for the sake of clarity, I just want to write this portion on the left, I am just going to call this region 1, the portion in the middle, which is common to time instant t and time instant t + delta t, I am just going to call this region 2 and the portion, which is left the control volume, I am going to call this region 3 okay and what I want to do is; I want to look at the rate of change of a particular property okay, associated with the control mass.

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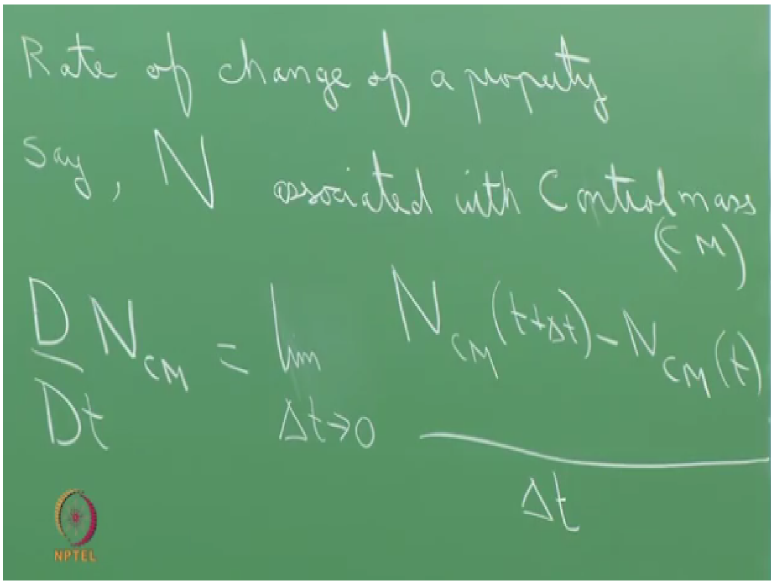

So, you can have the property associated either in the control volume or with a control mass, so what we will do is; we will look at rate of change of a property, say I am just going to call it capital N, okay, associated with the control mass CM. Now, since it is associated with a control mass, you need to take into account the fact that is going to occupy different positions at different times.

And so what we are looking at is the total derivative or the material derivative okay, so now we going to use the same notation as last time and although, this may be need extra cautious, I am telling you, the property associated with a control mass, I am telling you it is a substantial derivative because I am looking at the rate of change of the derivative of the rate of change of the property with time.

And this as you all know is nothing but in the limit of delta t tending to 0, the property of the control mass at t + delta t - the property of the control mass at t/ delta t, okay that is just the definition of the derivative of the property with time and this is the left hand side, which is the time derivative, which I would get when I use my Lagrangian approach and what I want to do is; I want to use this right hand side and try to relate that to changes in my control volume.

So, that is the plan okay, so let us go back a little bit here, I want you to recall that the control mass; the property associated with a control mass at time instant t is the same as the property, which is present in regions 1 and 2 because at time instant t, the property of the control mass; is the control mass occupying the regions 1 and 2 and so the property NCM is the property associated with what is present in 1 and 2, okay.

And a time means t + delta t, the control mass is now occupying regions 2 and 3, so the property associated with the control mass at time instant t + delta t is the property of; present in the region 2 at time t + delta t and the property present in the region 3 at time t + delta t, okay. So, you can see that I am shifting from control mass to a region in space and then we will see how things change in the limiting process.

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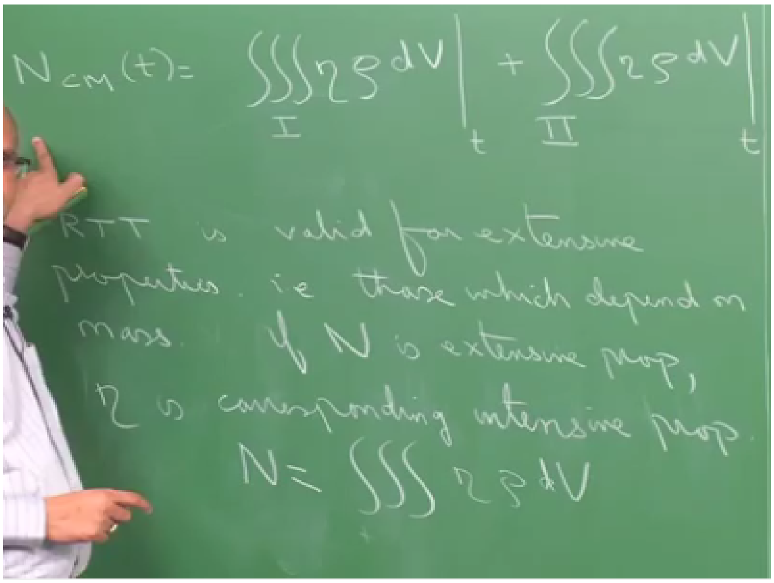

So, now we come back to NCM of t, this is nothing but the property associated with the control mass, which is now occupying regions, 1 and 2, okay. I am going to write this as the volume integral okay, of evaluated at t + the volume integral over region 2 evaluated at v. So, I am just indicating that at time is in t, I am evaluating this at time t okay and what I have done is; I am taking into account the fact that I can actually have variations inside my control volume.

I can have spatial variations of the property inside my control volume okay. Now, before I proceed, I want to mention to you that when we are using Reynolds transport theorem, it is basically going to be applied to properties, which are extensive okay, which are going to be basically changing whether they are going to be depending on the mass present. So, for example, a property like momentum that is an extensive property because momentum depends upon the mass okay.

So, what does eta represent? Eta represents a specific property or an intensive version of a same property okay. So, basically what I am trying to tell you here is; before I proceed, RTT is valid for extensive properties that is those which depend on mass and if N is my extensive property, the corresponding intensive property is eta, okay. So, if N is the extensive property, eta is the corresponding intensive property okay.

So, what you can do is; you can look at NS in a particular region as eta rho dv, so dv is the volume element okay that I am looking at, rho dv represents the mass element, I am going to allow for variations of density inside my region, so I want to keep this here, so this gives me my mass element and this is property per unit mass and therefore, this is going to represent my total property in that region okay.

And I am going to integrate over the entire region 1 and region 2, I get the property associated with the control mass okay. So, the triple integral just tells you that this is a volume integral, let me just say that it is a volume integral okay because the control volume is 1 and 2 together. Now, I want to look at what is going to happen at time t + delta t.

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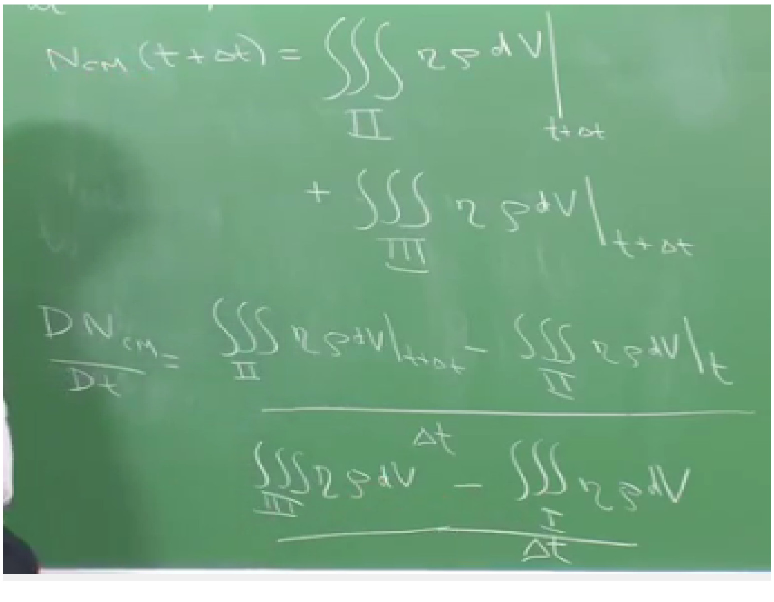

At t + delta t, we have NCM of t + delta t that is going to be the property present in the system at time instant t + delta t in regions 2 and 3 okay because at time t + delta t that is the region, that is being occupied and I am going to go from CM to control; to volumes now to region in space and I write this as a triple integral over region 2 of eta rho dv but this is at t + delta t and then this is; I am forbidden to go past that line, so I come back here, 3 okay.

So, what do you want to do is; you want to subtract and get the numerator NCM of t + delta t - NCM of t and so we just go back and rewrite Dt of NCM of t as integral; I'm going to combine say what we observe is that the region 2 is common here as well as in the other term, so I am going to club those two together, I am going to Club the other two together okay. So, the numerator becomes of my substantial derivative of my material derivative becomes integral over region 2 of eta rho dv of t + delta t - integral over region 2 again of eta rho dv evaluated at t/ delta t, okay.

That is one of the contributions and then the other contribution is coming from this term, which is integral eta rho dv over region 3 divide by; and then - integral over region 1 eta rho dv/ delta t, okay. So, all I have done is rearrange those terms keeping regions 2 together and 2 and 1 separately. Now, clearly these volume integrals are both over the same region in space and remember this is all happening in the limit of delta t tending to 0.

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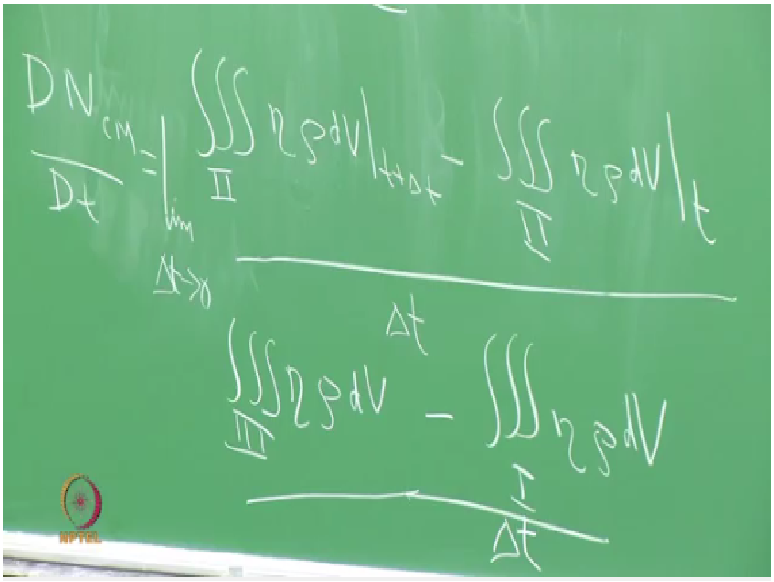

So, I need to remember this otherwise, I am going to get into trouble okay, this is a limit of; in the limit of delta t tending to 0, what is going to happen is; this corresponds to what, the region 2 is going to coincide with my control mass sorry, my control volume because at time t and time t + delta t, when you only have delta t changing incrementally; infinitesimally region 2 is going to coincide with the control volume okay.

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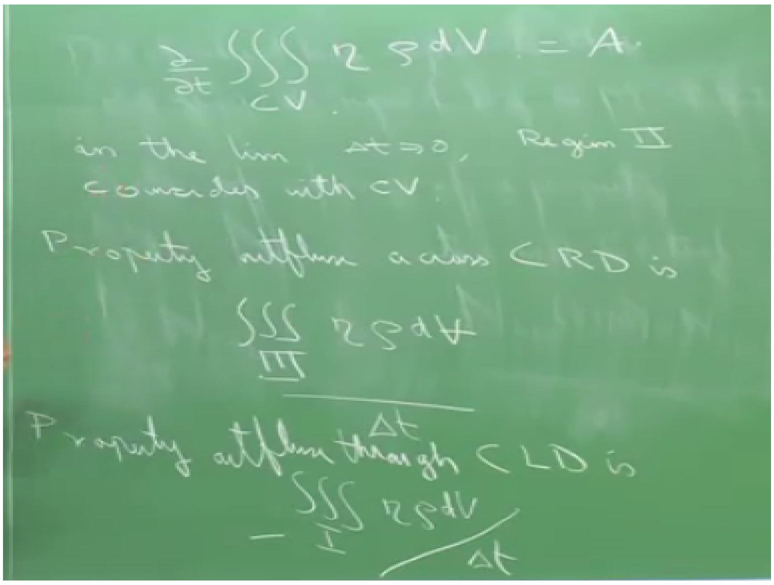

So, this particular term; this first 2 terms basically represent the change in the property in my control volume okay and so I am going to write these 2 terms as the partial derivative with respect to time of integral of eta rho dv, okay over the control volume. So, let me just tell you; write down what I just said in the limit of delta t tending to 0 region 2 coincides with my control volume okay.

And so this region 2, which I had; I have just written it as control volume. So, I am just going to say that this particular term; the first 2 terms, let me call this A and the next two terms, let me call this B. So, I have just written A here, okay. Now, we need to worry about this other 2 terms. What does this represent? This represents eta rho dv in region 3 at time t + delta t represents the property, which has actually left, the control volume through this part of the control surface over this time interval from t to t + delta t.

So, this represents the efflux or out flux of the property from the right hand side of the surface, so maybe I should just; A, this R and B, I have already got an A and a B there, so maybe I should not use A and B, just use C and D okay and this is L, so R is for right and L is for left. So, C, R, D is a portion of the control surface and what I am talking about is the integral that you have over region 3 here represents the property, which has left the control surface through the fraction C, R, D okay.

So, I am saying that the property efflux or out flux across C or D is given by the integral over region 3, okay of eta rho dv/ delta t. So, this is the rate at which the property is leaving okay. So, this particular term represents the way that means property has left. Now, what about this term here; eta rho dv and remember this associated with a minus sign, so this is basically going to be evaluated.

This basically represents in this time t; from t to t + delta t, the property which has come into the system okay. So and what is come into the system in the region 1 and it is occupying the region 1 is represented by the last term here and since there is a negative sign associated with it, it effectively along with the negative sign represents a net efflux, net out flux okay. So, what I am saying is this is the out flux through that; property out flux through C, L, D is over region 1 of eta rho dv/ delta t, okay.

And since I have written out flux, I must put the minus sign. So, the region 1 basically tells you what has come into the system because of the flow process and that has the property we just entered the system through this portion C, L, B to this portion of the control surface CLD, region 3 tells you what is the property, which has left the system because of the flow. See, because of flow, what is going to happen is some property maybe momentum or a mass is entering the system okay and leaving here.

So, this represents what is left, this represents what has entered okay and so basically what this means this that these 2 terms here combine, which gives you B represent the net rate of efflux, net rate of out flux across the entire control surface okay.

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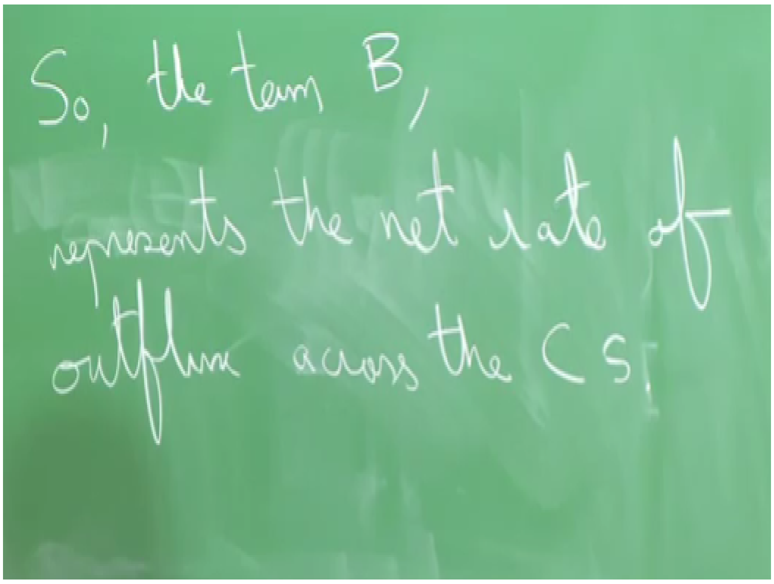

So, the term B represents the net weight of out flux across the control surface okay. So, what I am going to do is; I am just going to summarize whatever we have done.

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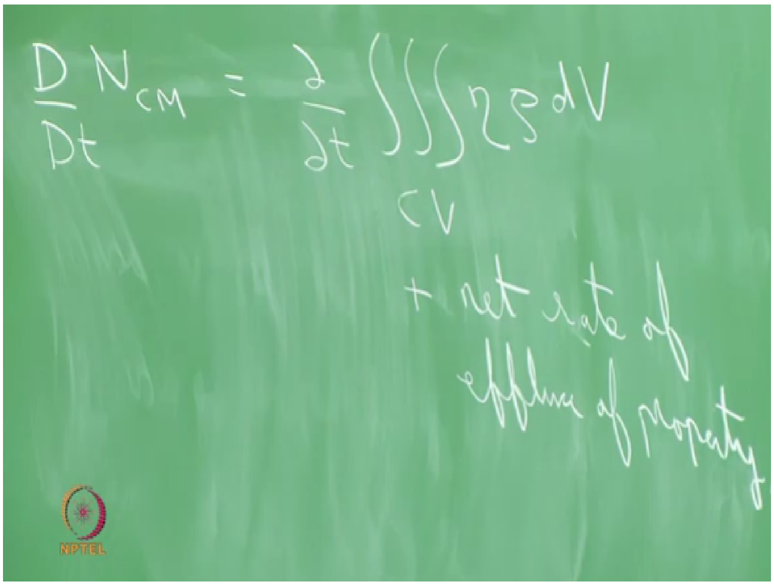

And write this as saying that DDt of NCM, the rate of change of the property associated with the control mass is = dV, the rate of change of property associated with the control volume and now remember at a particular time instant, the control volume and control mass are the same okay, plus the net rate of efflux of the property, okay. Now, since we are all mathematicians, we do not want this English.

And I am going to now convert this English to some mathematical expression, okay. I just want to tell you physically what is happening okay. So, how do we go about writing this net rate of property efflux in the form of a mathematical expression?

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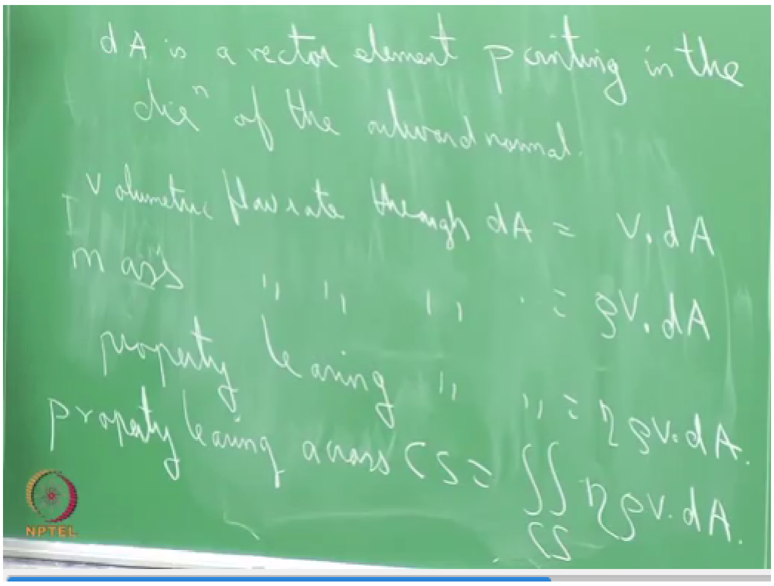

So, if you look at this control surface and you take a small element okay, this particular area element will be a source is a vectorial quantity and the direction of the vector is going to be that of the outward normal, okay. So, this is my area element dA and this is the normal direction okay. So, dA is a vector element pointing in the direction of the outward normal. Now, if I wanted to ask, what is the volumetric flow rate?

Because we have a flow problem, what is going to be the volumetric flow rate through this area element dA, you would find that the volumetric flow rate through dA is going to be given by V dot dA is the dot product of the velocity vector and the area vector okay, you want the mass flow rate is going to be given by rho V dot dA and if you wanted to know; what we are interested in is the rate of flow of the property.

So, along with this mass associated with this mass the property that is going to be leaving is going to be eta multiplied by rho V dot dA, okay. The property leaving through dA is going to be eta times rho V dot dA and remember what we are interested in is; the property leaving across the entire control surface. So, in order to find what the property leaving across the entire control surface, I need to take the surface integral of this term.

And that will tell me, what I really I am interested in, which is the rate of effects of the property and maybe I should write this here, across the control surface okay. So, to find the property leaving in fact, remember this is a rate, okay leaving across the control surface, I need to do the surface integral and so there are only 2 integral signs of eta rho V dot dA, okay. So, now I am in good shape.

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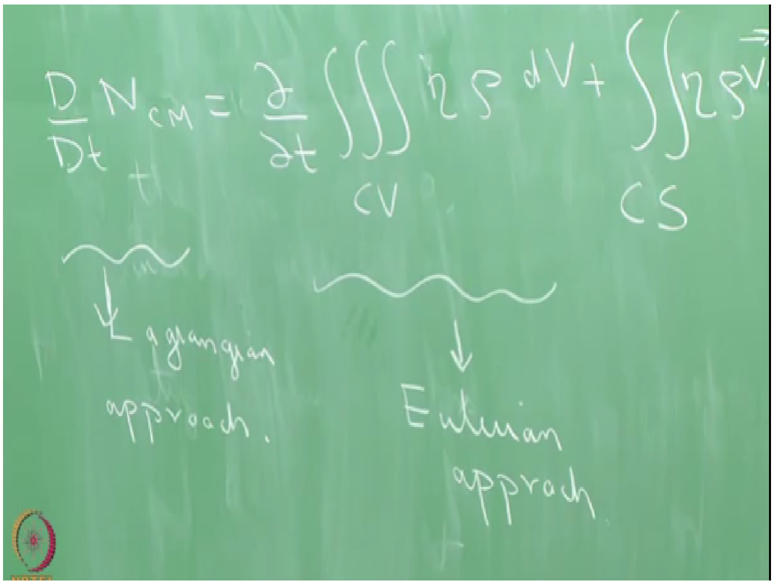

I can just write the mathematical expression that I wanted to derive, which is DDt of NCM equals the partial derivative with respect to time of eta rho dV + integral over the control surface of eta rho V dot dA, so this is Reynolds transport theorem okay and what it does is; it associates the rates of change of a property of a control mass to the rate of change of the same property occupying the control volume, where the control volume is taken to be that which is been occupied by the control mass at a particular instant of time.

And the relationship is given by the fact that you need to account for the fact that property can enter or leave, this is the control surface okay. So, this basically is what you would measure, when you are doing using a Lagrangian approach that is the term you would be measuring because you are talking about a control mass okay and this term is what is associated with the Eulerian approach.

And this basically tells you that the rate of change associated with the control mass is equal to the rate of change associated with a control volume plus the net rate of efflux okay, net rate of efflux of the property across the control surface. So, of course all this is fine, it is just some mathematical expression but maybe one or two illustrations will possibly clarify things. So, let us apply this to a couple of properties.

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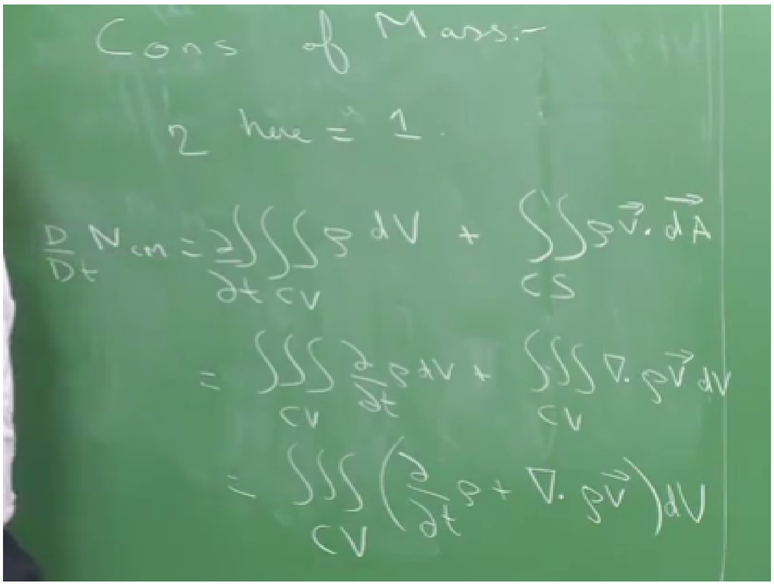

And okay, let us apply RRT to conservation of mass, now mass is an extensive property, right because clearly it depends upon the mass present. So, now if you are applying mass, the conservation; the Reynolds transport theorem to conservation of mass, what is going to be eta? Yeah, so eta here equals unity, okay. So, let us go back and write; somebody said 1 and this correct, the control volume rho dV okay, plus the double integral over the control surface of rho V dot dA.

Now, what I like to do is; I do not like the fact that I have a volume integral here and I have a surface integral here and this is where the mathematics you will learn comes in useful because you know how to convert a surface integral to a volume integral okay, you can use one of the theorems you have learned Green's theorem, Stokes theorem okay and what we can do is; we can write this as a divergence over the volume.

And I am going to do 2 steps together now, I am going to make use of the fact that my control volume was fixed in space okay, so it is not changing with respect to time. What this means is this time derivative I can actually pull it inside my integral, I can interchange the order of my differentiation and integration because my control volume is constant it is not changing with time. So, I am going to do that.

And I am going to convert this to a volume integral okay, rho dV + integral over the control volume of divergence of rho V dV, okay and now I can combine these two guys together and I can write this as integral over the control volume of d/dt of rho + divergence of rho v times dV okay. Now, since we are talking about conservation of mass, I am going to ask you, what is the left hand side?

The left hand side is identically 0 because by definition, control mass has a fixed collection of particles; the fixed mass, okay, the mass of the control mass is not changing with respect to time, the total mass present by definition does not change with time, so the left hand side is 0 okay.

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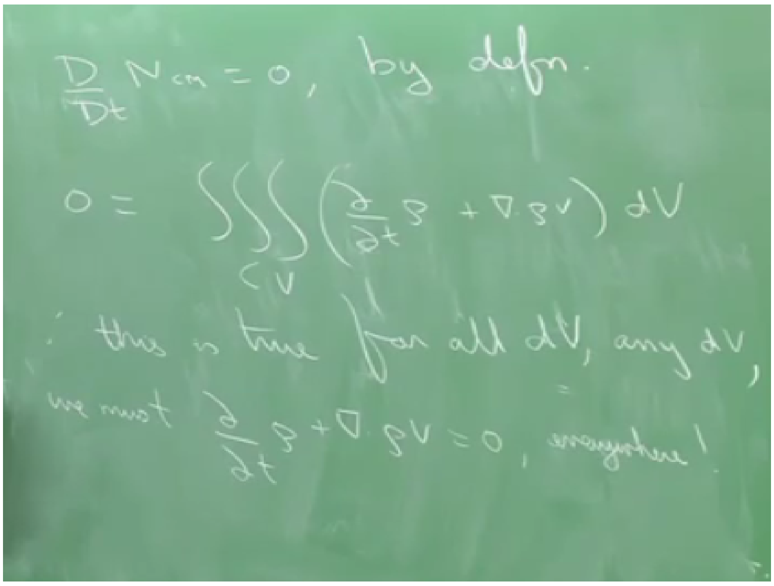

So, DDt of NCM is 0 by definition and what this means is; I have 0 equals control volume d/dt of rho + divergence of rho v DV but remember this is true for any control volume; any control volume element dv, which means it has to be true for all points. This particular; the only way this integral can be 0 is if this term identically vanishes everywhere inside my flow field, okay. Since this is true for all dV, I should actually use any dV.

We must have d/dt of rho + divergence of rho V = 0, identically everywhere, okay and that is the equation of continuity which you have seen before in the courses in fluid mechanics and transport phenomena, just couple of points, this particular DDt of NCM can be nonzero, under what circumstances? If you have a nuclear reaction for example, because then you have a material mass, you have a radioactive element.

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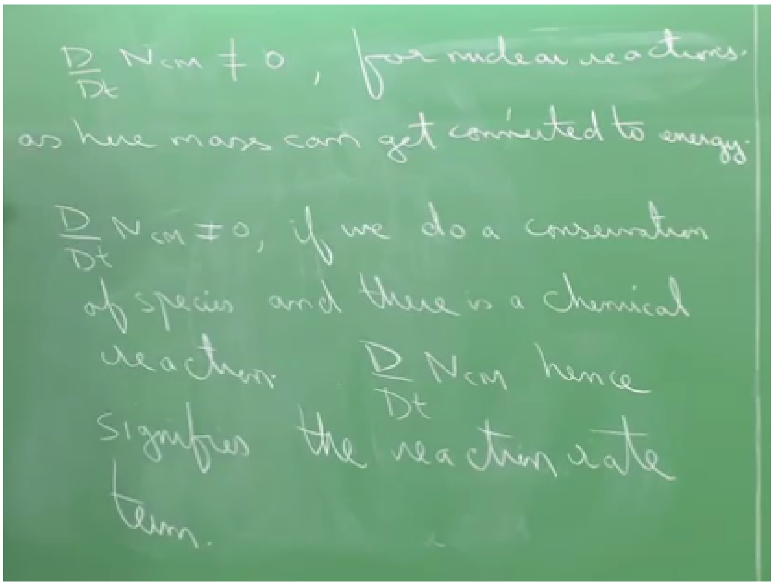

And what is going to happen is; it can convert; mass can get converted into energy and that can be a possibility of this thing decreasing, so I just something I wanted to mention to you okay, so DDt of NCM equals; is != 0 for nuclear reactions, since as here, mass can get converted to energy okay. There is another situation, where DDt of NCM can be nonzero and that is when you are not doing an overall mass balance.

But you are doing a species balance and when there is a chemical reaction taking place okay, so DDt of NCM is != 0, if we do conservation of species and there is a chemical reaction. So, if you put your molecules inside, so the total mass of the system will be constant but now you are not interested in the total mass, so as chemical engineers we are interested in sometimes following what is happening to a particular species, okay.

It could be a reactant or it could be a product, so now if focusing on the products you have a batch system essentially, there is no flow, because you are looking at fixed collection of molecules, fixed collection of particles okay but the reaction can take place and you can have the reactant getting converted into a product okay. So, basically what I am trying to tell you here is that if you have a chemical reaction taking place, this particular term is going to be represented by your reaction rate term, okay.

So, this DDt of NCM hence signifies the reaction rate term okay because I think there is normally some kind of a confusion which people have, when they are; you know doing these mass balance equations because we sometimes say that the time derivative is a reaction term sometimes we say a time derivative is an accumulation term, okay. So, now the point is the reaction term is coming from the material derivative okay, of associated with the control mass.

Whereas, when you are talking about the other the partial derivative term associated with the control volume that is associated with the accumulation term. So, now I will do what I promised you earlier that you have already seen Reynolds transport theorem in a different form earlier and maybe I will now write the Reynolds transport theorem in form which you are comfortable with.

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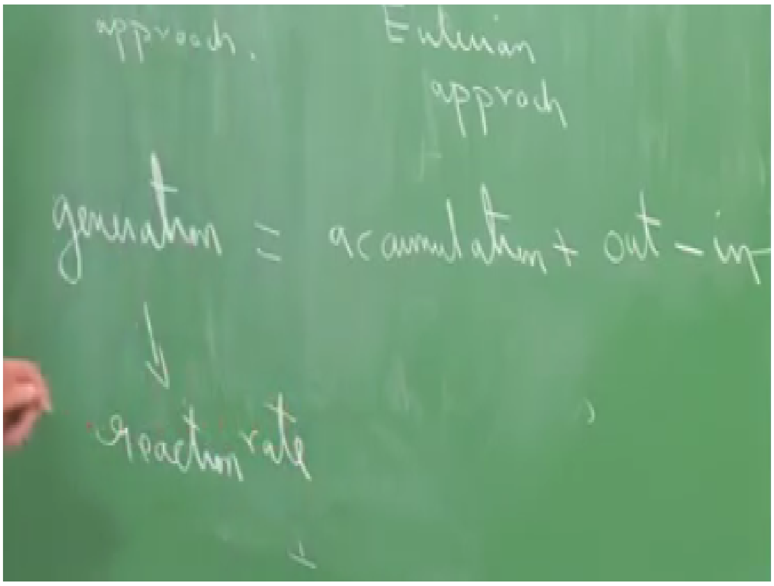

This term remember is nothing but that associated with the reaction term, rate of change of the property, so that is what is called in undergraduate classes as a generation term, okay, rate of generation and this here is associate with a fixed volume in space and that is my accumulation term and plus, this is minute efflux, so that is my out – in. So, most of you have actually seen this in different form, sometimes some books have accumulation equals in - out + generation.

Some books write it this way okay, so the point I am trying to make here is this is a form, which you may be already comfortable with that the Reynolds transport theorem is nothing but a similar equation as what you have already seen. Only thing is it has been derived in a slightly different way, okay in a more general way and like I said earlier, if you have a chemical reaction, this guy will be associated with the reaction term, this is a reaction rate.

And this is accumulation; what is happening in the system and this is what is leaving coming in and out okay, so this is the net rate of efflux, which is out – in, region 3, which show me what was going out, region 1 which is coming in okay, those are the 2 terms. This is my accumulation term associated with region 2, which we had in the derivation and this is the material derivative okay.

So, because there is always some confusion which people have rather to put the accumulation term as the rate term or this term, so what do you do when you actually are doing a batch reactor is; you have nothing coming in nothing going out, the rate of accumulation becomes equal to the reaction rate okay and so I want you to understand things from this perspective rather than you know putting things (()) (45:04) some, time derivative equal to a reaction rate term okay.

It is possibly necessary that you do some problems and we will do that or I will put it; we will make you do that and then we will hopefully clarify some of these concepts but one assignment, which comes to my mind immediately is yesterday we had the derivation of the Euler's acceleration formula right, I told you that is for a particular point, for a particular particle and this is just a macroscopic version of the same thing.

I want you to tell me if starting from the Euler's acceleration formula, you can actually get to the Reynolds transport theorem from an infinitesimal particle, from an infinitesimal region in space to a macroscopic region. So, that would just be a little bit of mathematics, maybe a little bit of physics.

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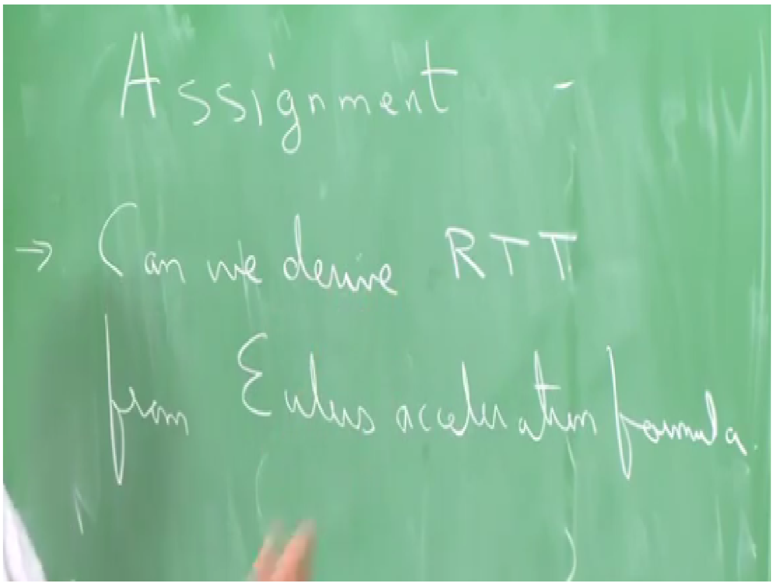

So, let me write down this; can we derive the Reynolds transport theorem from the Euler's acceleration formula. I mean you know, if you are doing Euler's acceleration formula, we did for velocity then the property would have been momentum but it does not matter, you understand what I am saying, you can just say any property eta and you can try and do this. This is just to make things different.

Because some people I can see a little bit bored because they see some of these concepts earlier, so I need to make it challenging for them okay. What we want to do tomorrow is basically do the conservation of momentum there again unfortunately, will be a small repeat of what we have done but I will just tell you what we are planning to do, okay.

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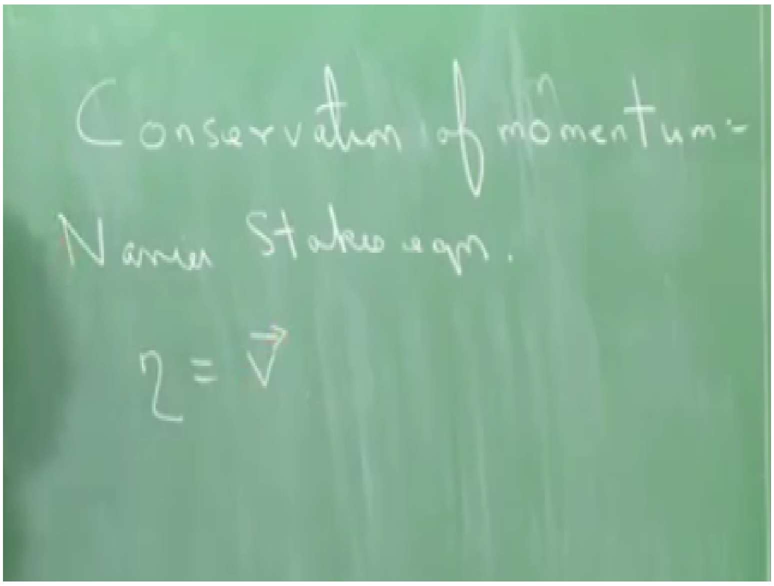

So, tomorrow we are going to see conservation of momentum that is the Navier stokes equations and here the intensive property eta is going to be the velocity okay but what we will do now is; we would write the material derivative of the momentum that is; that associate with the control mass, we will use Newton's law of motion to relate that to the forces, which are acting and then relate the forces to the changes in the control volume