Multiphase Flows: Analytical Solutions and Stability Analysis
Prof. S. Pushpavanam
Department of Chemical Engineering
Indian Institute of Technology - Madras

Lecture – 05
Scaling Analysis: Worked Examples

Good morning everyone. I welcome you to lecture 5 of multiphase flows. So today we will continue a bit more on scaling analysis. Last class, we had seen a basic introduction to how to use scaling analysis to analyse the physics of the problem under different limits. So today I will be looking at a transient problem, a problem which evolves in time. So in the last class we had a problem where in the flow was kind of steady, so there was no timescale involved.

Today, we are looking at a problem which is unsteady and it evolves in time. So the problem that we are going to look at today is a fluid which is confined between 2 walls.

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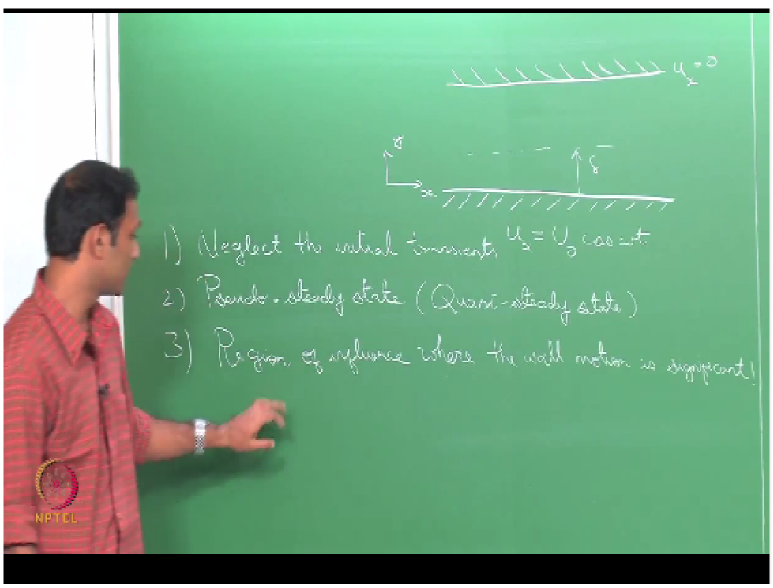

The top wall is stationary and the bottom wall is going to be moving periodically given by Ux in the x direction, U0*cos omega t, okay and the top wall is stationary and my coordinate system is x in this direction and y in the perpendicular direction, okay. So this is a problem that we are going to look at. So it is a transient problem. So the time t=0, the fluid is at rest and we look at the case wherein the bottom wall is set into motion periodically and right now, there are 3 scales, timescale in the problem, which we will be seeing shortly.

So the thing that we will be discussing today is essentially 3 things. The first one is then we can neglect the initial transients, okay. So we will be finding out conditions under which we can neglect the initial transients. The second thing that we will be looking at is a case called as pseudo-steady state or something called as Quasi-steady state, okay. So we will find out conditions under which we can assume that the flow is essentially at an approximation which is called as pseudo-steady state approximation.

And the third thing is we will find out the region of influence where the wall motion is significant. So essentially what I am saying is that in the third condition I am saying that the flow of the, the motion of the wall is going to affect the flow only till a particular region of influence, something called as delta. So these are essentially 3 cases that we are going to discuss. The first is wherein I have the motion of the wall set into motion at some time t=0.

And we are going to find out the conditions under which we can neglect the initial transients. In the second case, we look at the case where we can say that the Quasi-steady state approximation holds true and the third is the case where we will be looking at where the motion of the wall is only influencing till a particular depth, it is called the region of influence delta which we will be finding out through scaling, okay.

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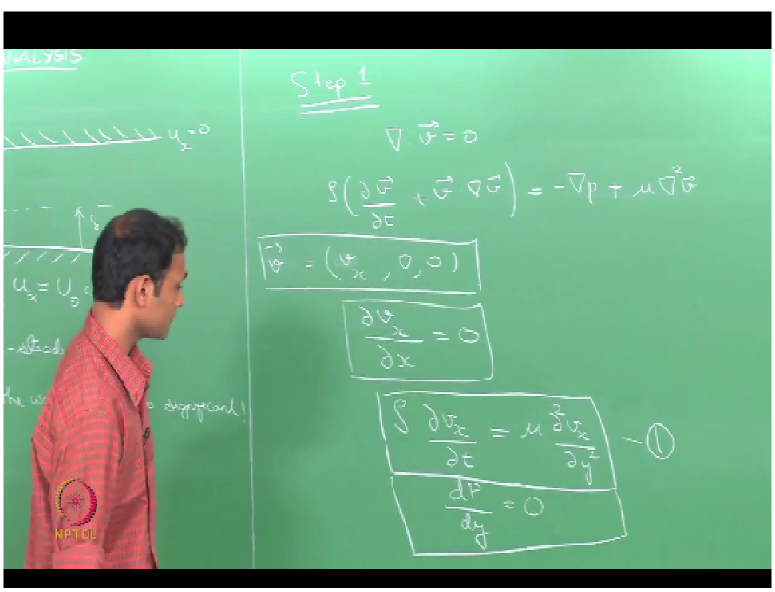

So we will go to step 1 as was discussed in the last class which is nothing but writing down the governing equations in the dimensional form. So as in any fluid flow problem, the continuity equation and the Navier-Stokes equation are the governing equations. So we have the 2 equations given as these, okay. So we have the continuity equation which is nothing but mass conservation and we have the Navier-Stokes equation which is nothing but Newton second law right for the fluid, okay.

So we will write it down in the expanded form for the 3 directions. So we fundamentally assume that the flow is 1-dimensional. So you only have an x competent of velocity, okay. So we assume that you only have Ux, Uy and Uz are 0. So this is my U vector or the V vector basically in this problem, okay. So we have V given by this particular vector, okay. Now if I substitute this into the continuity equation, I would get that my Vx is not varying in time, varying in space x.

So dou Vx/dou x=0 which basically means that my flow is fully developed, whatever I have at x = 1 point, is the same as it would be at any other x, okay. So this is my first equation, the governing equation, the continuity equation. If I substitute Vx and Vy and Vz to be 0 in this particular governing equation, the Navier-Stokes equation, you will see that very easily we can find out that the governing equation would be dou Vx/dou t given by mu * dou square Vx/dou y square, okay.

And this is the x component of the Navier-Stokes equation. The y equation in the absence of gravity would eventually be very easy and you would get dP/dy=0 which basically means that pressure is not changing in y direction. This is in the absence of gravity. So we are neglecting gravity in this problem, okay. So this is my equation that I have, okay. So we can basically neglect this equation, it just tells me that pressure is not changing in y.

And this just tells me that my velocity in the x direction is not going to change in the x direction. So the basic equation that we are left with is this one which I call as 1. So we have finished our step 1 where in we got the governing equations in the dimensional form, okay. Now we go to step 2.

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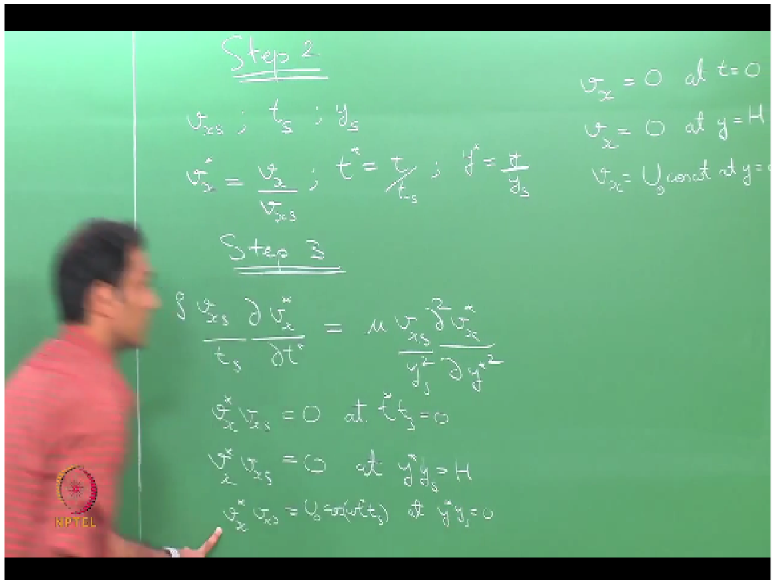

Step 2 essentially involves choosing reference scales to nondimensionalize the problem. So we choose Vx scale to be the scale for the x direction velocity component. Then we have ts which is the scale for time and then we have the y direction. So we choose a scale in the y direction to be given by ys, okay. So our nondimensional velocities is given by Vx/Vxs, nondimensional time is given by time/ts and the nondimensional y directional length is given by y star being y/ys, okay.

So these are the non-dimensional variables that we are going to use. We substitute in step 3. Now we just substitute these variables into the governing equations that we had got. We neglect the first and the last equation, we just directly look at equation 1 because that is the one which is helping me to find out how my Vx is going to change in time. So I rewrite down rho*Vx scale Vx star/dou t star, so I have a time scale coming in here, multiplying by mu*Vx*dou square Vx/dou y star squared, okay.

So these are the governing equations that we get once we have nondimensionalize it. So what we have accompanying these governing equation is the initial and the boundary condition which basically means my velocity is 0 at time t=0, okay and my velocity Vx=0 at y=H where H is the distance between the 2 plates and my Vx is U0 cos omega t at y=0. So these are the initial and boundary conditions which are used to solve this problem that we have.

So nondimensionlizing the boundary conditions and the initial conditions we get Vx dou Vx scale=0 at t times=0 that is the initial condition we have. Next we have the boundary condition at y=H which is given by Vx star Vxs=0 at y star*ys=H, okay and then the third condition we have, Vx star*Vxs=U0 cos omega t star*ts at y star*ys=0, okay. Now next we just rearrange the equation and the initial conditions and the boundary conditions.

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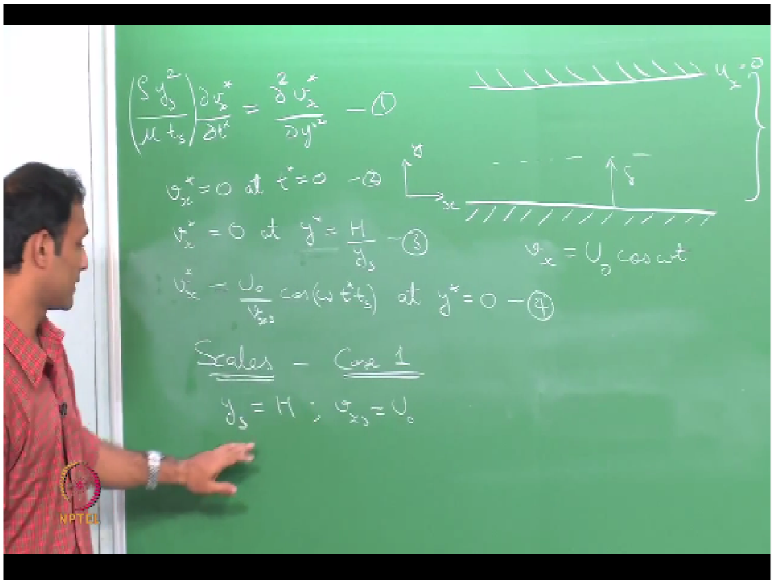

So this was our Vx and this height is given by h, okay. So we just rewrite the equations and we get rho*ys square/mu*tx dou Vx star/dou t star=dou square Vx star/dou y square. So I just rearranged the governing equation, got all the coefficients on to the left-hand side to get a term here which multiplies my dou V/dou t, okay. Now I do the same thing for the initial condition and the boundary condition.

So I would get Vx star=0 at t star=0. We get Vx star=0 at y star=H/ys and we get Vx star=U0/Vx scale cos omega tx at y star=0, okay. So this I get just by rearranging the terms in the initial condition and the boundary condition, okay. So one timescale that is very evident from the problem is the timescale associated with the periodic motion which is given by omega. So omega basically tells me the frequency at which the plate is moving.

So we have a timescale directly given by omega and the other problem, the other timescale that involves is the motion of the wall should be diffusing into the fluid and reaching the entire, the domain that the fluid is present in. So we have a time which is called the viscous timescale which is basically the timescale associated with the motion of the wall being transported because of viscous forces throughout the domain, okay.

So one timescale that we very evidently have is the periodic motion of the wall. The another timescale of the problem is the timescale which is associated with the motion of the wall being transported throughout the domain, okay. So these timescales we can directly get using the scaling analysis. So we look at the equation, this is the governing equation 1, the initial condition is given by 2.

We have the boundary conditions given by 3 and 4, okay. If I look at 3 and 4, we directly can see that using the order of 1 magnitude analysis, we can choose our ys to be H, okay, so that y star goes from 0 to 1, okay and then from here we see that we directly have a velocity scale which is given by Vxs being U0 so that Vx star goes from 0 to order of 1. So we can choose 1 particular scale.

So this is case 1 where I am choosing my y scale to be H, my velocity scale to be U0, okay. So I am going to say that my velocity is going to vary from y=0 to H. So the motion is being driven by the wall throughout the domain. So that comes by choosing the scale y=H and then I have my velocity scale which is given by Vx=U0 which tells me that my velocity of the wall is important in this problem. So I have retained both the physics, I am telling that my velocity of the wall is important and that the flow will be affected throughout the domain when I choose ys as H, okay. Now we will just substitute these into the equation.

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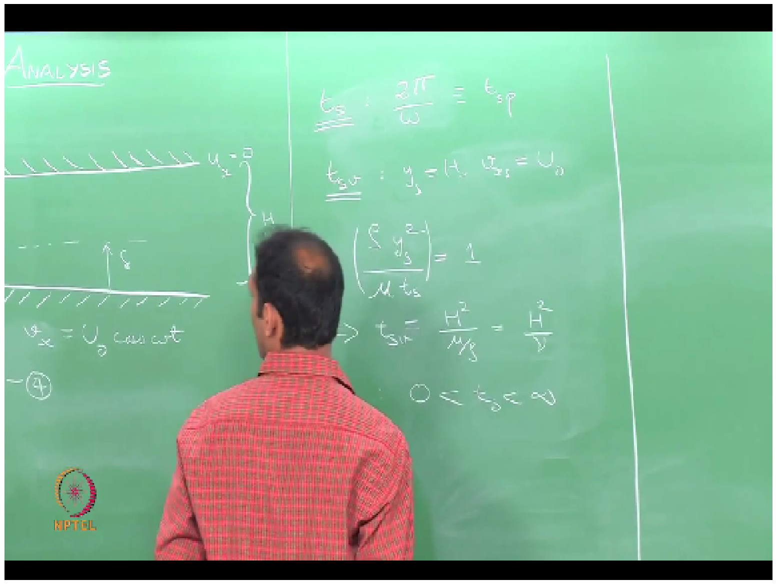

Before that we know that we have to choose a scale for time 2, okay. So we have a choice for timescale from the problem here. We could look at equation 4 and we could make this particular term of order 1 and get a scaling for the time directly which basically means that I could choose my timescale to be given by 2pi/omega which means the frequency of the wall motion is the timescale that I am choosing.

So that can be called as something as tsp wherein is the timescale associated with the periodic motion of the wall. So I have one timescale which is given by 2pi/omega, okay. Now once I have chosen ys as H, Vxs as U0, I could substitute these into the governing equation 1, make this term 1, then I would get the timescale if I just substitute Vx here, the ys here. So which means that I could get one more timescale which I would right now call it as tsv, okay which can be made by just substituting ys, ys is H and Vx it is U0, okay.

And I am making that term which is the coefficient in equation 1 to be 1. So that means rho ys squared/mu*ts can be made 1, okay. If I do that, I will get a timescale associated with this particular scaling to be H square/mu/rho or basically H squared/mu, okay. So what we have essentially done right now when we have said that this particular term rho y squared/mu ts is 1 is that we have said that dVx/dt is important in the problem.

And the right-hand side was basically the viscous term, we have got it from the second derivative velocity coming in the Navier-Stokes equation. So that was the viscous term and this is the transient term. So right now when we have found out a timescale by equating these 2 terms, what we are saying is that the timescale is the one which is associated with the effect of the motion of the wall to penetrate throughout the domain H through the viscous forces.

So that is why I have called this as tsv which is nothing but the timescale associated with the viscous transport. So the fluid has transported the motion of the wall throughout the domain and this timescale is the timescale associated with that transport. So how much time it would take for that motion of the wall to be transported throughout the domain, so that is given by this particular tsv.

The another timescale which is present in any transient problem is the timescale associated with the observation timescale which I will just call it as the t0, okay. So we have a timescale associated with this, the problem being transient. Which means that this could essentially vary from 0 to infinity. So I could start observing the flow from any time t=0 to an infinite time. So this is a timescale which is present in any transient problem.

The other 2 timescales that we have observed, the periodic one and the viscous timescales were obtained because we have a problem which is periodically being driven. So that is the timescale associated with the periodic motion of the wall and then the viscous timescale is the timescale which is corresponding to the motion of the wall being transported throughout the entire domain. So we could scale the problem using any of the 3 timescales.

And we will see that we could get different physics out of the same problem using the different timescales, okay. So I will just go to equation 1 again.

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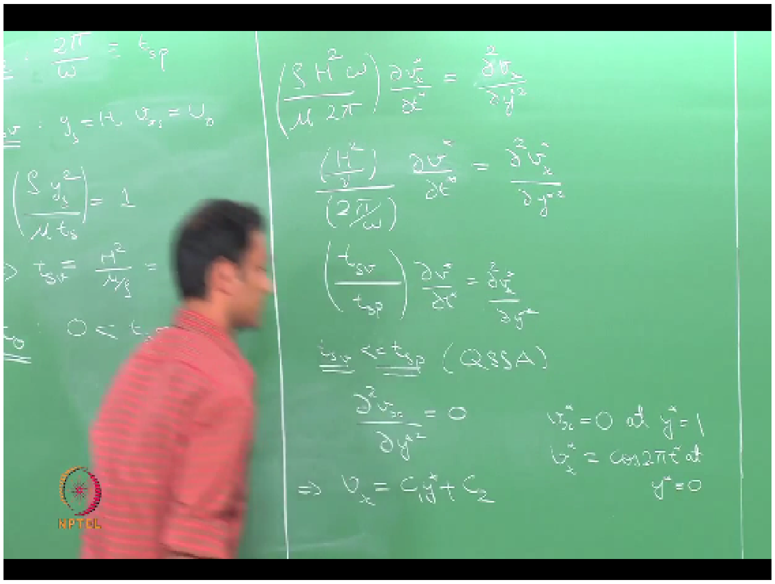

So I have choosing ys to be H, I have rho H squared/mu, by timescale is the timescale which I would take to be the one associated with the periodic motion of the wall. So I would get something like this, okay. This is multiplying dou Vx star/dou t star=dou square Vx/dou y square, okay. So what we see here, the term that is multiplying dou Vx/dou t star is nothing but we could rewrite this particular term as H squared/nu, which is mu/rho, /2pi/omega.

So this thing H square/nu was the timescale which was associated with the viscous transport in the problem which we had got here, tsv. So we could rewrite this term again as tsv/tsp dou V star/dou t star giving dou square Vx star/dou y star square, okay. So if you look at the left-hand side of this equation, we have tsv/tsp, okay. We could neglect this term only under the condition that tsv is much much < tsp, okay.

If this condition is satisfied, then we could essentially neglect the left-hand side, okay and then we would get an equation which was dou square Vx/dou y star square=0, okay. Now if you see the governing equation, it is a simplified form of the actual governing equation that you had started with. You had a transient term. Right now under these conditions, we know that there is no dou V/dou t which essentially means that the problem is independent of time at least in the governing equation, okay.

So this particular case is the Quasi-steady state approximation. So you can have a quasi-steady state approximation only under the condition that the viscous timescale is very less compared to the periodic motion of the wall, okay and then this can be easily solved like you know the second derivative of Vx with y is 0. So you could just integrate this. So you will get Vx..., okay.

And to solve this problem, we have the 2 boundary conditions which were Vx star is 0 at y star=1 and Vx star is cos 2pi t star at y star=0, okay. So if you use these boundary conditions, you could easily solve and get the velocity profile directly, okay. So we have looked at the problem and found out the condition under which we could use the Quasi-steady state approximation and get the simplified solution to the flow, okay.

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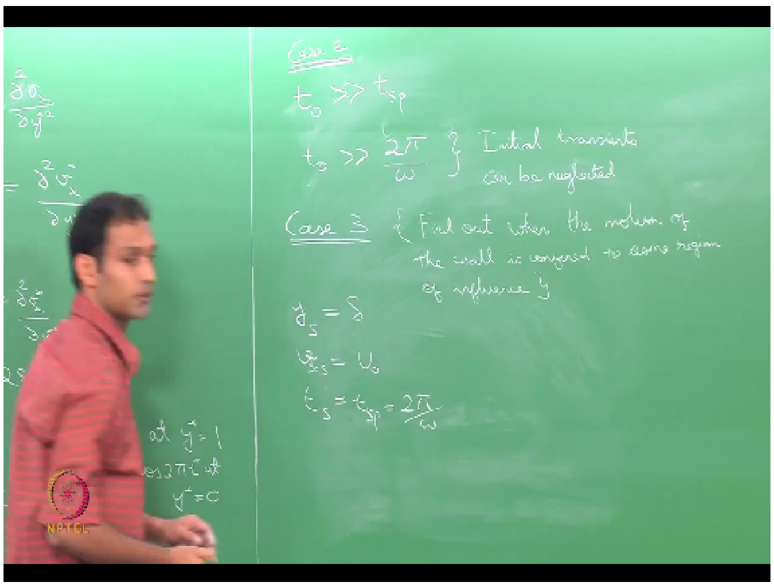

The second case is essentially we will be finding out the condition under which we could say that the transience of the problem have been died down, okay. So that essentially corresponds to a timescale t0 be much much greater than the timescale associated with the periodic motion. So I had the 3 timescales in the problem. So the t0 was the observation timescale. So I have gone till what time, I have waited till what time, is the one which basically tells you that t0 is, basically the physical time associated in the problem.

So if I have waited long enough compared to the timescale of the oscillation, then I could say that the initial transience has kind of died down. So this condition or t0 is greater than 2pi/omega, okay. So this particular condition is the one which helps you tell, under these conditions, the initial transience can be neglected, okay. So the case 3 was the case where we are going to find out when the motion of the wall is confined to some region of influence, okay.

So we have the governing equation and the initial and boundary conditions as was given in the case 1 which we just looked at. So what we are going to say is that in the case 1, we had chosen ys to be the entire flow domain which was going from 0 to H. So we had chosen a length scale in the y direction to be H. So here right now what we are essentially saying is that the flow, the motion of the wall is confined only till a region of influence.

So in this condition, we will choose a ys which is given by some delta which right now is unknown, okay. So I have chosen a scale ys which is given by delta, my Vx scale is the one which is the motion of the wall. So I have that given by U0, okay and the timescale I am going to choose the timescale associated with the periodic motion, okay. So that is nothing but 2pi/omega, okay.

So I am right now very close to the wall, okay. So there is a small region of influence where the motion of the wall is felt which is given by delta and i do not know what this delta is right now. So what we will be essentially doing is finding out using scaling analysis the value of delta. So we can find out what region will be influenced by that motion of the wall, okay.

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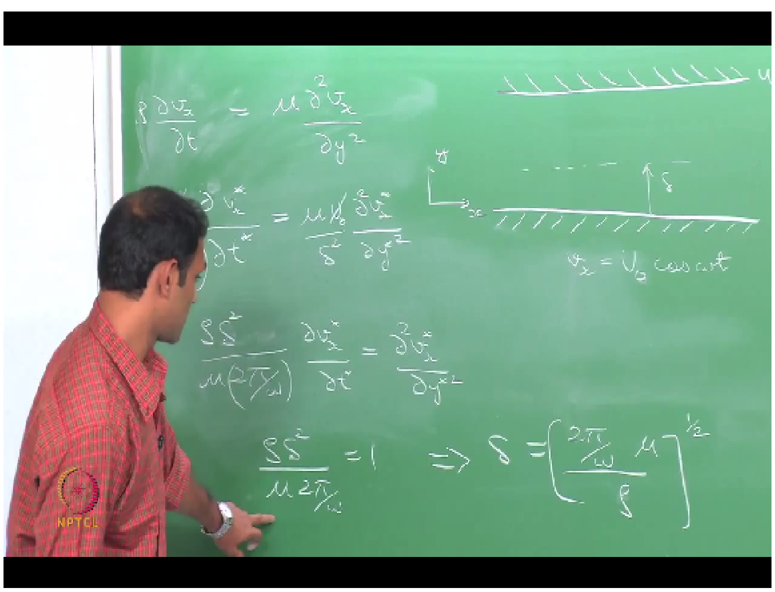

So we have the same equation which is given earlier. It was dou vx/dou t, okay. This was the governing equation that we had initially, the transient term being balanced by the viscous term. So right now this was a dimensional equation. We used those scales which were given there to nondimensionalize this particular equation. So we get rho*U0 here*dou V star/dou t star. So the timescale was 2pi/omega.

So we have 2pi/omega here, okay. This is given equals to mu*U0, which is the scale used for Vx, dou square Vx star/dou y star square, okay and y star is scaled with delta, so we get delta squared * y star. So I just have used the corresponding scales and nondimensionalized my equation, okay. So if you see U0 gets cancelled, okay. So if I rearrange my terms, I would get rho delta squared mu/2pi/omega, is that correct?

I think mu comes below, * dou Vx star/dou t star= dou square Vx star/dou y square, okay. So I have just rearranged my terms. So what I am saying is that right now I am interested in the condition wherein I am looking very close to the wall and the motion of the wall is confined to a region of influence and when that is happening, when the motion of the wall is confined only to a region of influence, then problem is always going to be a transient problem.

So I have to retain the dou Vx/dou t in the problem, okay. So the way you retain is by making this term which is associated with it to be 1, okay. So you make the term which is associated with dou Vx/dou t, 2 pi/omega=1. So only if this term is of order 1, so only if I make this guy 1 will this LHS also become order of 1. So from here I can see that I can get my delta solved which is nothing but 2pi/omega/, is that correct? Root under, okay.

So I have taken it onto the right-hand side, okay. So if I look at this, delta... So I could divide delta with H, okay to get me delta/H.

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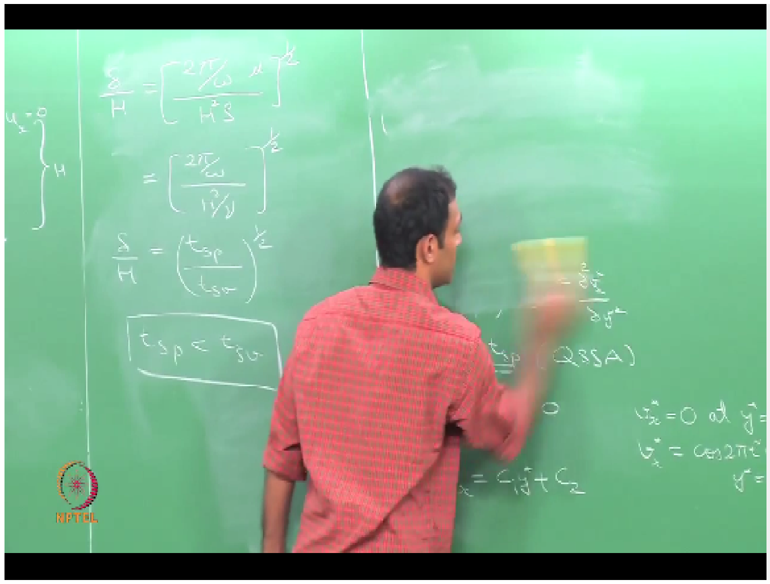

I just divide delta on the left-hand side with H, so I have to do again divided H square here because this term is under the root. So I will get delta/H to be 2pi/omega, okay and here I would get H squared and there was a mu here, okay. So I have delta/H given by this term. So if I look at this guy, I could rewrite it as 2pi/omega/H square/nu to the power 1/2, okay. So this is nothing but delta/H being given by a timescale which is periodic 2pi/omega/ts viscous, okay.

So which essentially tells me that the region of influence which is the fraction of the region, basically delta/H is how much of it the fraction of the channel which is influenced. So delta was the region till which the motion of the wall was affecting and H is the total height of the domain. So delta/H is basically the fraction in which the flow is being affected, is directly given by the timescales ratios, the timescale of the periodic motion by this one.

So if the timescale associated with the motion of the wall is very low which means, if tsp is less which means that the plate oscillates very fast, then you will see that the region of influence would be small, okay. So if tsp is < tsv, then delta would be small which means that if I have a fast moving plate, the region of influence would be very much close to the wall. If it is slowly moving, then I have enough time for the fluid to respond because viscous timescale is given by the H square/nu.

Once I fix my H and I have chosen my fluid, H square/nu remains fixed, okay and then I could make the entire delta more, occupy the entire region or confine it to a small region, choosing my omega correctly. So if I choose very small omega, the motion of the wall would be confined to a very small region and if I choose large omega, then the fluid has enough time to actually respond throughout the domain.

It could be carried out by viscous forces, okay. So for the case 3 problem which was the region of influence problem, you could use these scales and get the equation directly.

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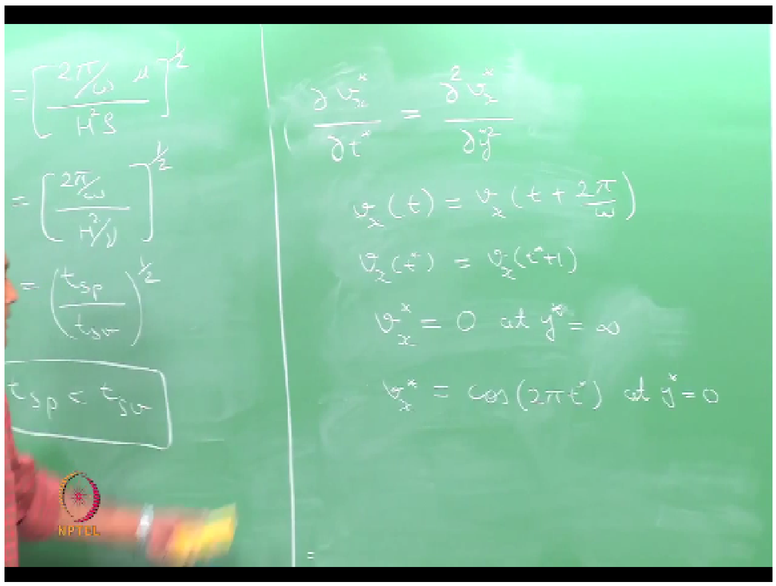

So under these scales, you would get that the problem is transient and you would get the governing equation to be dou Vx/dou t star=dou square Vx/dou y square, okay and when you are looking at this problem, you could give the initial condition. Instead of initial condition, you could give a region which means the motion of the wall would remain exactly same after 1 cycle of oscillation.

Instead of using the initial condition which was the wall being at rest, you could use a periodic condition in time which means that U or the velocity in the x direction at t, okay, is the same as velocity after some 2pi/omega, okay. So this would essentially in the nondimensional frame because my scale right now I have chosen as 2pi/omega, would be Vx at t star would be same as Vx at t star+1, okay and then my boundary conditions I already have which means my wall is not moving.

The top wall is not moving at y star=0, okay and then I have the other condition which is Vx star, that was at infinity at H because the top wall is not moving, y star is infinity basically at that point because this would come as delta/H, okay because... Basically I have a condition y... this is the condition at y=H. So in the nondimensional frame, it would be y star*... H/delta. So y was H, so y star*delta is H, so y star would be H/delta and delta is very small right now.

So it basically corresponds to a fact that it has infinity. So I have a velocity which is going to 0 as I move away from the wall and Vx star would be the one which is the periodic motion which is given by cos 2pi t star at y star=0. so you could easily solve this problem and then get the velocity profile for this conditions. Thanks.