Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture - 42
Shallow Cavity flows

So in todays lecture what we will do is we will look at another application where we can possibly get a simplified solution for a fluid flow problem okay and the idea is we are not going to be doing stability analysis we are trying to get a solution now to a problem which rather we can exploit the presence of different length scales.

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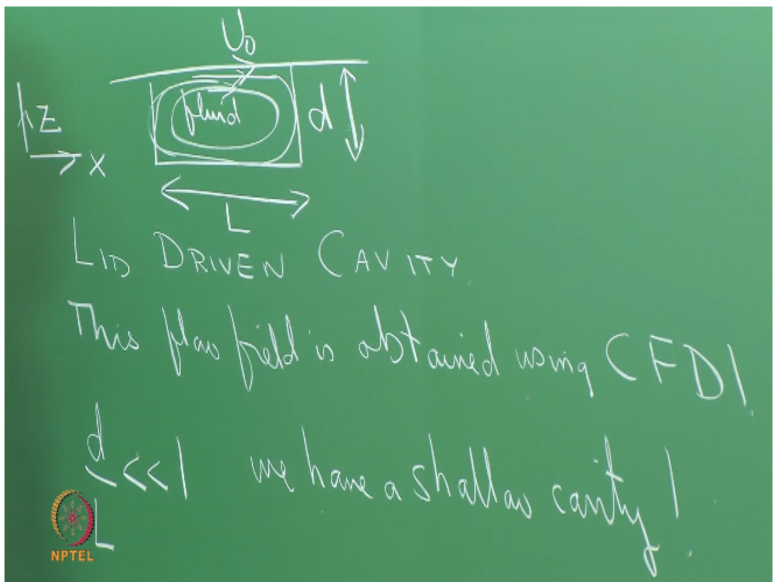

So, this particular problem is going to be called a shallow cavity flow. So, as far as shallow cavity flow is concerned first of all what is exactly cavity and just think of a rectangular channel okay and when I say there is a top plate which is moving with a particular velocity u0. Okay so there is a fluid which is confined here and this is the rectangular channel and let us say it is always it is extending to infinity outside the plane of the board.

So, we are looking at the story in this 2 dimensional plane okay so no variations in the direction perpendicular to the board. So, what we expect that the liquid is going to be dragged here by the top wall and it is going to circulate definitely can penetrate the wall. So, you have this kind of a situation. So, actually there is a very classic problem this problem is called the Lid driven cavity and this is one of the first problems people solve in computational fluid dynamics.

So, you write down the equation of continuity equation of momentum in x and y directions and then you solve using some numerical method. Okay this flow field is obtained using computational fluid dynamics. Okay so the question of course is whether we can actually make some kind of a simplification and get some idea about the flow field okay under some conditions.

So, when you talk about a shallow cavity we are talking about a cavity whose depth let us say this is the z direction and this is x direction. And let us say the extent in the z direction is d that is this distance and the extent in the x direction is L okay. So if d/L is very much <1 okay and we have a shallow cavity. Okay so what we will like to do is analyze this problem and see if we can get some idea about the flow field inside. Okay by exploiting this factor d/L very much <1.

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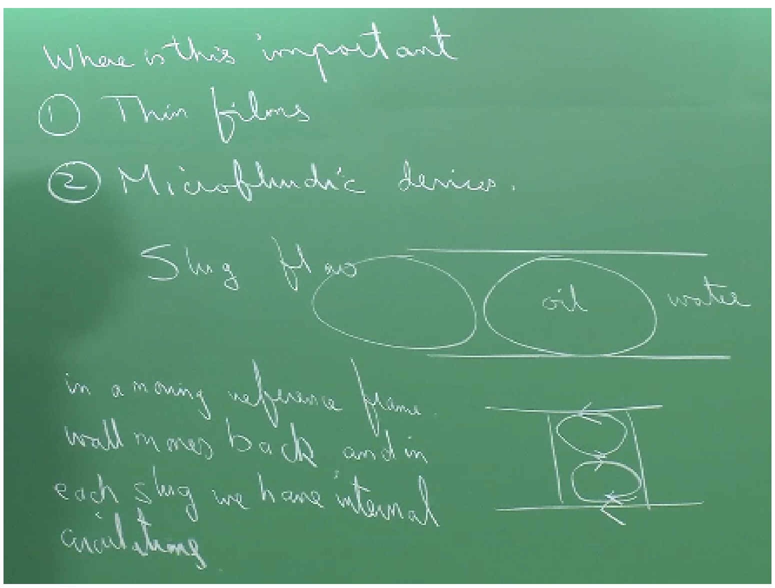

Now where is this important so for example you can have thin films okay or we can have maybe even microfluidic devices? So, supposing you have a very thin film of liquid on a solid surface and either the lower surface is moving and the upper surface is exposed to atmosphere Okay you would have a situation where this thin film and that is the direction which is being dragged for example and this of course is a solid solid wall that have a solid gas wall.

But again the business of these 2 length scales is going to be present and the basically the presence of the length scales d/L is being very much <1. d/L is a very small parameter you can think of already an epsilon coming up here and how I can possibly use this epsilon on for simplifying equations I am getting some insight that is the idea okay and microfluidic devices for example me long back spoke about something a slug flow.

And the basic idea in a slug flow situation is supposing you have a 2 phase flow and let us say this is oil and this is water and you would have things like oil slugs almost occupying the entire channel okay separated by water. So, what is going to happen the oil is going to be flowing okay and there is going to continuous stream of oil drops or oil slugs flowing and of course there is also water present here.

And this is also flowing again if you do what we did yesterday that is work is a moving reference frame. That we are sitting on the oil drop and moving along with it would be like the oil is stationary and the wall is moving backwards okay sitting on the water occupied portion and moving again the wall is actually moving backwards. So, what we can see is when you have this kind of relative motion.

Between the wall and this you are going to have internal circulations inside the slug. Okay in the moving reference frame you have let us say if we are going to move the film this guy is moving backwards. So, this something like a Lid driven cavity problem okay only thing is these are solid walls I have a liquid liquid interphase I am just approximating this to this and what I am going to observe.

I have some kind of a flow pattern of that kind and a flow pattern of that kind here so somewhat this is induced. Okay., so the point I am trying to make here is when you have this kind of a slug flow regime you would have because of the viscosity you would have vortices induced okay and the one way to understand this to look at this problem in a moving reference frame. So, well then look at the because actually this is unsteadily state problem.

Because at any instant time you will have one particular portion occupied either by oil or by water. So, just like yesterday we went to a moving reference frame. I go to a moving reference frame and I say look the walls are going backwards liquid is stationery and what this mean this you have these internal circulations. Now what we want to do is going to understand these internal circulations because that is going to help in mass transfer and heat transfer and things like that.

So, when we are trying to do some kind of a reaction and let us say there is some species here which has to be transported from the aqueous phase to the organic phase or vice versa. This flow is going to actually help in moving the species okay so one wants to do is wants to also understand how these what this is actually going to be developing and whether we can get some idea about the magnitude of the velocity.

Of course one approach is do CFD and the other approach is see if you can get the some insight without doing CFD. And like I told you at the beginning what we can do is we can get these analytical solutions get some idea. And if we are really interested we can do a CFD and get a more rigorous solution and in some limit the CFD can be validated by this analytical solution because at the end of the day CFD can always give you some nice pictures some nice graphs.

And results but the numerical accuracy of these results can be verified only by verifying this. And some limit for example that will give you some confidence in your CFD results okay. So, the idea is the moving reference frame the wall moves back okay. And in each slug we have internal circulations okay internal circulations and this is important to understand mass transfer for example in reactions.

So, typically what will happen is there will be some species here and another species in this phase both have to come together to 1 phase for the reaction to occur. Okay and this convection is going to be deciding the whole thing. So, how do you go about analyzing this so the point I am trying to make here is this problem is very similar to that problem. Okay you have a rectangular wall.

And the only difference is the boundary condition here here I have a solid wall but here I have a liquid liquid interface but if you neglect it the frame is going to get reflected that means again perpendicular component of velocity will be 0. But you will have to compare velocity for example okay so how do we go about this this is just for motivation for doing this kind of a shallow cavity problem.

Okay typically in micro channels the slug length can be 5 to 10 times that of the diameter. So, maybe we can push our luck and try to get some understanding of the flow field here using the shallow cavity limit.

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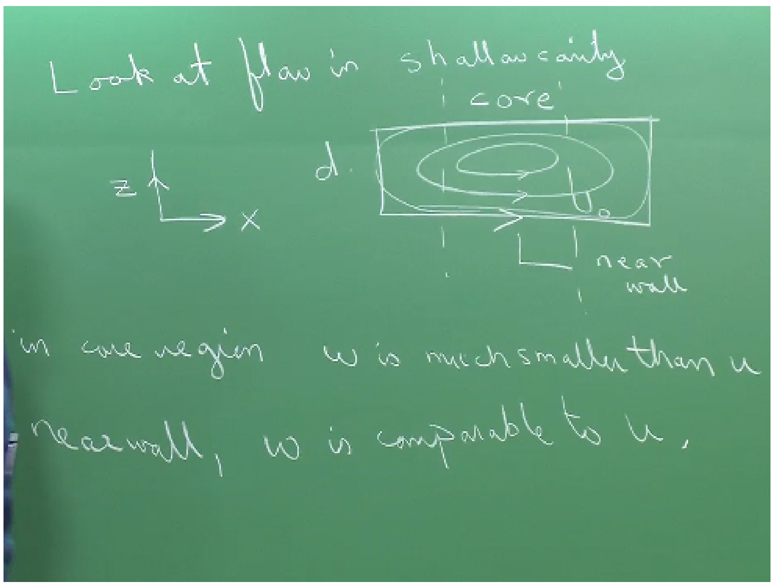

So, what we will do is look at the flow in the original problem that we had in the shallow cavity this is the problem dL I will do what has been done in the lead using. So, Gary Leal he has this business I think this is the x direction and that is the z direction the lower plate is the one which is moving in the u0 with the velocity u0 in the positive x direction okay clearly what is going to happen is liquid is going to get dragged.

And I am going to form somewhat of a vortex okay now this whole problem can be analyzed by dividing this and domain into 2 parts. Okay the central portion here this is the core region and the portions on the other 2 sides which is the near wall region. Okay so I am dividing the entire domain into 2 portions the central portion which is core and here where the fluid is actually going to be getting backwards.

So, there is a difference in the physics in both these regions for example if you focus somewhere in the central portion of the core region you do not have the effect of these walls. Okay for all practical purposes you can view the flow as being almost parallel only thing is the flow is the positive x direction in the lower portion and in the negative x direction in the upper portion Okay whereas here.

And therefore for all practical purposes you can imagine that the vertical component of velocity is 0 I mean in some limit of course there is a smaller vertical component of velocity but that is going to be negligible. Whereas here the vertical component of velocity is going to be comparable to the horizontal component of velocity. So, in the core region the u and w sorry w is much smaller than u whereas near the wall w is comparable to u.

Okay we are also going to look at microfluidic applications in a very small channels and the flow through these small channels the characteristic Reynolds number is going to be very low okay we are talking about very very slow flows. So, what we will do is we will try and analyze this situation using the lower Reynolds number limit. So, rather than write everything and then put Reynolds number=0.

I am just going to at the beginning itself put inertial term=0 write my equation of continuity and equation of momentum okay, so let us do that.

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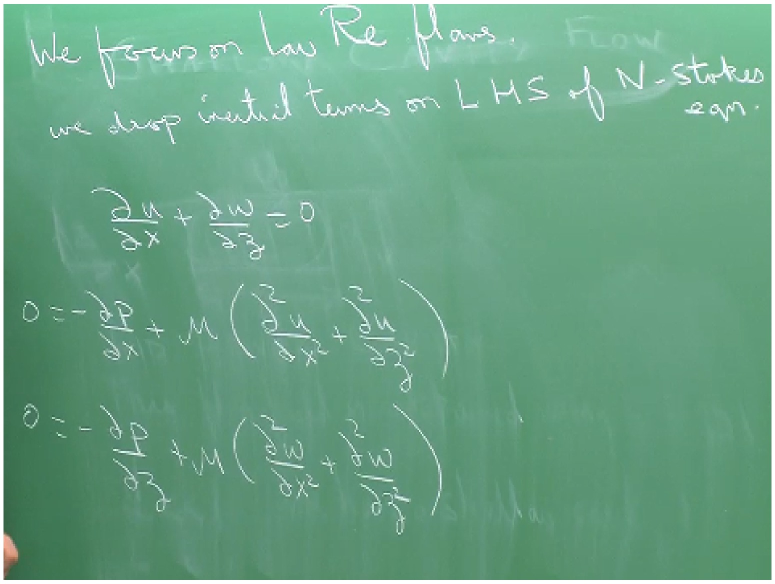

So, we focus on low Reynolds number flows which means we drop the inertial terms on the left hand side of the Navier stokes equations okay and what I am left with the equations of continuity with us du/dx+dw/dz=0 and then 0=-dp/dx+mu d square u/d x squared+d square u/dz square okay and 0=-dp/dz+mu. So, what I have done and gravity I am not worried about just imagine the gravity is actually absorbed into one of these specific terms okay as a gradient.

So, this is the modified pressure. So, that is my equation of continuity the y direction is outside the plane on the board I am neglecting. So. this is the regular thing that we have so what we want to do is we want to simplify this make it dimensionless okay and then exploit the fact that d/L is very much lower than 1 and see what kind of simplification we are going to get. So, see this business of the length scale in one direction being much smaller than the other.

Is something this is you have seen in boundary layer flow for example okay the thickness of the boundary layer is very small. So, what is the argument you make over that that w s smaller than u we need to have an estimate clearly what is the characteristics scale of the velocity in the x direction it is u0 whereas the lower plate velocity. What about the characteristic velocity in that is going to be decided by the problem here?

In the sense that both these terms have to contribute okay and I am going to scale the length in the x direction with L and the length in the z direction with b. Those are the characteristic phase in the respective directions. Okay so u characteristic in the x direction is u0, x characteristic is L and y characteristic is oh sorry z characteristic is d okay and we need to know what w characteristic is.

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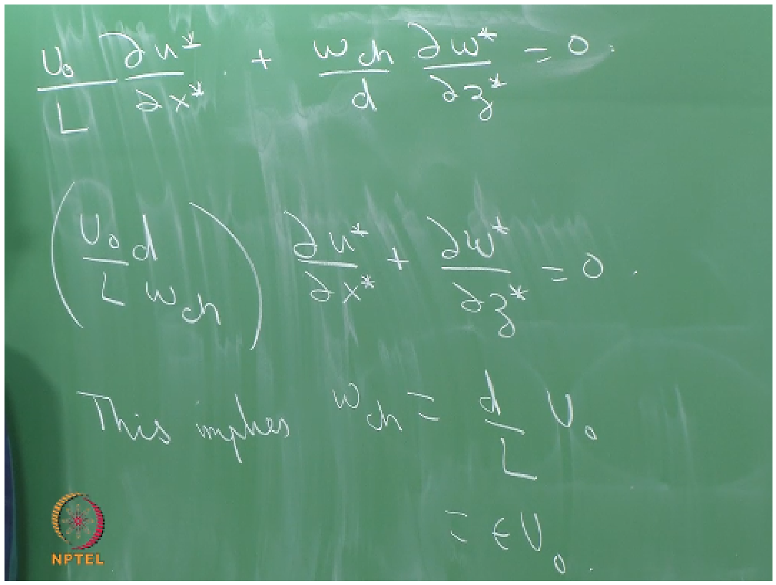

Okay so clearly I am going to substitute here, I am going to get du star/dx star and this is u0 and that is the u characteristic and L+w characteristic/d 0 0 okay so the idea is that these 2 terms have to balance each other they can balance each other only if this is of the order of magnitude 1 and so this implies that w characteristic is d/L times u0 it may be called epsilon u0. So, it is epsilon times the u characteristic.

Okay so what I am going to do now I am going to use this in my momentum equation make them dimensionless. Okay and get an idea about what is happening what I want you to understand is you know the regular flows when you have a pipe flow the pressure drop is something you impose experimentally okay whereas in this particular problem there is going to be a pressure gradient which is going to be decided by the flow.

Okay so dp/dx and dp/dz is something which I do not know is something I have to find out earlier in the second Poisson flow you said because it is a pressure driven flow that is controlled by you experimentally. So, you impose dp/dx and then you find out what is the velocity field you get the parabolic profile okay remember the flow is going to be driven by the wall. So, dp/dx and dp/dz is something that which I need to find out.

Okay so let us do this I also do not know what is p characteristic because the characteristic pressure is going to be something that is decided by the flow then it is taking place and the fluid properties there are things like this viscosity okay and the dimensions of the channel. So, P characteristic is also an unknown. So, what I am going to do is I am going to take the next momentum equation make it dimensionless.

And idea is to might be x star times p characteristic/L+mu times d square mu/d x square will give me u0 squared okay I like to get d/L out. So, I am going to take out d squared and the z direction I have d that comes out. So I have d squared/L squared here. Okay remember this is made dimensionless L that is coming here this is my dimensions less with respect to d. So, that comes here I am taking out d so that d comes there.

So, clearly this d squared/L squared is epsilon squared so if I have chosen my characteristic scales properly what this tells me is that the second derivative in the x direction is much smaller than the second derivative in the z direction see the z direction distance is very small so that the variations are much sharper. So when I have to compare these 2 I can actually neglect this in comparison to this.

Okay so basically this is an order of 2 orders of magnitude lower than this depending upon epsilon, this is Epsilon square term but what I need to do is I need to do is I need to choose my p characteristic so that this pressure gradient in x direction is going to balance the term in the z direction, the second derivative was the z direction okay then only this will balance that. So, I am saying I am going to neglect this compared to this.

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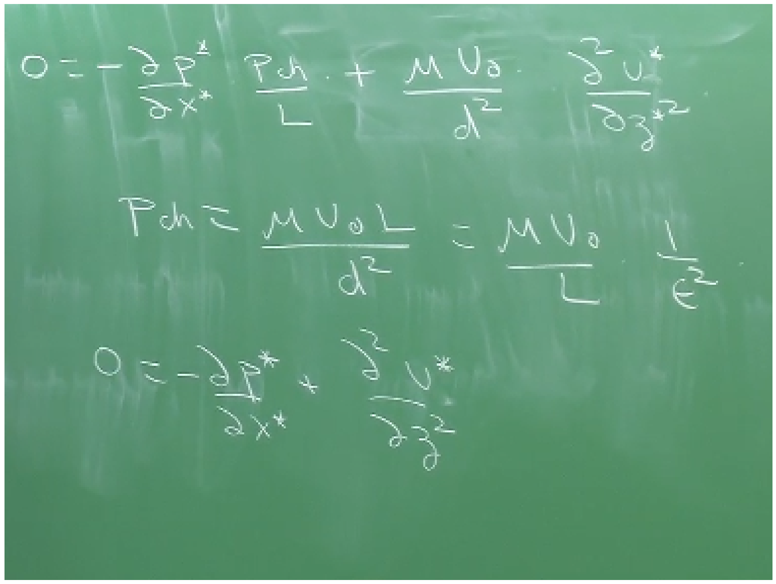

Okay this is epsilon squared that is epsilon squared and I have 0=-dp/dx and p characteristic+mu u0/d z squared star square. Now p characteristic is chosen as mu u0 times L/d squared or mu u0/L times 1/epsilon squared. So, basically the characteristic pressure which is actually developed inside the cavity is given by this. Okay so if you choose this then this becomes equal to that.

And you are simplified equation of momentum x direction is –dp star/dx star +d squared u star/ dz squared that is our dimensionless equation of momentum. So, basically the 2 important forces are the pressure force and the viscous force okay gravity any way you are neglecting inertia is gone so that is something very similar to what do you have seen in your second Poisson flow you have pressure and you know your viscous force.

Only thing is I need to account for this guy bending back and all that and dp/dx is not known to me. Okay so now let us do the other direction 0 that is the z momentum equation okay that is what we have to do and what I will do is go to the other side of the board.

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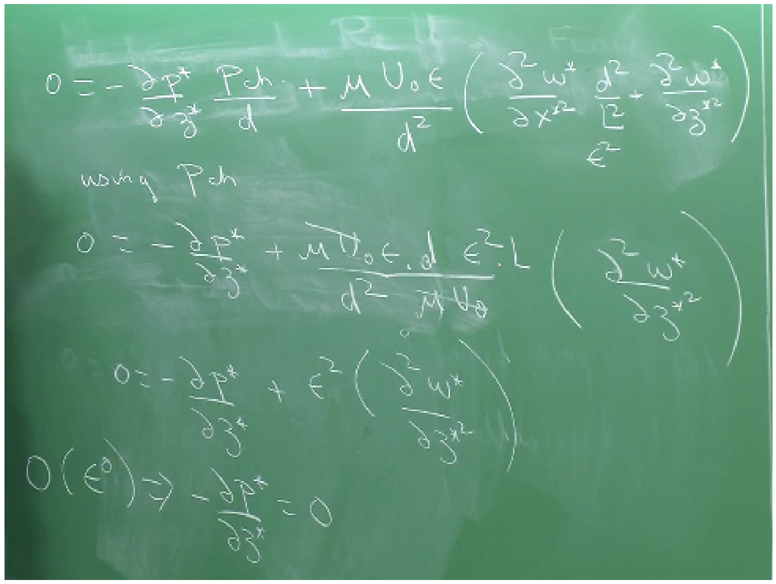

I am going to start with this my d is z star this is p characteristic which I know and this is which I found from that and this is going to be give n by d+mu times w characteristic is u0 times epsilon that comes out and I will do my usual stuff just bring a nice s d squared outside here right and I am left with okay. So, I have the same stuff L coming as the characteristic length scale here so that L comes there this give same d d comes out and d goes there okay.

So, now remember what is p characteristic I already found out what P characteristic is I am going to substitute that here and I am going to get using the p characteristic we get 0=-dp star/dz star+mu u0 epsilon/d squared times d which goes there and p characteristic is coming to the bottom which means I have epsilon square/ mu u0 and there is a L here okay times between these 2 terms this is much smaller than this this d squared w star/d z star squared.

Because this is epsilon squared times that okay so let us simplify I get d/L as epsilon d/L is epsilon right yeah epsilon goes off and I get 0=-dp star/dz star+epsilon square times d square w star/dz star squared. So, in the limit of epsilon tending to 0 of if you actually did a bottom measuring series solution and if you found out the solution in terms of a power series okay so what you would get is in the limit of epsilon to 0.

This term is going to ne negligibly small compared to this okay. So, if you look at the terms of order epsilon to the power 0 this is going to be-dp star/dz star=0. In other words, the pressure variation in the z direction is not there clearly the region is so thin in the z direction for all practical purposes we actually neglect the pressure gradient in the z direction. Rather the pressure gradient in the x direction is given by this momentum equation which we have wrote over there.

Okay so basically what we have done is if we use or find the epsilon very small and rather than do every formal power series expansion what you should do now you should seek p as p0+epsilon p1 w as wo+epsilon and then equate coefficients but those equations are independent of epsilon the epsilon is occurring only here. So I am just getting the 0 th order solution directly by putting epsilon=0.

Okay I mean your 0 th order solution or a base solution put epsilon=0 I am getting the base solution. So, I am going to get the base solution just by putting epsilon=0 okay you can do a more formal thing like we did earlier and then look at the first term then you will get this.

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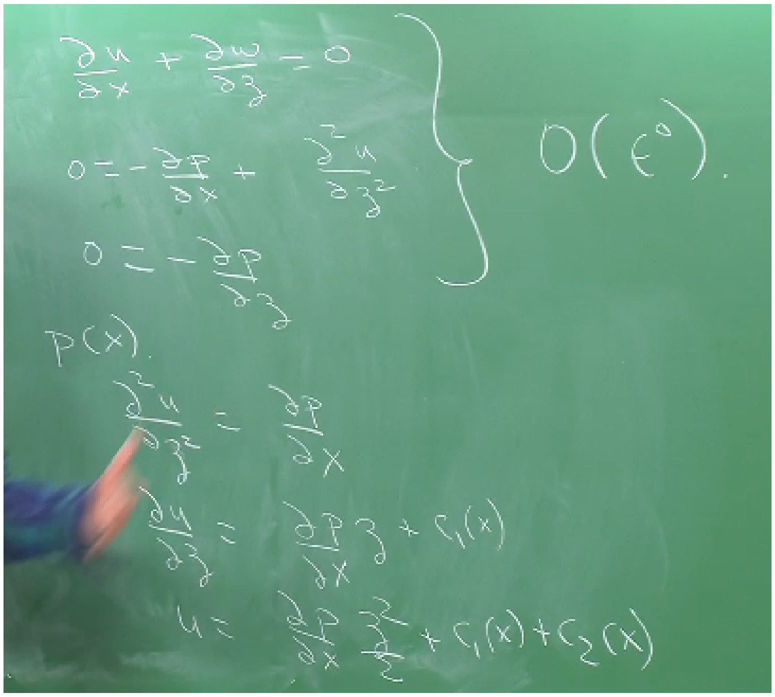

This is the term we need to solve now I am going to drop all the stars okay o=-dp/dx 0=-dp/dz these are the terms of epsilon=0 these are the 3 equations which we have. So, let us look at how we can proceed what does this imply dp/dz is 0 implies that the pressure is a function only of x okay it is too thin in the other direction so you neglect the variation in that direction this implies that pressure is a function of x okay if pressure is the function of x alone.

Okay then I can integrate this d squared u/d x squared dz squared=dp/dx from this equation okay from my equation of the momentum in the x direction d squared u/d z squared is dp/dx this is a function of x. I am going to integrate this twice but since this is a partial derivative in the z direction want to have this these constants are going to come they can actually be functions of x. Okay so integrate this once this is du/dz=dp/dx times z+c1 of x.

Okay integrate this one more time and you get u=dp/dx times z squared/2+c2 of x these are the functions of x. Because I have a partial derivative my total data have been constant okay so remember u can be a function of x because the dp/dx can change with x that is what this means dp/dx remember the function b the function of x. So, since pressure if a function of x I am allowing for the fact that u can change.

“**Professor - student conversation starts”** Yeah that is z I thought I wrote z I guess I did not okay “**Professor - student conversation ends”** Now you have to put the boundary conditions what are the boundary conditions at z=0.

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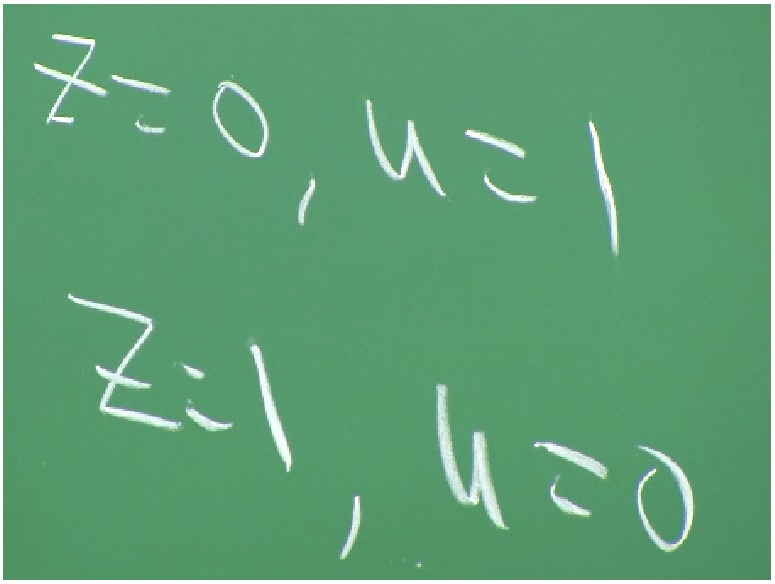

u is 1 because that is how I made it dimensionless and that z=1 I have u is 0 because I have upper wall is stationery right when I put z=1 when I put z=0 I have u=1 okay that gives me c2=1 c2 of x is1.

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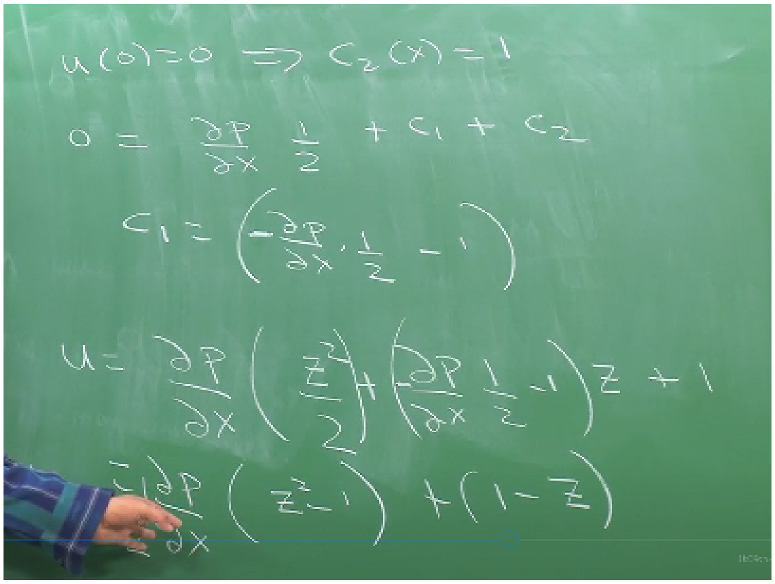

U00 o implies c2 of x=1 okay and z=1 u is 0 and o =dp/dx times 1/2+c1+c2 so c2 is 1 and that was c1 is okay that is c1 so now actually I have the expression for my velocity u is dp/dx times z squared/2+c1 is c1x is dp/dx times 1/2-1 times z+1 okay I am going to group my dp/dx terms together which one. **"Professor - student conversation starts"** which one –dp/dx here okay thanks yeah yeah I always do it right man this – right.

So, I need to put a –here I think that is it. **"Professor – student conversation ends"** so that is the velocity field it is something like it is a parabola of course and remember but how do you solve the problem yeah we need to find w but we know u we do not know u we do not know dp/dx dp/dx is something which I do not know. I need to find out what is dp/dx remember it is not I am imposing a pressure gradient a wall is moving.

How do I find out dp/dx? I need to use some condition about the flow. So let us look at to give an idea and to prompt you yeah.

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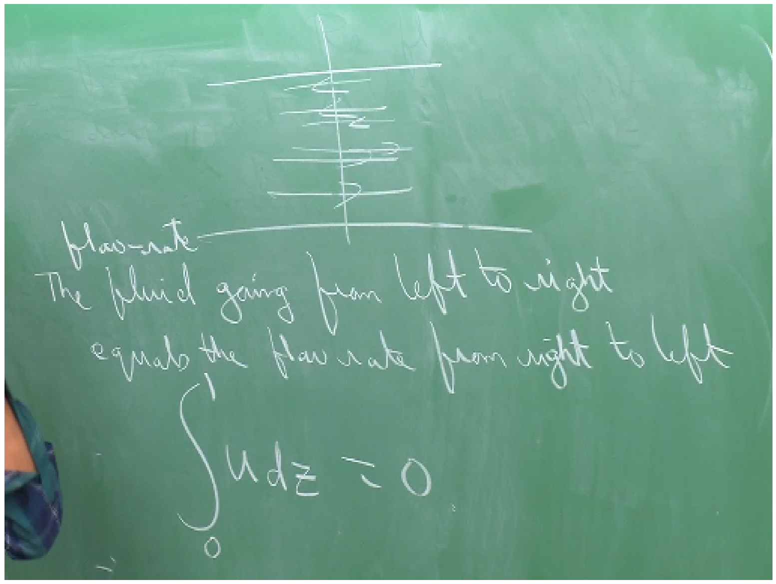

So, I am looking at the core region right and in the core region how is the flow somewhere on the top it is going from right to left the bottom is going from left to right and for all practical purposes we have this kind of flow. Now what do we expect is that a net flow across this line you have a confined liquid liquid its confined whatever liquid is going to go from left to right you have a semi state situation.

Okay whatever liquid is going from left to right must be the same as a liquid is coming from right to left okay so what that means is the volumetric flow from left to right must be balanced because there is nothing the liquid is not leaving of 2 walls you have 2 rigid walls the liquid cannot go out. So, whatever liquid is coming from left to right must actually come back from right to left. So the total volumetric flow rate across this line must be 0.

Okay and that means the fluid going from left to right I cannot say the fluid the flow rate = the flow rate from right to left okay which means the net flow rate must be 0 or integral udz from 0 to 1 must be 0 because I have confined liquid and then I am just moving this flow. So, I am going to use that condition and of course I know u I am going to use that and find out dp/dx so put this here.

And if you do the algebra you will get this is used to find dp/dx=6 okay I mean we can just use the integration and you can find out that dp/dx=6 what that means is dp/dx is indeed a constant and that is something similar to what you had for your Hagen Poisson flow when when you say that the pressure gradient is a constant. So, if dp/dx is 6 now you go back to that equation earlier we said p was a function of x but I know it is linearly varying. So dp/dx is constant which means u is not a function of x u is a function only of z.

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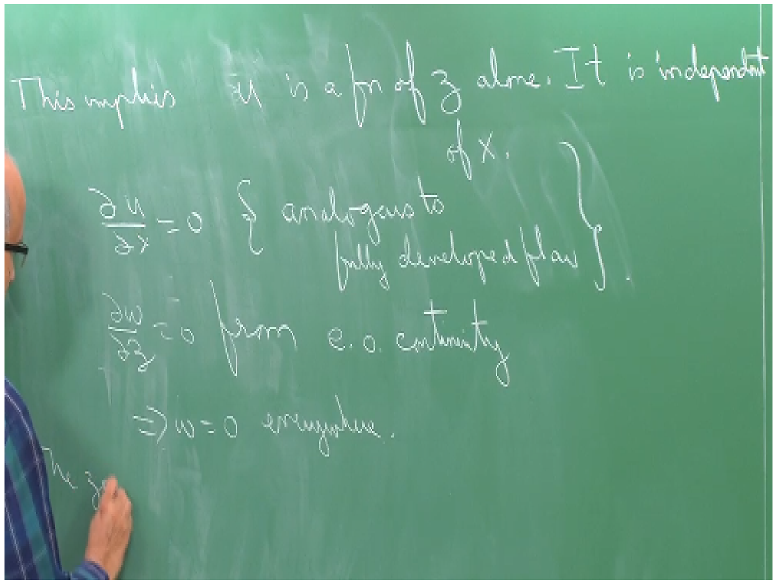

Okay this implies u is a function of z alone it is independent of x okay so u is independent of x so therefore what it means is you have something like a flowy developed flow situation du/dx is 0 okay that means du/dx is 0 I just said analogous to fully developed flow and from the equation of continuity dw/dz=0 from equation of continuity and since you have 2 solid walls w0 at the walls then w0 everywhere applies w=0 everywhere.

So, what we have done we basically got an idea about the flow field is at the 0 th order but the epsilon is very very small epsilon is 0 okay so the 0 th order solution tells you there is no vertical component of velocity in this core region when I am focusing okay the velocity is almost fully developed so there is no change in the x direction and the exact dependency on z is given by that and just put dp/dx=6 you get them.

So, you will get something like I mean if you really did the and plot it or something we put z=1 that has to be 0. I have put z=0, you get 1 or something okay now dp/dx is 6 3 **"Professor - student conversation starts"** yeah this is it, is it yeah thank you, yeah this is z ok yeah yeah then I am okay that is why it was not satisfying the boundary conditions out of all of this yeah now z=0 it is 0 and z=1.

Yeah, right it is happening that is good if that is happening then everything is fine yeah z squared-z+1-z "**Professor - student conversation ends"** yeah the 0th order solution is like this and this is 1 actually thought of a profile I will get something like a parabola with a bending back normally you have a parabola which is 0 at the 2 walls but here you have a parabola which is 1 here drags and then actually goes back to 0.

Okay let me see if I can draw this properly if I draw this properly it is 0 here and then it is something like +1 is it yeah something like this this is 0 and this is+1 of course this is+1 that is the value of the velocity okay very close +1 to 0 moves right and then bend back this is an awesome outbreak this is also not right I think I need to plot this function **"Professor - student conversation starts"** this is not right why is it not right yeah this is not correct.

This is not correct so what is the right thing clearly this is wrong nothing goes beyond 1 that is one reason and the velocity has to be negative and the way I have drawn it everywhere it is positive okay yeah. **"Professor-student conversation ends"** So you need to draw it right.

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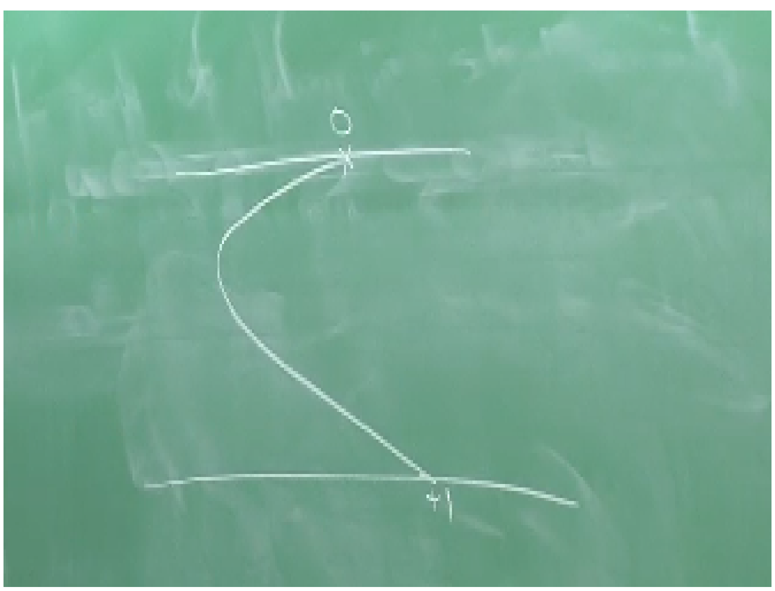

And let us draw it right for a change which is going to be like this 0 and this is +1 okay. So, that is basically one application for shallow cavity flows and then you can do this also for the near wall region and then you can do it for different boundary conditions and stuff like that I think with that we will stop.