Multiphase Flows: Analytical Solutions and Stability Analysis

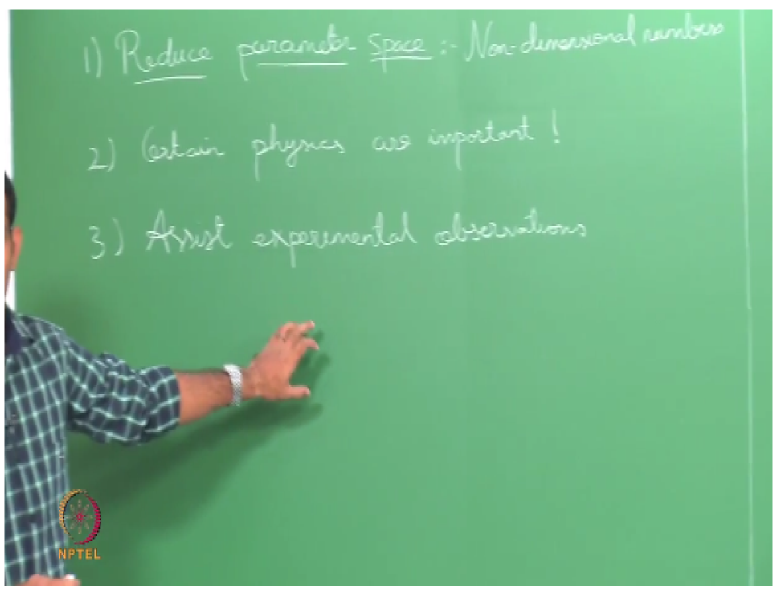
Prof. S. Pushpavanam
Department of Chemical Engineering
Indian Institute of Technology – Madras

Lecture – 04
Scaling Analysis: Introduction

Good morning all. Welcome to lecture 4 of Multiphase Flows. Today, we will be dealing with something called as scaling analysis. I am sure like most of you would have gone through what scaling analysis is in certain part of your academia. So, today we will just go through a systematic approach about how to use scaling to solve certain problems in fluid mechanics. So, I will be first going through what the motivation for scaling analysis is and then I will solve a problem which can be solved using 2 scales.

And then I will tell you how you can get certain physics out of the problem using one scaling and certain other physics from the other scaling. So, I will start with the motivation.

(Refer Slide Time: 01:06)



Scaling is important because first it helps to reduce the parameter space. So, by this what I mean is that if you see any fluid mechanic problem, obviously the flow field would depend on the viscosity, density and other pressure that is applied, the length scales near the channel, how large the diameter of the pipe is and things like that. So, what scaling helps us is to reduce the parameters that are affecting the flow. So, you can club all the parameters to essentially lesser number of non-dimensional groups.

So, for example the famous numbers like Reynolds number, Weber number, they all are group of all these individual parameters that affect the problem but they have been clubbed and then essentially you can determine the flow if you the Reynolds number. So, if you know the Reynolds number to be low, then you know that it is a kind of (()) (02:04) limit and if the Reynolds number is high you know it is a kind of (()) (02:10). So, the first use of scaling is to reduce the parameter space.

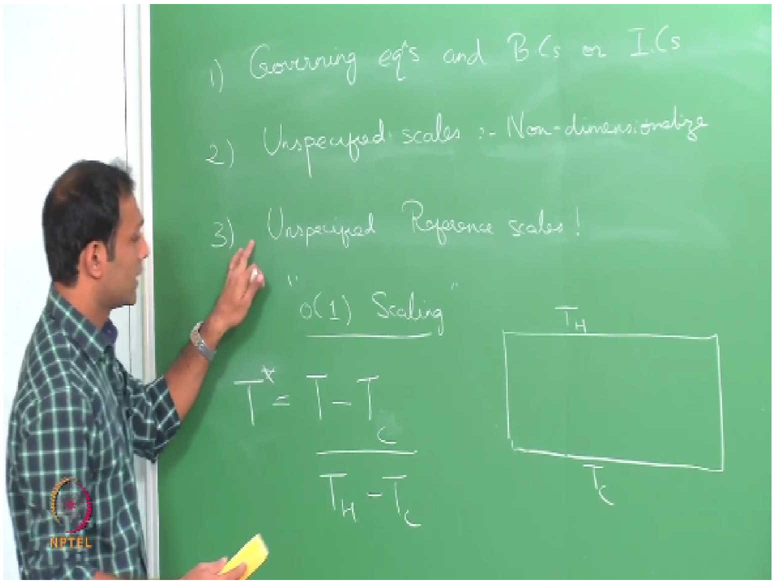
So, essentially results in obtaining non-dimensional numbers, okay. The second advantage of using scaling is to say when certain physics are important. By this what I mean is that say for example, if you have fluid flowing in a pipe, then you know that if the Reynolds number is low, then the viscous forces are important and inertia can be neglected. These things the understanding of physics when you can say that certain forces are not important and certain forces are dominant that comes after like scaling the problem appropriately.

So, we will see the same thing in today's example that we will be looking at. So, scaling essentially helps us to find out again non-dimensional numbers using which we can say which of the physics can be neglected under what conditions, okay. The third important point is to assist experimental observation. By this what I mean is that see an experimentalist is doing a particular experiment wherein he sees certain physics happening in it.

So, once you know the problem and then you calculate the non-dimensional number and then you find out what its value is, then you can say that to explain the physics of whatever we are observing, you can neglect certain forces and take care only the other forces which are important. For example, it we see something happening in the lower Reynolds number regime, then you know that inertia basically not important. It is something to do with viscous forces and things like that.

So, it basically helps experimental observations to be explained correctly. So, once you scale the problem correctly, you have these advantages that makes your life easy, okay. Now, I will be just taking it through what the essential steps are involved in scaling given problem. So, these are fundamental steps that you can use for any problem that will be encountering in fluid mechanics or any other problem for that matter. So, this is a systematic approach which you can just apply once you know you get a feel of it.

(Refer Slide Time: 04:55)



So, the first thing is obviously write down what the governing equations and the boundary conditions or initial conditions are. So, any description of physics in mathematical terms would have a governing equation and associated boundary conditions or initial conditions and the governing equations generally are PDs or ODs or something like that, okay. So, the second step involves defining certain unspecified scales by which I mean that scaling would essentially be you having a governing equation, boundary condition and you making it non-dimensional, okay.

So, the scale that you choose to non-dimensionalise the equations, the reference scale that you are using to non-dimensionalise the equations could be infinite. Like you could choose any velocity to make your velocity in the pipe dimensionless because you could divide it with whatever velocity. Say, if the wall is moving with some velocity, you could divide that velocity to make it non-dimensional or you could even divide it by us earth velocity for that matter, even then it would be non-dimensional.

But then the point is you want to scale it in such a way that it is easy for you to analyze the problem. So, right now we would just say that there are certain unspecified scales that we use to non-dimensionalise the above equations, okay. The third one is something called as unspecified reference scales, okay. So, before going into the details of what a reference scale is, what kind of scaling that we are going to use is called as an order-of-one scaling.

By order-of-one scaling what I mean is that we are going to non-dimensionalise the boundary conditions, the governing equations and things by certain scales. When you are non-dimensionalising, you are non-dimensionalising it in such a way that they become of order one which means that if you have velocity, you choose the scales such that it changes between 0 and order-of-one. By order one I mean that it could be of any order of magnitude around one.

So, it could be varying from 0.1 to 10 but then not more than that. So, it is not exactly 1, it is around 1 but not orders of magnitude higher than 1. So, you choose scales to make the governing equations, the boundary conditions and the variables and their derivatives of order one, okay. So, this is what makes it a unique scaling. So, when I said you know you could non-dimensionalise the governing equations with any infinite possibilities of scales.

You could choose the velocity of earth, moon, anything of a truck which is going around, but then we are choosing a velocity which would make the equations, boundary conditions, initial conditions everything of order one. So, this would essentially from the physics tell you certain unique scale that you can choose, okay. So, that is what we will be looking in today's class. Now, I come to the third point which is the unspecified reference scale.

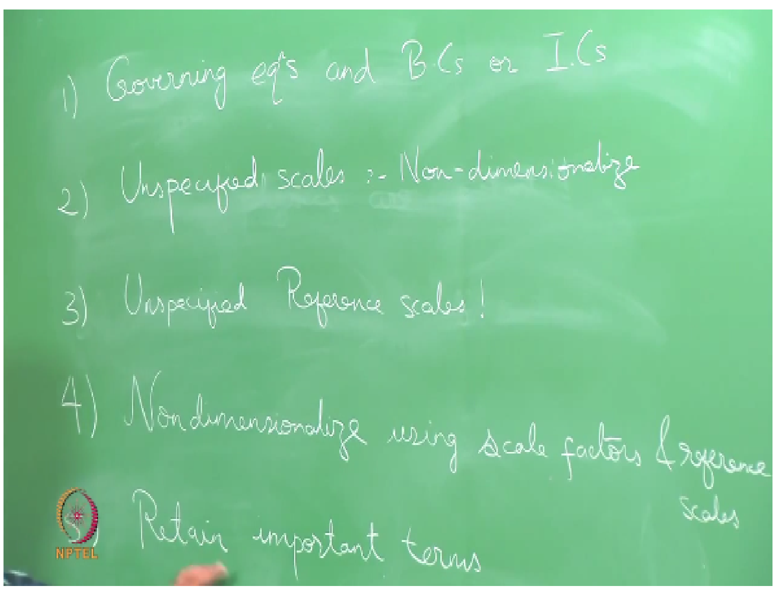
So, unspecified reference scales are those which basically help you to bound the variables between zero and making it order-of-one. So, for example, if you have certain variables in the problem which are not by default going from 0 to 1. So, making it order-of-one is very easily because you just have to choose a correct scale to make it reach that order one. So, maximum value that could reach is of order one but the minimum value should be zero, it should be bounded between zero and it should go to some order-of-one value.

So, referencing basically helps you to make the minimum value go to zero, okay. This is still the third point. So, for example if you have a rectangle a slab minutes when heat transfer is happening and you have an equation which governs the temperature in the slab. So, you have a boundary condition which would basically tell you there is a hot temperature on top and say some cool temperature at the bottom, okay and the temperature inside this by diffusion is going to only change between Tc and Th, okay.

Now, if I would have chosen any scale to non-dimensionalise the temperature, it would not start from zero because the lowest value which is there is Tc, okay. So, it would go from Tc by some Tc scale the temperature scale to Th by temperature scale. So, to make it go from 0 to order-of-one, we choose a reference scale. So, for example, in this case wherein you want the non-dimensional temperature to vary from 0 to order-of-one, you could define your non-dimensional temperature to be T-Tc/Th-Tc, okay.

So, what this tells you is that if I had not used Tc as my reference, then this would not have gone to zero at the bottom. So, it is basically this reference which helps me to make it go from 0 to 1, okay. So, that is what we say when we have the third step as an unspecified reference scale basically helps you to go from 0 to order-of-one magnitude.

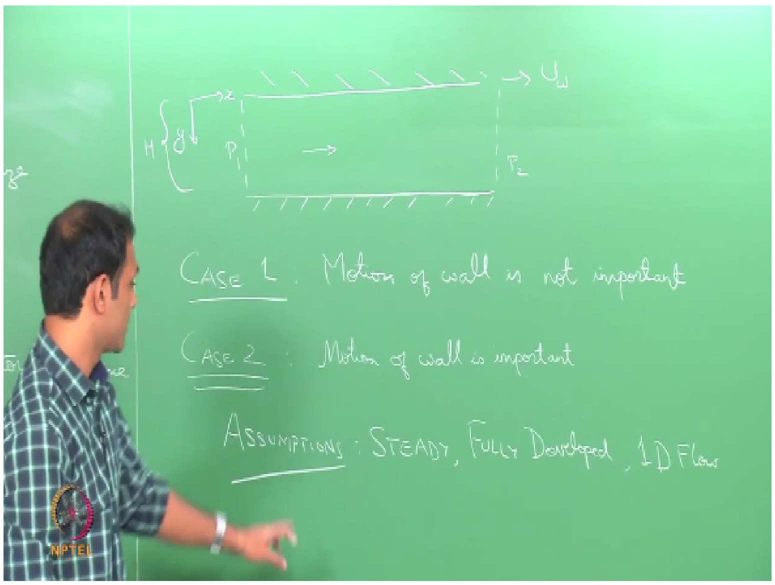
(Refer Slide Time: 11:06)



The 4th step would be to non-dimensionalise using the scale factors and the reference scales, okay. Once you do this, at this step you basically would have governing equations, boundary conditions, initial conditions, non-dimensionalised and varying from 0 to order-of-one, okay. Then, comes the fifth one which basically helps you to retain important terms. So, this will be clear to you when we do the example, that we will be following soon.

So, it basically just helps you once you do the non-dimensionalisation using the correct scales and references. There would be certain terms which becomes very small so that you can get neglect them. So, you can tell that for these conditions you do not have to keep certain terms and things like that. So, that is what I mean by retaining important terms in the problem, okay.

(Refer Slide Time: 12:37)



So, the first problem that we will be looking today is fluid confined between 2 infinite plates and the fluid is driven by pressure plus the top wall is moving. So, I have the top wall moving with a velocity U, okay and I have a pressure which is also driving the flow. So, I have some pressure here P1, some pressure here P2 and there is a pressure drop which is driving the flow along with the motion of the wall, okay.

So, this problem can be completely solved analytically, but then there are 2 things which is importance. So, the flow is essentially driven by 2 forces and it is being restricted by viscosity, okay. The viscosity is the only thing which is offering resistance to the flow and the thing that is driving the flow is the pressure which is applied plus the motion of the wall. So, you can solve this problem completely but then there are certain problems where you would not be able to keep all the physics together.

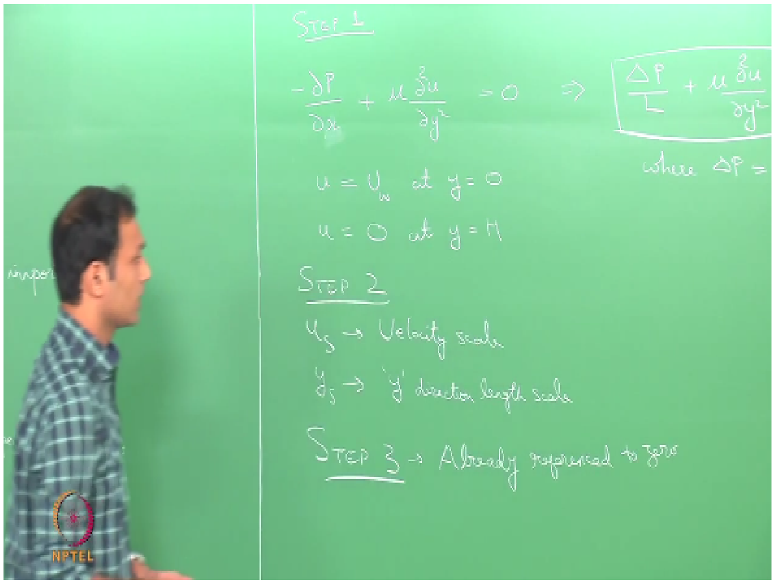
So, you would have to solve the problem into limiting condition, which means that you have to solve the problem when you inherently assume that the motion of the wall is not important, that is one case and the other case would be the motion of the wall is important, okay. So, those are the 2 extreme limiting cases under which we can solve this problem; one wherein you say that it is basically the pressure which is driving the flow and the motion of wall is not that important.

And the second case is where it is the motion of the wall which is important or at least we are looking at the regions close to the wall where the motion of the wall is the one which is driving the flow, okay. So, those are the 2 cases that we will look today. So, there are 2; case 1 wherein motion of wall is not important, okay and case 2, where motion of wall is important, okay. So, first we will take case 1 wherein we are going to solve this problem for the condition where the motion of wall is not important, okay.

So, I will write down the governing equations for this particular problem. So, you have a fluid which is flowing in the X direction and I choose y to be downwards. So, this is y=0 the wall and y=some H (()) (15:45) bottom all is located and this is not moving, so the velocity here is 0. So, the governing equation for a one-dimensional flow, so the flow is in one-dimension in the X direction and we assume that it is steady which means it is not changing with time and we assume that it is fully developed which means it is not dependent on X coordinate.

So, whatever velocity field here is the same everywhere along X. So, there are 2 inherent assumptions looking at steady and fully developed one-dimensional flow, okay.

(Refer Slide Time: 16:37)



So, the governing equation would be something like this which comes from the Navier-Stokes equation, okay. So, this equation is dimensional which means that all of them have their own dimensions, so pressure has a dimension, the lengths have dimensions which are of length. So, this is the first step that we looked here wherein we are writing the governing equations and boundary conditions.

So, since it is a steady state, we do not have initial conditions. It is independent of time. We are just looking at what is going to happen after all the transients have died down, okay. So, we have an equation which is like this and then we have 2 boundary conditions which is the xcompetent of velocity, U=Uw at y=0 and U=0 at y=H, okay. So, this is step one where we have written down the equations and its boundary conditions.

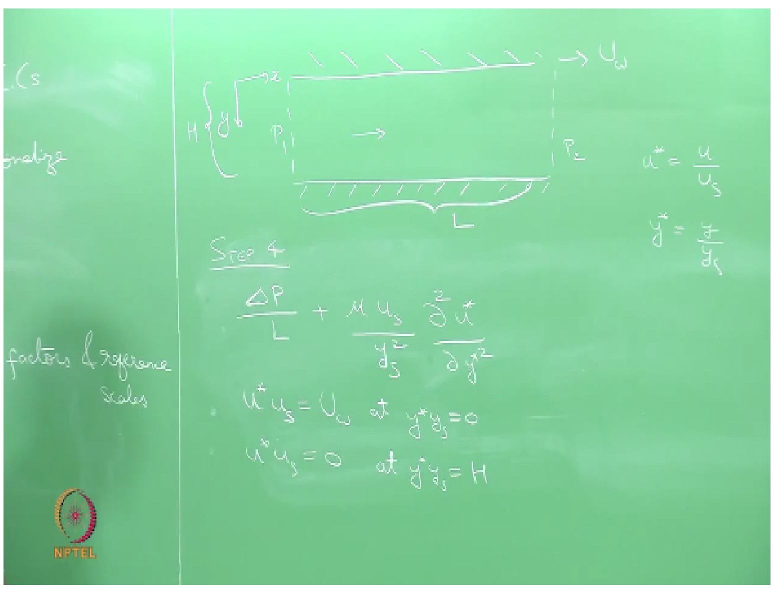
So, step 2 would be to look at the unspecified reference scale, the unspecified scales which are used to non-dimensionalise the equations and the boundary conditions, that is step 2. So, I will just say that I have a velocity scale which is Us is my velocity scale, okay and let say I have y and x. Let say y is my length scale in the y direction. So, it is my y direction scale, okay. So, this equation can be integrated in the x direction, which means that this term is basically independent of x, so it is like a constant.

This term can be integrated in x and then you can just put the boundary conditions that the pressure is P2 or P1 at x = some distance between them being L, okay. So, when you do that, you will get this equation where delta P is minus of, okay. So, that comes by just integrating it in x direction and then putting the limit. So, let us say that this is my governing equation that I am looking at and the boundary conditions.

Now, I have got my velocity scales and the length scales written down but they are unspecified, that is what I meant by just writing down unspecified scale factors. Now, the step 3 here is to make it referenced to 0, so the variables that you are looking at is u and y, u is already referenced to 0 because it is 0 at the bottom wall, okay and y is also referenced to 0 because one of the walls is at y=0. So, in this problem, we do not have to put any reference scale.

So, step 3 is basically redundant in this problem, so already referenced to 0. So, we have essentially completed 3 steps for this particular problem. So, the step 4 is to non-dimensionalise the equation with the given scale factors that we have chosen, okay. So, I will write it here.

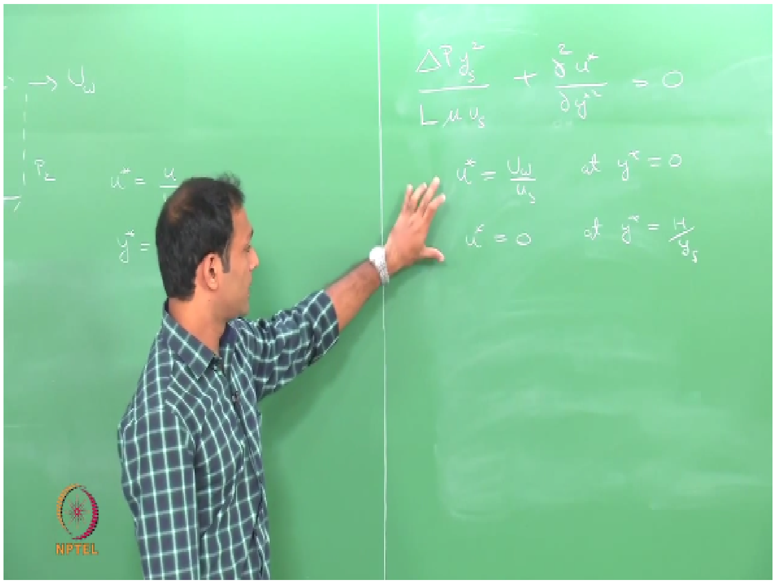
(Refer Slide Time: 21:06)



So, I am going to now non-dimensionalise my equation which is my step 4 to give me the governing equation would fall something like wherein by u star=u/us, y star=y/ys, okay. So, I have non-dimensionalised my velocity and length using the corresponding scales that I had chosen, us and ys and just substitute for you as u star us, y as y star ys in the governing equation and then I would get this particular equation, okay.

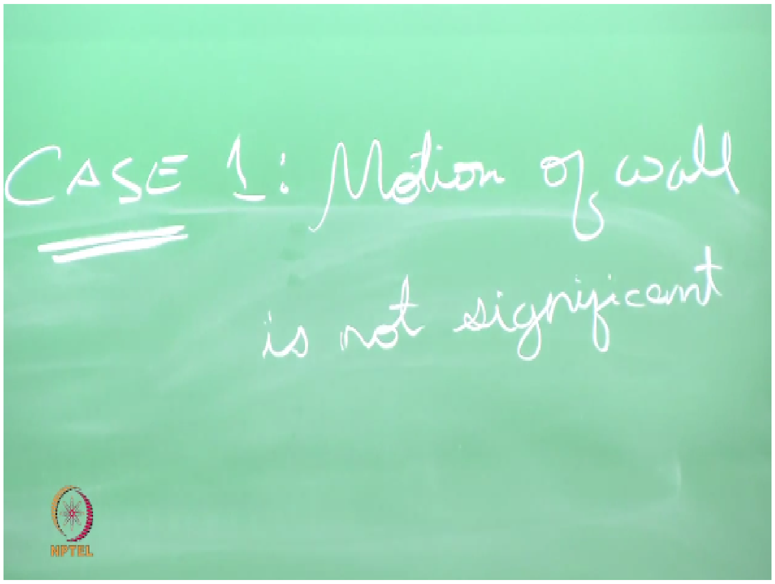
Same I should do for the boundary conditions also. So, I would get u star*u scale=uw at y star*ys=0, okay and the other boundary condition is u star*us=0 at y star*ys=H, okay. So, I am just going to rearrange the terms and then write it down again.

(Refer Slide Time: 22:45)



So, I will just rearrange everything to give me delta P ys square/L mu us + dou square u star/dou y star square=0, that is my governing equation and then the boundary conditions are u star=uw/us, okay at y star=0 and u star=0 at y star=H/ys, okay. So, these are my equations and the boundary conditions. So, the basic funda was that we want all the variables to be going from 0 to order one, okay and in this particular problem the case 1.

(Refer Slide Time: 23:52)



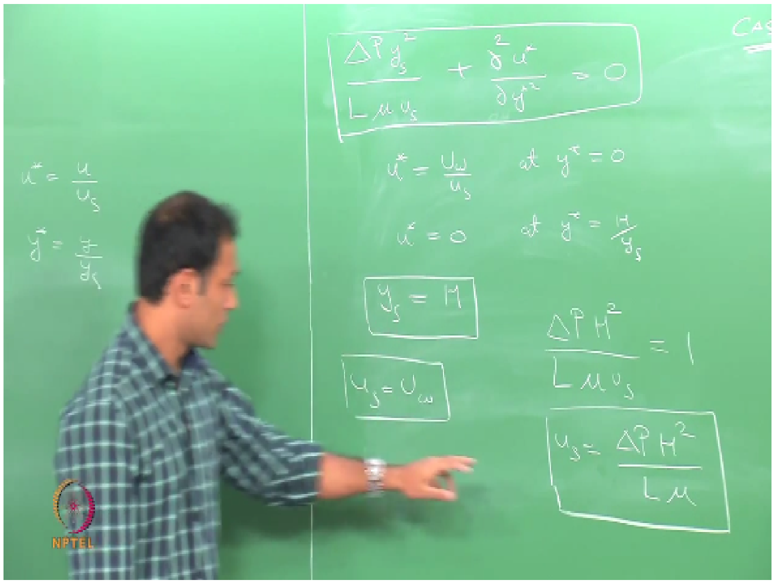
This is case 1 that we are still solving. Case 1 is the one that we are looking at wherein the motion of wall is not significant, okay. So, using scaling, we are going to find out the conditions under which we can assume that the motion of wall is not significant. So, till now I have just orally told that we are going to look at a case where the motion of wall is not significant.

But what does it mean mathematically, under what conditions can you say that the motion of wall is not significant, that is going to come out through scaling analysis and that is what we are going to look at today. So, till here it is clear, right. So, I just non-dimensionalise my equations using the scales. Now, the case where I am looking at wherein the motion of wall is not significant, so the velocity is going to be affected by the change in pressure throughout and the viscosity is the one which is resisting the flow, okay.

So, you have pressure which is driving and viscosity which is resisting the flow. So, the length scale is very much evident from the physics of the problem as well as from the body condition here that ys could be chosen as H so that y star goes from 0 to 1, okay. So, that is also from the physics because I know that when my pressure is driving the flow that is going to affect the flow throughout the entire domain, okay.

If it was the case wherein the motion of the wall was significant when it is not that evident that the motion of wall is going to pervade throughout the domain, okay but a pressure which is driving the flow would be pervading throughout the domain, so we can directly take ys to be H, okay. So, from here you know that if I choose ys to be H, y star would become one and that is what like order one scaling also would ask you to do.

(Refer Slide Time: 25:58)



So, you know that y star is going from 0 to 1 now. So, I choose my ys to be H, okay. So, if I look at this guy here, I could choose my velocity scale to be motion of the wall itself uw. So, I have an option of choosing us to be uw, okay. If I do this, I am telling that the motion of wall is important because I am going to divide all my velocity throughout the domain by that velocity of wall.

So, when you choose a scale, you should choose a scale so that you want to get the physics also correctly. So, right now when I am saying that I am looking at the case where the motion of wall is not significant, I am not supposed to be scaling with the motion of the wall. So, I will choose the velocity of wall to be my velocity scale. Even though I could do that, but then if I do that then it means that motion of wall become significant because that would always come through the boundary condition.

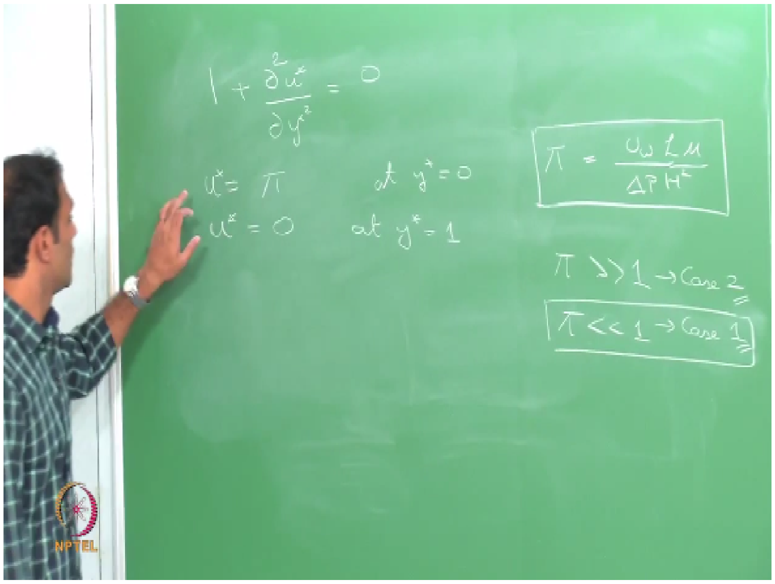
I will never be able to neglect this term if I choose my velocity scale to be uw because if I choose us to be uw, at the wall the velocity is always going to be 1, okay. So, what I will be doing now is trying to get a velocity scale which is coming from the problem directly. So, I will just look at the governing equation here, okay. This is the pressure term which was driving the flow and this was the viscous term which was resisting the flow, okay.

So, what I have done is, I know what ys is. I put it here, okay. I want to look at the case where the pressure is important, so I would retain this term, okay. I cannot neglect this term because right now I am looking at the case 1 wherein the motion of wall is not significant. So, I retain this term, so making it order-of-one would give me delta P, ys is H, H square/L mu us, I make it of order one which basically means that I equate it to 1 so that I get a scale which is us out of the problem.

So, right now I can just manipulate this to get us, is that correct, okay. I will get us as delta P H square/L mu, okay. So, I have 2 scale for velocity basically because there are 2 things which are driving the flow, the motion is because of the wall being moving and also because the pressure which is driving the flow. So, this particular scale, the reference scale that we have got here, is the one which corresponds to the pressure driving the flow because has a delta P/L, so it is the one which corresponds to the velocity scale which is given by pressure driving the flow.

Whereas this particular scale gives you the velocity scale which is coming because of the motion of the wall, okay. So, in case 1, we will non-dimensionalise the equation using this scale because this is what we are looking at wherein in case 1 the pressure is the one which is important and not the motion of the wall, okay.

(Refer Slide Time: 29:33)



So, we will do that here. If I choose us to be given by that delta P H square/L mu, I would see that the first term in the governing equation becomes 1, because the velocity scale by making that term as 1, so if you put back velocity scale there we will get that term to go to 1 and you will get dou square star/dou y star square=0, okay. So, this is your governing equation and the boundary condition would be u star being uw/us, so uw is as it is and us would be delta P H squared/L mu, okay.

So, this is my velocity at the wall which is at y star=0 and I have the other boundary condition which just means that the wall is not moving at y star=1, because ys was chosen to be H, so I have y star=1. So, if you look at this particular term here uw by another term here which is us, we will see that it is the ratio of 2 velocities. So, the numerator basically tells you that it is the motion of the wall and this tells you that it is the motion which is driven by the pressure which is imposed on the fluid, okay.

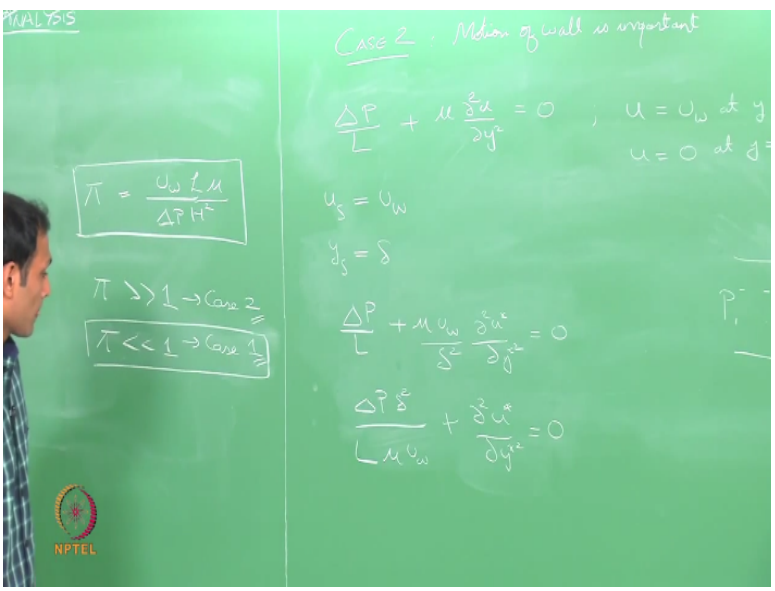
So, this is a non-dimensional number because it is the ratio of 2 velocities, okay. So, this can be called as pi, we will call this as a non-dimensional number pi which basically is the ratio of 2 velocities uw/delta P H square*L mu, okay. This is clear. So, I just took this on top and delta P H square was at the bottom. So, this particular number, the pi, is a non-dimensional number and this is what tells you when you can neglect the motion of the wall and when you cannot neglect motion of the wall.

So, if pi is very large which means uw is much high compared to what is the velocity determined by the pressure, then it is case 2 because the motion of wall become significant, okay. When pi is < 1, okay that is corresponds to case 1 which means the motion of wall is comparatively insignificant compared what is driven by pressure. So, for case 1, we basically have that pi is much much < 1, okay and this is 1 because we are looking at an order of scaling one wherein you have everything of order one, okay.

So, this is my equation, so I can just replace this boundary condition as pi and I can very well solve this problem. This is a simple problem which we you can just integrate. It is actually only an in second order. So, you just integrate and then you can solve for velocity very easily, okay. Now, we will look at the case 2. Before going to case 2, you can just see that if you are looking at the extreme case wherein you do not want the wall to be important at all, you can just put u star=0 and then you would get a parabolic profile.

But this is possible only because you have scaled it in this way. If you had scale it wrong, this would not have been possible. You would see that the equations become inconsistent which we will see at the end.

(Refer Slide Time: 33:58)



So, right now I will look at the case 2. Basically I am looking at the case where motion of wall is important, okay. So, I have the governing equation to be the same which is my delta P/L was balanced by mu*dou square u/dou y square, okay. The boundary conditions where the same thing again, u was uw at y=0 and u=0 at y=H, okay. The basic difference is in the choice of scales that we are going to do.

Right now, in the previous case we were interested in the motion of wall being insignificant. Right now, we are going to look at the case where the motion of wall is important, okay. So, earlier, we had seen that we had 2 choices of velocity and things like that. So, we will do something similar. So, we had the motion of wall and some pressure which is driving the flow, okay. So, what we will be doing is choosing the scales correctly right now.

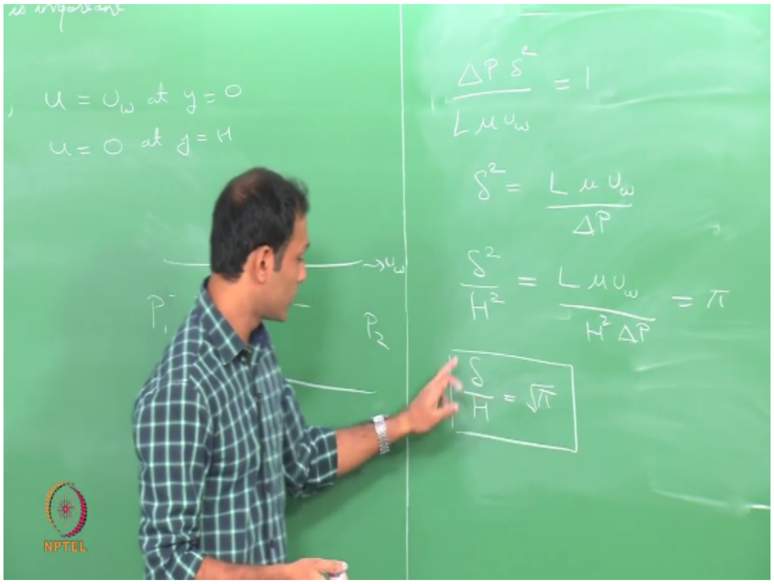
So, you know that since the motion of wall is important, you could directly choose the velocity scale given by uw, okay. So, the velocity scale is pretty evident but the length scale is not evident. You know the y scale is not very evident because when you are looking at the case where the motion of wall is driving the motion, it is not that evident that the motion of wall would permeate throughout the domain.

So, there could be just a small region of influence delta only till where the motion of wall is pervading the flow and after that it is the pressure only which is driving the flow, okay. So, we will choose the reference scale for y to be delta which is currently unknown. So, ys would be delta, okay. So, this is something very important because this is what basically makes your problem different in analysis.

Earlier, we had the y scale given directly by H and the velocity was found out from the physics of the problem. Right now, you have velocity scale given by the motion of the wall and the length scale is being found out from the problem. So, we will just do the same thing that we had done in case 1. We will non-dimensionalise it using this. So, delta P/L remains as it is and then we have mu*uw/delta square and dou square u star/dou y star square=0, okay.

We will just bring this terms on to the other side and we will get delta P, delta square/L mu uw < dou square u star/dou y star square=0, okay. So, what was not known to us was delta. So, we can use again making it order-of-one to get the value of delta.

(Refer Slide Time: 37:48)



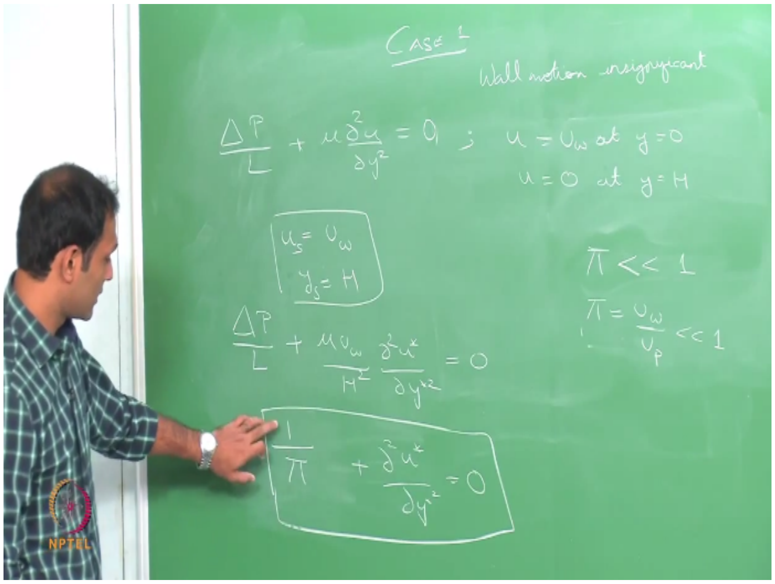
So, making that term to go to 1 we get delta P and delta square/L mu uw=1 and this would give you delta square to be L mu uw/delta P, okay. I could just divide it with H square on this sides to give me delta square/H square=L mu uw/H square delta P, okay. So, if you see this term which is here, it is nothing but the same pi that we had obtained in case 1, okay. So, that is exactly same as pi. So, you could write delta/H to be root of pi, okay.

So, the pi which is the non-dimensional number which is the ratio of 2 velocity scales is not only helping me to find out under what conditions can the motion of wall be neglected. It is also giving me the area of influence which is the delta. So, I know if I can calculate from my experiments, if I have an experimental setup wherein I know what the pressure drop is between the 2 ends of the pipe and I know what the motion of the wall is, I can easily calculate what my pi would be because I know what the values are.

So, if I know L mu uw H square delta P for a given experimental conditions, I can find out pi and using pi I can determine whether the motion of wall is significant or not significant. Using pi, I can also find out the fraction of the channel in which the motion of wall is important because delta was the region into which the motion of the wall was important, delta H gives me the fraction of the domain which is being pervaded by the motion of the wall.

So, that is one important thing and this problem can be easily solved. Solving the problem is not hard. It is to understand like how to solve it under 2 different extreme conditions.

(Refer Slide Time: 40:25)



So, before going, I will just look at the problem are for the condition wherein I would have used wrong scaling. So, for example, I would look at case 1, okay wherein the motion of wall was insignificant. So, I had wall motion insignificant, okay. So, I have my governing equation given by delta P/L + mu dou square u/dou y square=0, okay.

So, I am going to look at case 1 now and the boundary conditions where u=uw at y=0, u=0 at y=H. Now, what I am going to do is choose a wrong scale. I am going to choose the velocity scale which is u to be uw, okay and the ys is H. So, I have case 1 wherein I have purposely chosen the velocity scale to be uw which is not correct because right now I am interested in the case where the motion of wall is insignificant, okay. So, I will just do this and put it up into the governing equation and see what comes up.

So, we have delta P/L+mu * uw/H square (()) (42:09) and then you have dou square u x star/dou y star square=0, okay then I rearrange again to delta P H square/L mu uw + dou square u star/dou y star square=0, okay. This thing as we had found was nothing but pi, right. So, this term could be replaced by 1/pi, okay. So, this term was nothing but 1/pi. So, you have an equation of this form, okay and for case 1 where the motion of wall was insignificant, we found that pi<<1 because pi was nothing but uw by u pressure because that big term which we had is nothing but the velocity scale determined by pressure.

So, this thing for case 1 is < 1, okay. So, if you put that here, then this term is going to blow up. So, 1/pi would be something very huge compared to what this term is here. This term is of order one because we have non-dimensionlise to make it of order one. So, it is no way possible for these 2 terms to cancel each other and get me zero, okay because this guy is of order one and this guy is a kind of growing up.

So, from the final equation that you arrived at, you know yourself that there is something wrong in the basic assumption that I started. You ended up with this equation because of this wrong scaling, okay. If you had scaled it correctly, you have seen that this term would be 1+dou square u/dou y square in both the cases, because in both the cases we chose correct skills. So, we got 2 terms which are both being balanced and both of them are of order one.

And they could cancel each other, but then if you choose a wrong scale, the equations themselves will tell you that there is something wrong in the equation and it is not possible for them to cancel out, okay.