Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 40
Viscous Fingering: Darcy's law

So, welcome to today's lecture on multiphase flows. What we will today is discuss another instability problem in multiphase flows in a 2 phase flows, okay and this particular instability problem is called viscous fingering and it also goes by the name of Saffman–Taylor problem after the 2 people who first investigated this problem. So, what we will do is, we will discuss this problem today.

The Saffman–Taylor formulation and that is basically what we are going to stick to, we are going to make the assumptions which Saffman and Taylor did and then who analyze the problem, okay and formulation of viscous fingering that is the physical phenomena. Now, this particular problem has a lot of applications especially in the field of oil recovery, okay. So, you talking about enhanced oil recovery.

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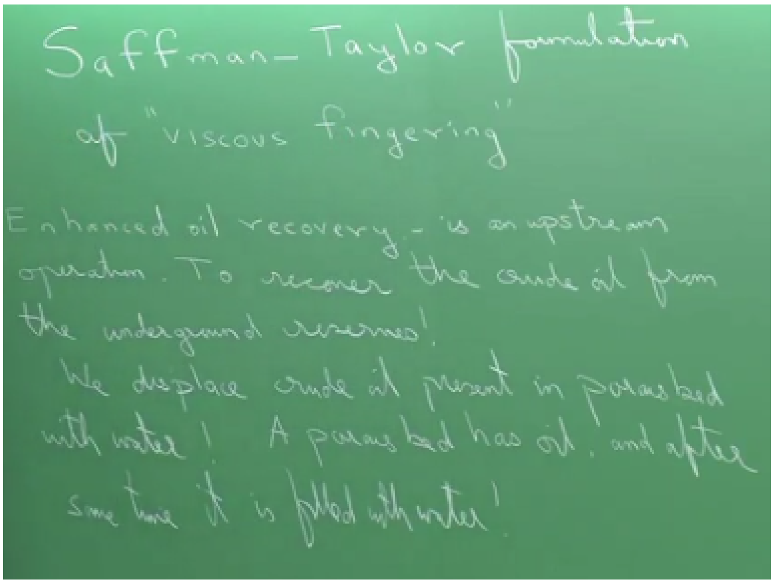

So, as chemical engineers or as engineers who intrusive energy, we are having sudden upstream operations wherein you have all these petroleum screwed reservoirs and one of the objective is to take out all this oil which is present underneath in a porous media, okay. We want to be able to extract it, so that then you can come and process it in your refinery and get you petrol, diesel etc, okay.

So, enhanced oil recovery is an upstream operation and what is our objective to recover the crude oil from the underground reserves, okay. So, basically what we are talking about is you have oil which is present in a porous bed, okay and so what you want to do is, we want to be able to pump it out. So, one way that is normally used is you are actually pump in water at high pressure and the water will actually displace this oil.

And what was originally containing petroleum, will now contain water and the petroleum which was present, the crude just present will actually come to the surface, okay. So, what we are trying to do is, we are trying to displace one liquid using another liquid. So, we displace the crude oil present in the porous bed with water, okay. So, what it means is porous bed initially has oil and after sometime it is filled with water.

Now, the problem that we are talking about if we really want to visualize this, let us say this is my porous bed, okay this drawing is horizontal and so, I mean I do not want to confuse things, but just imagine this is a porous bed. These are you soil particles, is there everywhere, I am just not drawing too many of them and you have oil here. What I am going to do is, I am going to pumping in water from the left.

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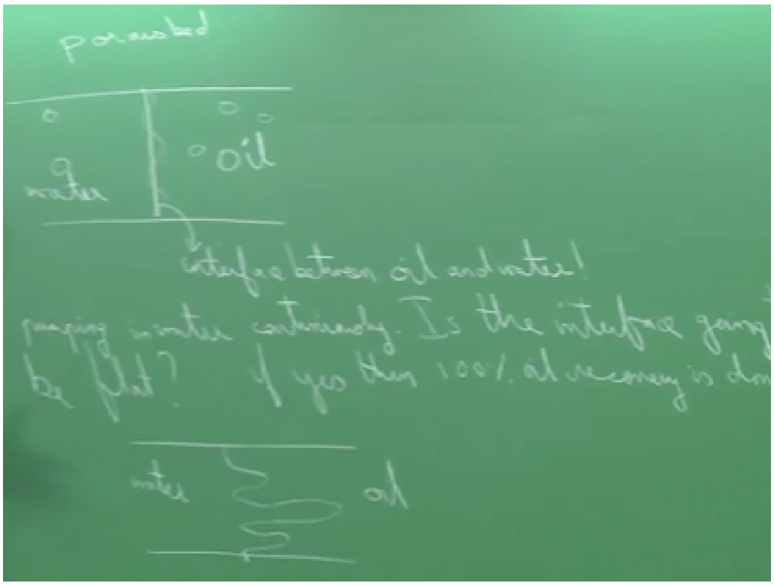

Of course natural situation, you do not have a well defined geometry, but since we are interested in the mathematical formulation for understanding, we are looking at a very nice rectangular geometry and here let us say this is water and this is the interface between oil and water, okay. Ideally, the question that what we are interested in asking is when I want to do this pumping of water to displace oil, will this interface for example always remains flat.

If it remains flat, then the entire oil which is present in the reservoir is going to be recovered, is going to be displaced because the interface continues to remain flat so far when we are doing a pumping, there are going to be disturbances. Disturbances can be because of some nonhomogeneity in the soil property, in the density of the soil packing things like that, the porosity for example.

The question is, is this interface going to remain flat as it moves. So, the problem is I am pumping in water continuously, is the interface going to be flat? If it is flat, then we have all the oil recovered, all the oil will be displaced and you have 100% recovery, okay. If yes then 100% oil recovery is done, but supposing the interface does not remain flat what is likely to happen is you give a small disturbance and this interface has going like this.

So, this is after the disturbance has actually evolved in time. So, first give a small disturbance, it may be sinusoidal, but what is likely to happen is, if the system does not stable, if the system is stable, the small disturbance is going to disappear and you get back your flat surface. Supposing this interface is actually unstable, then this interface is going to get deflected, okay.

And if it gets deflected clearly this when you pumping in water, this guy, this is water here and this is oil here this water is going to the tip of this finger as you can see this is all looking like fingers, okay and that is the reason you call it fingering and this is actually what to do with the viscosity. This is moving in a faster rate and that means essentially unstable and this guy will actually penetrate the bed and come out as water, okay.

And what that means is there will packets of oil which are going to be left, trapped inside the reservoir which you are not recovered. So, basically you know incomplete recovery of all the crude oil, understand. So, that is basically what the implication is. So, basically what this means is here, the velocity of the water tip is fast. It breaks through the bed, okay and petrol crude is going to be left behind partially.

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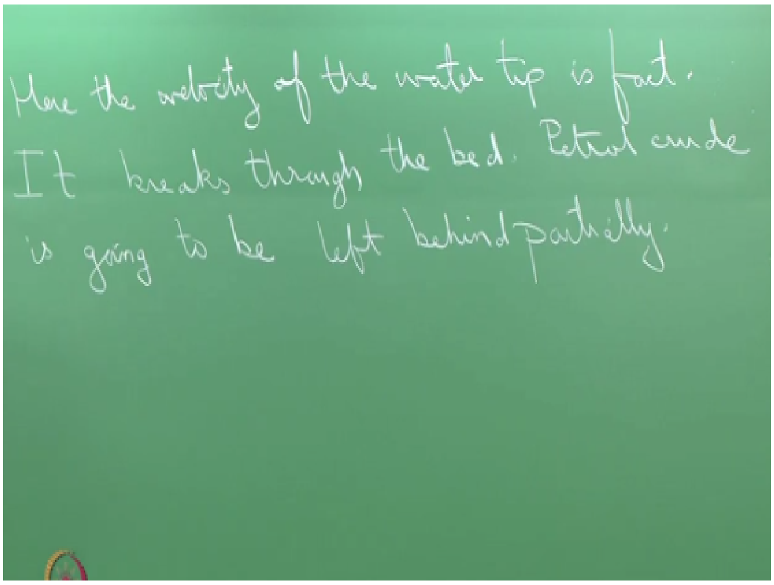

So, you do not have recovery of the entire petrol crude. So, one of the things we want to do is we want to understand what is it that is actually causing this kind of phenomena if this occurs. So, actually experimentally you can actually do this. So, all you have to do for example if you want to do a simple experiment, you can just take 2 glass slides, put small glass spherical particles, okay.

You can make a one porous bed and you can just pump, you can fill it with one liquid, pump another liquid and see if the liquid is actually going to be displaced and the interface is flat. The very simple experiment for you to do, clearly you can do an experiment underground right. So, basically what you want to do is, you want to do an experiment make a porous bed, which is hopefully transparent.

So, we can see this interface. So, we can make small glass beads, make this bed, fill it with oil of whatever properties you want and then pump another liquid and then see under what conditions is the interface going to remain flat, under what conditions the interface going to actually deform (()) (11:37). So, you can possibly put a small dye to be able to contrast between the 2 liquids.

Both of them are transparent, then you would not be able to see, how the interface is. So, the things like that. But this is the actual question that we are asking, okay and clearly, 2 things are important. One is the viscosity and of course, the surface tension because you are working about the interface. So, what Saffman and Taylor did is the submit the problem in the absence of surface tension that is what we will do, but then you can always include the effect of surface tension and then you can redo the analysis.

This problem is similar in some sense to the Rayleigh–Taylor problem. If you remember the Rayleigh–Taylor problem, I mean I think there is something which you people have to do we have done all these instabilities in the class, you have to be able to see what the similarities are and the dissimilarities are, that helps you understand things better. The Rayleigh–Taylor problem which you remember you have 2 liquids, one on top of the other.

And the 2 liquids were stationery, okay and then you ask the question when is the interface is going to be stable, unstable, okay. So, in that particular case, we consider the situation of inviscid liquids. Here, I want to include the effect of viscosity because it turns out that there is an extremely crucial parameter. It turns out that the viscosity ratio between these 2 fluids is the one which actually decides whether the interface is flat or not flat, okay.

And what we try to do is prove this mathematically. Over that we include the effect of surface tension, but we let go off viscosity. Here we are doing the opposite, we are going to keep viscosity, but right out to begin with we are going to let go off surface tension and then we can see and the only thing there it was verticality, here also we can put vertical in fact we have vertical geometry here also.

And of course, there the base velocity was 0 and we did the linearization, here the base velocity is not going to be 0 because of actually having continuous pumping of one liquid or the other. So, having said that what we want to do is rather than analyze this problem with a base velocity which is a uniform velocity, we will do the analysis of the problem in a moving reference frame, okay.

So, basically what I am saying is the base velocity, whose stability are interested in measuring we can here, this is the question we are asking the base velocity is uniform across the cross-section, okay and let us say that is capital V. Now, what is going to happen, what is the base state? Base state is the one which you are talking about is the one where the interface is flat and then we gave perturbations and we are asking whether it is stable or not stable, okay.

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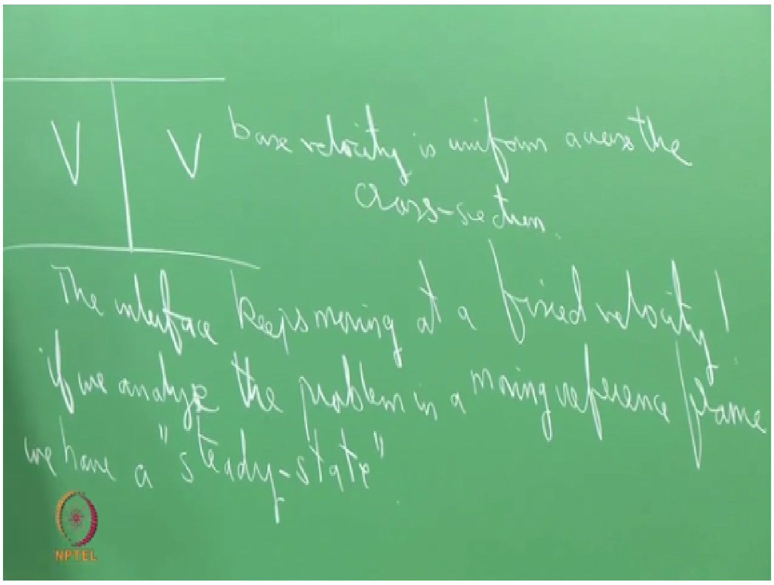

So, the base state is going to be 1, which is having a flat interface and continuously pumping one liquid, so what is going to happen this interface is going to keep on moving, so actually the base state is not a steady state, understand because the interface keeps moving. So, in order to make the base state as steady state, one way to do is just analyze the problem in a moving reference frame that is if the base velocity is V, you move along with the liquid.

So, in every point of time, it looks like you are at a steady state, okay. So, the interface keeps moving at a fixed velocity, okay and if we analyze this frame, okay and so the moving reference frame will be having a velocity V, capital V again and if you are moving along with this, it look like the interface of stationery, okay. So, you know for all practical purposes this is valley of multiphase problem because you have solid bed, interfaces and everything and then you have 2 liquids, right.

So, this must be the grand climax of this course, having so many different phases. Now, we have not done flow through porous media so far, but you will also use Navier-Stokes equations and one of the things which we want to do is flow through porous media, this is characterized by what kind of relationship for momentum balance, the one which relates pressure gradient to the velocity feel, can we use Navier-Stokes equation?

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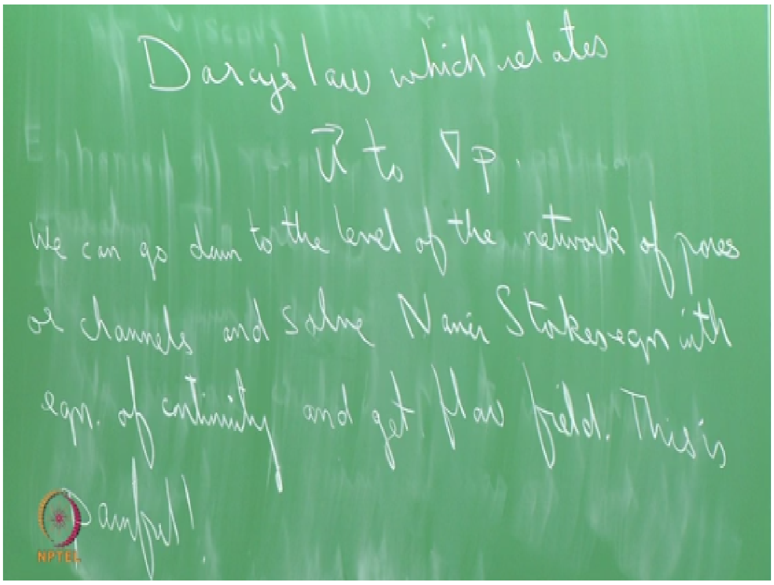

Of course, you can use Navier-Stokes equations, but what is typically used in porous media to describe pressure drop and velocity, the Ergun equation or a simplified form of the Ergun equation would be the Darcy's law, okay. So, basically what we are going to do is, we are going to use Darcy's law which relates velocity to the gradient of pressure, okay. Now, if you really see many people do a lot of extensive results in porous media, okay.

And depending upon the level of detail, you are interested in, you will decide you know go into detail and start doing the analysis of the problem. So, one approach is extremely detailed is, you would take a porous media and the porous media is basically going to be an interconnected channel, right. Basically, the liquid is going to flow through some interconnected ne2rk of channels or pores.

And if you are interested, you can really go down to finding out what this geometry of these channels is. It could be a random ne2rk and then in each of these channels, you should write down the equation of continuity, equation of motion and then you solve for the velocities and then you actually predict the behavior of the flow that is one approach. Clearly, what that means is, you need to possibly model the entire thing as a section of channels and final the velocity field.

The advantage is or one of the simplification is, that since the size of the channel is very, very small, your Reynolds number are going to be a very low and for all practical purposes, you can use the creeping flow limit, low Reynolds number limit. You can neglect all the inertial terms and you only have the pressure term and the viscous term, okay. So, that is one approach where you go to the detail level.

So, we can go down to the level of the ne2rk of pores or channels and solve the Navier-Stokes equation with equation of continuity and get the flow field. This is of course extremely intensive, right. This is so you already just say painful who is going to sit down and go through every pore and do this ne2rk, right. We most of the time what were interested in is, some kind of an average information, right.

We want to find out, so just like you do used a continuum hypothesis and talk in terms of velocity at a particular point, which means it is the velocity of a collection of molecules in the neighborhood of at that point, right. So, we are going to use something like a similar averaging approach to actually describe and that is what Darcy's law does. Darcy's law when it is talking about u being related to gradient of p, okay being linearly related to gradient to p.

Because you are in the creeping flow limit, okay he is talking about an average velocity and so, what is this averaging, this averaging is being done over length scales which are larger than the length of the pore, but not so large that the variations of velocity from one point to another in the system at lost, you understand. So, that is you have to do an averaging on velocity over bunch of pores.

So that you get some average information on the velocity, but you could take the averaging over the entire length, then you will get a uniform value across the entire length. Then you are losing the information of changes along the length of the channel, okay. You want that information to be present. So, you have to choose this averaging domain, averaging size, the length scale over which you are averaging properly and that is basically what we do in continuum also.

In continuum hypothesis, when you are writing down the equation of continuity, equation of momentum, you choose your length scales or your volumes, so that it is not too big and it is not too small. If would be sufficient to have enough number of molecules, so that you can actually get a good average. If it is too big, then you would not get the velocity variation, okay.

So, basically that is what we are doing. So we are not going to be using this approach because this is just not worth it, because you are going to get information, it just possibly not useful, who care what the velocity is in pore, you only want to know if the interface is going to be deflected or not. So, I do not want to waste my time, trying to get this detail velocity field information, okay unless you are graded the course depends on this, then you have to, okay.

So, what we do is, I know grade does not depend on that, so we can be sure, okay. So, we average over a length scale which is larger than the pore size, okay, but lower than the size of the system. So, it might contain several pore sizes, okay. So, basically your average velocity is, the velocity average over these 4 or 5 pore sizes, okay and ideas you take more, it is not going to change average value.

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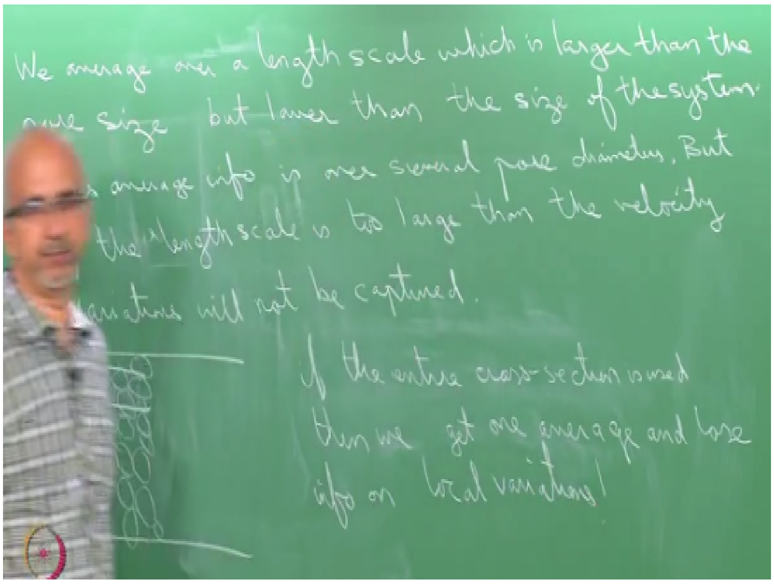

So, this average information is over a several pore diameters, but if the length scale is too large, then the velocity variations will not be captured. So for example, if you are having the porous bed, that is it. I am not going to draw more of this, but if you are interested, all the actual velocity field in each of these pores that is my first approach, that is this approach when talking about.

But I am going to take this length scale, take several pores into account, I am interested in the average velocity here, okay. Then, I can get the velocity at this portion, I will look at this region here and get the average velocity. This way, if there is a variation in velocity from one section to another, unable to capture it. If I take the entire thing and then do the averaging, I will get one velocity everywhere and then the variation in the velocity are losing, okay.

So, I want to get the variation in the velocity in this domain. So, I am going to have to keep it sufficiently small to do that. I do not want to go too small because then I do not have enough pores to do the averaging. So that is a kind of tricky basis and that is basically what the theory is for Darcy's law, okay and so that is, if the entire cross-section is used, then we get one average and lose information on the local variations.

So what I like to do is, I am not going to derive Darcy’s law, but I am going to give you some field for how Darcy’s law can possibly come from the Navier-Stokes equation. Once you understand that, then we can proceed further, okay. There is some small shuttle things, which I liked to mention. So, let me say an approximate derivation of Darcy’s law, not a derivation actually, okay.

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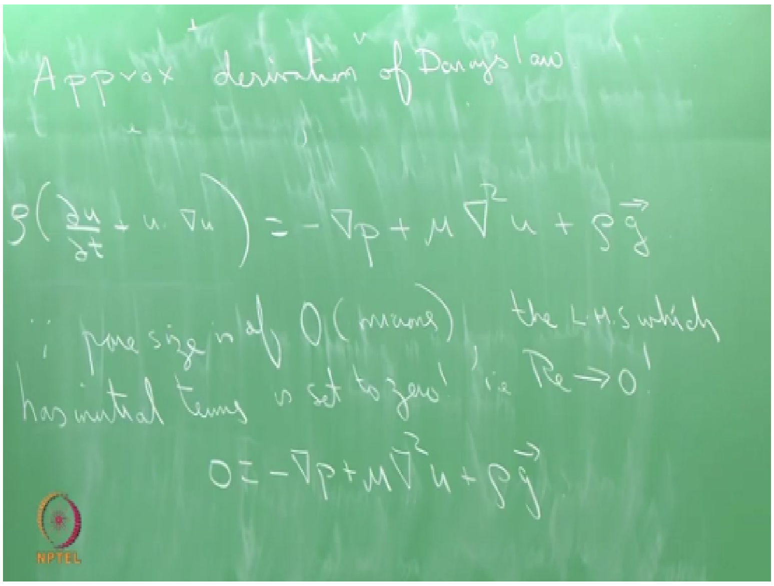

Now, what is the Navier–Stokes equation that you write, du/dt + u dot del u equals – gradient of p + mu del square u + rho g that is your Navier-Stokes equation and clearly, inside every pore, the Navier-Stokes equation is valid, okay, but remember we are talking about porous bed where the size of the pore is just order of some microns. So, the Reynolds numbers are going to be very, very low, okay.

So, essentially what that means is if the Reynolds numbers are very, very low, you have the inertial terms are very, very negligible. So, this is basically 0 for all practical purposes and that is your creeping flow limit, okay. So, since pore size is of the order of microns, the left hand size which has inertial terms is set to 0, low Reynolds number, i.e. Reynolds number tends to 0.

And then what are my left writhe, equation 0 equals – gradient of p + mu del square u + rho g, okay. So, now let us forget the fact that, I mean actually the 3 velocity components, so let us just look at one velocity component. So for each velocity component, it is going to be of the form, for each component, the equation is going to be of the form let us dp/dx, if I bring it to the other side equals mu del square u, this is for the velocity vector + rho gx.

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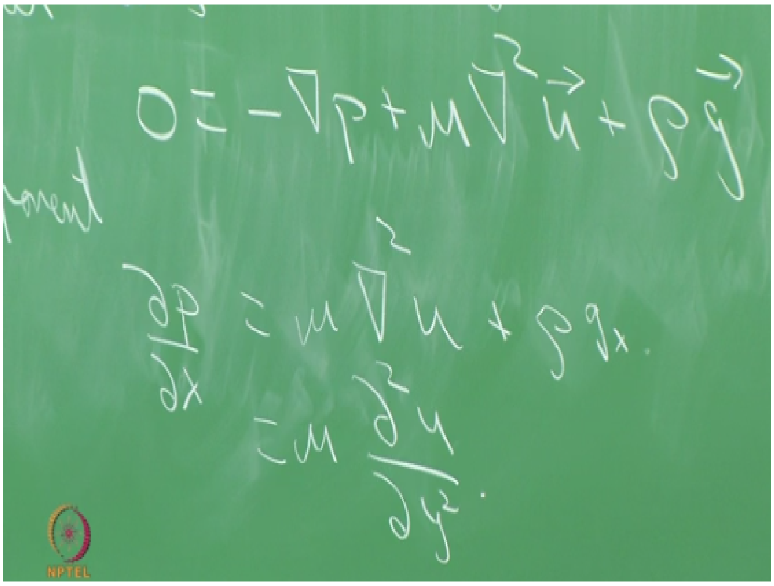

Now, del square u and like you have seen in the case of the laminar flow, for example, the flow is going to be laminar inside these pores, del square u is therefore going to be given by some kind of one dimensional velocity field, where you have something like a parabolic velocity profile. So, supposing you have a second derivative here which is what you will get in the y direction, - dp/dx is mu d square u by dy square.

You have a very thin channel and you have a second derivative inside this channel. One way for you to do this to approximate the second derivative by a simple finite difference scheme, okay, so this is flow through a small channel which is very thin just use second derivative approximation numerically. What you would get, ui – 1 – 2ui + ui + 1, okay. So that is what I am going to do.

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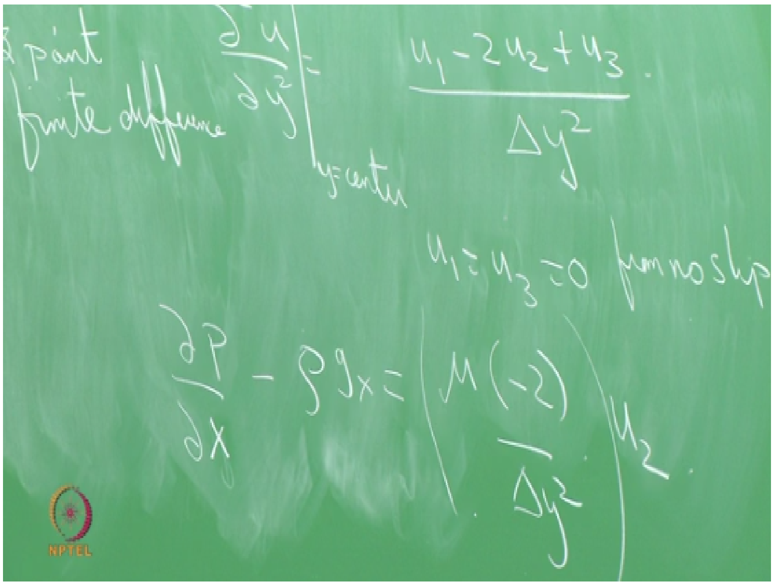

I am going to basically use a fact, so this is the y direction and this is x direction, d square u/dy square and rather than take you know 10 grades and 20 grades, let us just take 3 grades, 2 will coincide the wall, one will be at the center, okay. So, this u1, u2, and let us say u3. Clearly, what is d square u/dy square evaluated at y = the center, it is u1 – 2 u2 + u3 divided by delta y square, okay where delta y is this distance.

Now because of the no slip boundary condition, u1 and u3 are going to be 0, okay so basically what I have done is and this is a 3-point finite difference that is what I am using and from no slip boundary condition, these guys become 0. So, u1 equals u3 equals 0 from no slip and now, if I want to basically use this information over there, what am I going to get, dp/dy – rho gx.

Sorry, dp/dx – rho gx equals mu times this thing here, mu times – 2 divided by delta y whole square times u2, okay. Now, basically this is of course a constant and what I could do is, I can bring in this gravitational field also as a gradient. I can write this as d/dx of rho gx x, okay I can combine these 2 guys and I can get d/dx of some modified pressure, the thing that was describing the other day equals – of u.

And remember that is basically what I wrote Darcy's law is the pressure gradient is the linearly related to the velocity and that is basically what I am showing you here that this is the pressure gradient which includes the effect of the gravitational field, is linearly related to the velocity, okay. Of course, this is not a formal proof, but that is basically the idea that you have and the Darcy's law was actually found by experiments actually.

And then I know peoples are wondering about how to go about getting this. So, basically what I am saying is, I am going to write this as d/dx of p – rho gxx equals -2 – mu times 2/delta y whole square times u2, your u2 is some velocity inside the pore which has let us say average doubt, okay and this tells me this is d/dx of some kind of modified pressure equals – mu/k u and this is for one component which I have written.

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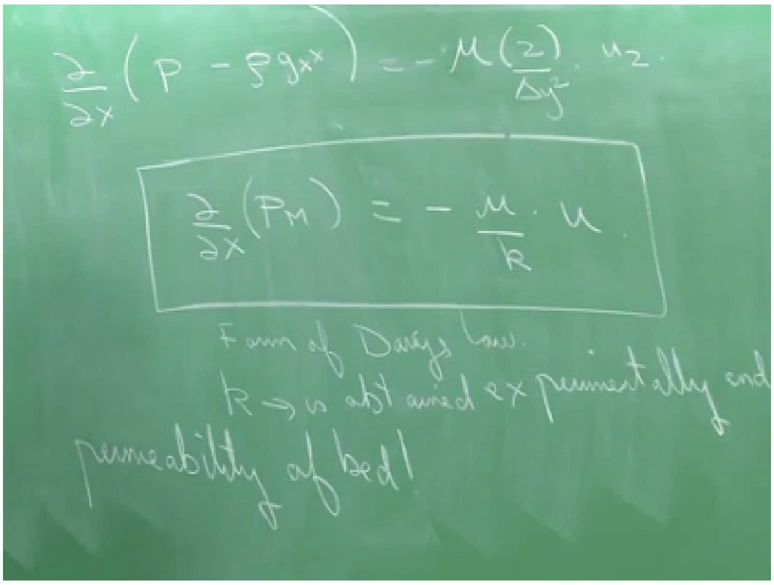

You can write the same thing for different component of velocities in the y and z direction, okay and you would get the same thing. So that is basically to try to tell you how Darcy's law comes up, okay and pm is including the natural pressure and the gravitational field, p is only the pressure. I want you to keep this in mind because tomorrow when I am applying the normal stress boundary condition, I am going to be the same things about the pressures are equal, okay.

Then, I have to use the actual pressure not the modified pressure. So that is an important shuttle point which I have to keep in mind. So this is basically the form of Darcy's law. Now, I would suddenly put in k here and the delta y disappeared, okay this k is actually a property of the bed because depending upon, so the clearly the pressure drop and the velocity relationship is going to be decided by number 1 the property of the fluid.

And it turns out that the property of the property of fluid which is important its viscosity, okay because that is the one gives you the drag force along the walls. So that is being retained and this approximate thing where is delta 2/delta y square disappeared, I just put in my k here, there is something like a proportionality constant that is going to be decided by the material of the bed, is it sand, is it clay, is it glass fears.

So depending upon that you will actually get different proportionality constant. So, this k is obtained experimentally and is the permeability of the bed, okay. So, you will say that the bed is very permeable that means the voids spacing is very, very large, the liquid can flow through very easily. If the pore space is very, very low, then the permeability is low things like that, okay.

So that is the proportionality constant which is the property of the bed. Now, that is the simple, so one thing which I want you to be clear about is that the viscosity plays an important role, okay this is the viscous flow, viscosity is important. However, the fact that velocity is linearly related to the pressure, okay is symbolic.

So for example, many people write this as, I can write the same relationship as u equals – k/mu times gradient of pm and I am tried to follow these notations as much as possible, okay and this can be written as gradient of c and I am using +, here is of – that is this is remember some kind of a scalar, the gradient of that is basically my velocity, okay. I mean k divided by mu multiplied by pm is my c, okay.

So, this p is I am using the simple c because c is normally used to denote potential, okay. So, if you go back to a fluid mechanics what kind of flows are actually going to be given by a relationship of this kind where the velocities, a gradient of potential of course you have these c called potential flows and if you remember, something you must have studied somewhere that potential flows essentially arise when you have an inviscid when viscosity is not there, okay.

Normally, you associate potential flows with inviscid flow. So, the point I am trying to make here this is something like a potential flow because I can view the velocity as the gradient of the potential but actually I am not saying this inviscid and saying viscosity is present, so I mean I am trying to incorporate in Darcy’s law formulation, we are including the effect of the viscosity, okay.

Because that is how I have u equals -5/mu gradient of pm, but it is also similar in some sense to a potential flow and potential flow is normally associated with inviscid, okay and inviscid, I am not going to say inviscid when I am doing my viscous fingering problem, because we say inviscid, viscosity is 0, viscosity is 0, then I mean I am not able to be say you know when this is going to be fingering and that is going to be no fingering.

So, if viscosity = 0, I want to keep the effect on the viscosity, okay. So I am going to use the fact that something like a potential flow, but I am going to keep the effect of viscosity. So that is something like a vey shuttle point, which I wanted to emphasize here. So, there are many people who mistake potential flows to be flows which are inviscid, I am saying here we have a potential which is not inviscid, which is actually viscous.

In fact, the people who are actually distinguishing between potential flows which have the effect of viscosity included, okay. So for example, so to summarize what I am saying is Darcy's law has the effect of viscosity, okay, it can be viewed as a potential flow, but this is an example of a potential flow which is viscous. Normally, potential flows are synonymous to inviscid flows.

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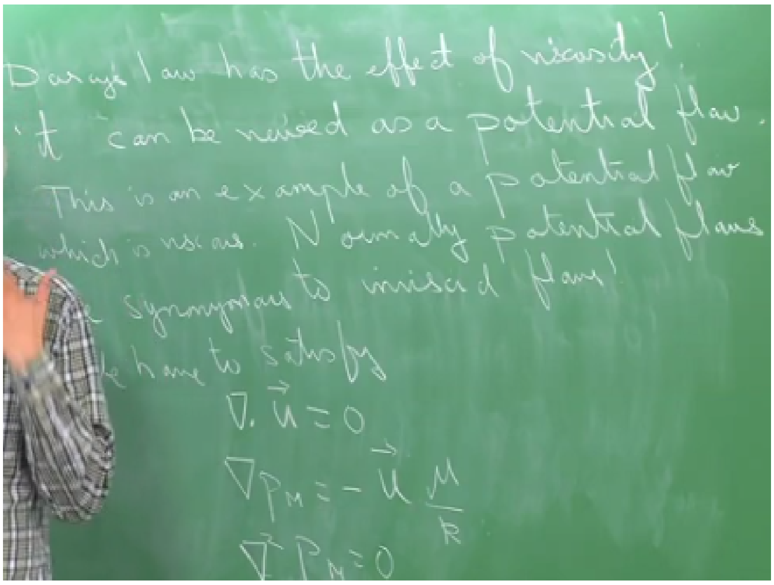

I mean that some kind of conception this people have, potential flow that means inviscid, it is not necessarily true. It is a potential flow, you can also have a viscosity effect, okay and there is something which is extremely important which I want you to understand. So now, life is a little bit more simple in the sense to just give the outline of what we are going to do, we have 2 equations have to be satisfied.

One is the continuity equation and the momentum equation, right. So, we have to satisfy the emergence of u = 0 that is the continuity equation and the momentum equation which is my Darcy's law, okay. My Darcy's law which is gradient of pm equals – u, I mean you can put this k and mu and all that, where do that come, multiplied by mu divided by k that is my Darcy's law in the general form, in the vectorial form, okay.

So, the simple thing to do is and this approach is going to use tomorrow. I am going to take the divergence of this equation, we take the divergence of this equation, I can divergence of u here, I can divergence of del p, which is del square p, okay = 0. So, basically what I am saying is the flow field satisfies del square pm = 0 and that is a Laplacian equation which you know everybody know how to solve.

Because you have done this course in calculus and you can do by separation of variables whatever, okay. So this is the linear equation which you can solve and that is one of the advantages of this potential flow, the relationship is linear, then you go back you solve for the pressure field. Once you know the pressure field, you can go and get the velocity field, you can put those boundary conditions.

The other important point is although this has the effect of viscosity in it, okay what is the order of the equation for the velocity in the Navier-Stokes equation. The regular Navier-Stokes equation is second order, okay, but when I am going to not do the Navier-Stokes equation, but I am going to use Darcy's law, I have the same problem in the sense that I do not have my second order term for my velocity, my second derivative term d square u/dx square.

Normally, I would have d square u/d y square in this case, I need to have 2 boundary conditions. I have only the first derivative term in the equation of continuity, okay. So, again I have the same problem as that of inviscid flow, inviscid flow what you do, you should look my equation which was second order the momentum equation has now become first order, so one boundary condition I have to let go off and the boundary condition which we let go off is, the tangential stress boundary condition.

So, here again we can see mathematically is a first order equation in velocity, again it is not a second order equation velocity. So, you have to let go off one boundary condition. Again, the boundary condition which you go let off is the tangential stress boundary condition. So, although you have viscosity present in it, you are going to let go off the tangential stress boundary.

Because you will not be able to use the extra boundary condition which you can solve for the constant, because the order of the equation is actually reduced. So that is again something which you have to keep in mind tomorrow when you are doing it, although ideally you need the tangential stress boundary condition, the normal stress boundary condition, we are going to let go off the tangential stress boundary condition because mathematically we do not require one boundary condition.

The normal stress boundary condition is important because the pressure difference is the one which is actually deriving the flow and I want to keep that guy and if I let go off that nothing is going to happen, okay. So, let us some small points here, we will do the actual derivation and get the condition for stability and try to understand things tomorrows, thanks.