Multiphase Flows: Analytical Solutions and Stability Analysis

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Lecture - 37
Flow in a circular curved channel: Solution by regular perturbation

So good afternoon, today we will be carrying on with the second half of our work on curved channel flow. So just to recap we are looking at single phase flow through a curved circular pipe and we have adopted a coordinate system where we have a polar coordinate system in the cross section.

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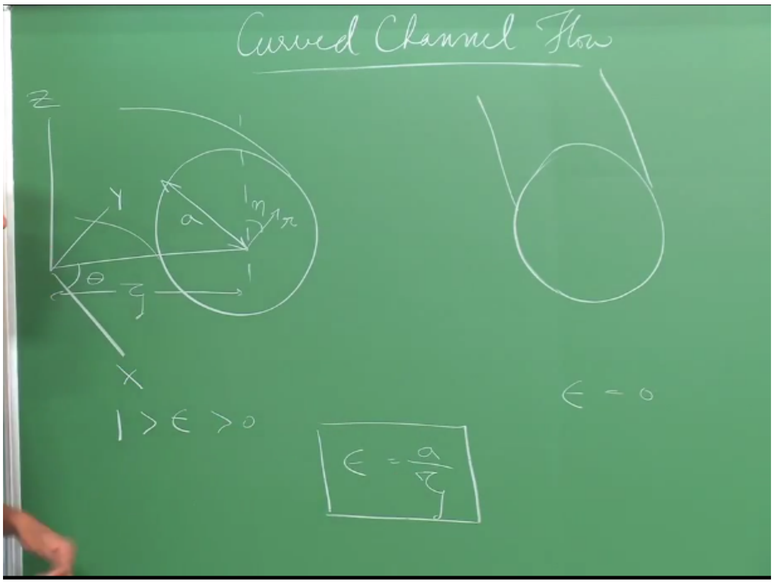

And the positive of the cross section is given by the angle theta from the x axis. So we quantified the curvature of the channel with the curvature ratio epsilon and that was given by the radius of the channel a/radius of curvature zeta and so we realize that if the quantity zeta tends to infinity then we will approach the case of a straight channel, which means that when epsilon is 0, curvature ratio is 0, I will have a straight channel.

And when epsilon > 0 but of course < 1, I will have a curved channel. So this one bound is physical characteristic of the geometry. This radius cannot be more than this radius. So that is the general idea. So realizing this fact we wanted to apply what we have studied in the course basically perturbation theory and try and understand flow in a curve channel as a small approximation of perturbation about the case of epsilon=0 which means about a straight channel.

So yesterday we had derived the Navier-Stokes equation in the toroidal coordinate system, scale the equations and got the governing equations in terms of epsilon. So today what we are going to do is carry or forward the usual procedure whereby we have taken asymptotic expansion in epsilon substituted back in and derived the equations for the 0th order problem, first order problem and so on.

So before I go to that I want to make a point here that I have said the straight channel case we approach when zeta tends to infinity but even for a finite zeta for a reasonable zeta you could still approach in the case of a straight channel if the radius became very small. So why is that important?

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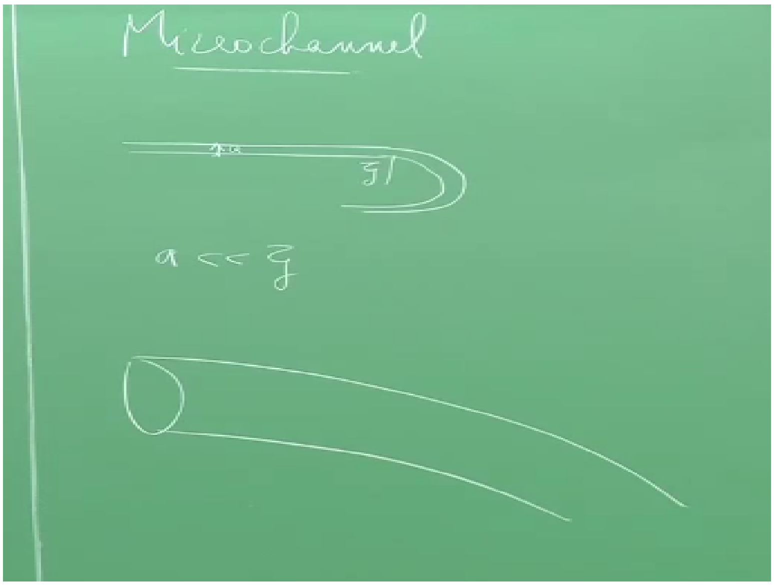

You imagine you have a microchannel, which has a bend like this. Now the curvature ratio of this bend zeta is definitely not infinity, it is a very finite value, it is quite a sharp bend in fact, but you could still approach the limit of epsilon tending to 0 provided the a is small. So even though one way of looking at the problem is a gently curved channel, one way of looking at small epsilon.

Another way is a finite curve curvature but a very small channel or a narrow gap approximation. So this where scaling comes into picture. So even though epsilon is not absolutely tending to infinity if it is relatively large compared to a that is good enough because a length scale in the problem is a so with respect to a zeta should be large.

So this is once again where the question is what is the physical scale that is important that is what we do and we scale the problem essentially. So that is what by scaling with a throughout the problem we got epsilon. So you put epsilon to 0, it could be in either that a is very small or that zeta is large and that does not really matter as far as the equations are concerned. So I could basically have a thin channel with sharp curvature or a large channel with a very gentle curvature.

And both these guys will have the same epsilon kind of geometric similarity. So with this in mind what I will do now is directly move on with the perturbation calculation.

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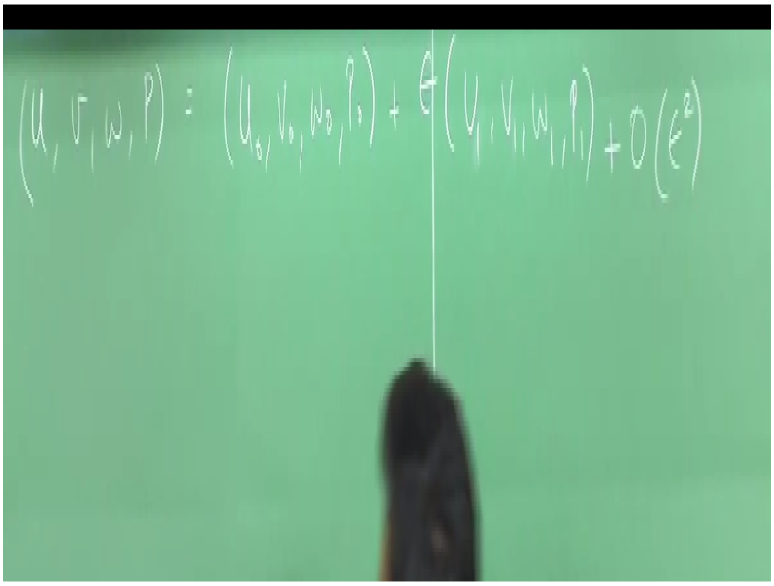

So let us consider order of epsilon 0 and in the previous class we have written down the equations the full governing nonlinear equations. So the first step of course is to consider the solution for u, v, w, p. So that is our standard power series expansion where we have expanded the vector field as the 0th order field+epsilon times first order variables.

So we take the asymptotic expansion and substitute back in those governing equations that we had yesterday and we will arrive at the equations for various orders. So I will start with the 0th order problem and that we can when you do the calculations to the equations and even by inspection you will get simply the equations for the straight channel and that simply Hagen-Poiseuille flow.

So what we will do is I will now write down the full set of equations that you can derive but the main idea is in those equations you will have all the inertial terms and the viscous terms but that is when we say that we are interested in looking at unidirectional flow through the channel as the base case so in that case all the inertial terms will get knocked off and we will be left with the normal Stokes equations that we get when we solve Hagen-Poiseuille flow.

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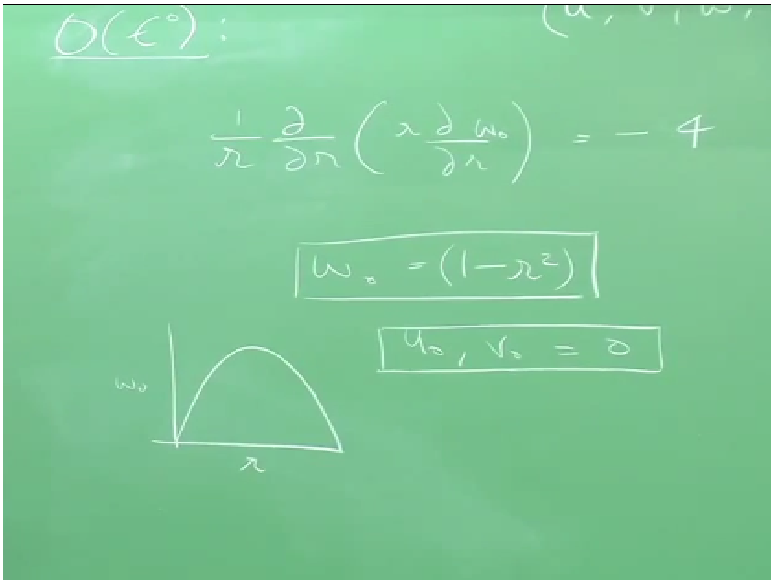

So the equation at order epsilon will simply be that is because you remember we scaled the equation with g so we would have got a -4 over there and only the radial variation of w0 persists. So u and v have gone to 0 because we were saying that the flow is unidirectional. So this is the standard Hagen-Poiseuille approximation. So finally we will get the solution at 0th order to be 1-r square.

Actually, this is -4 because delta p and del square v are in the same side, taken to the other side is -4. So that is the standard parabolic velocity profile if you plotted that we get in Hagen-Poiseuille flow through a straight pipe. Most importantly u0 and v0 are both 0, which means that there is no cross velocity in the pipe at all. There are no circulations, nothing is happening, it is just flowing straight.

So that is the straight pipe solution. Now what you are going to do is when we go to order epsilon 1, this axial flow w0 is actually going to move round a bend so w0 will give rise to centrifugal forces that will come into the problem at order epsilon 1. So this is the typical stepwise procedure in perturbation calculations. The 0th order effect is just flow and at first order we will start feeling the effects of centrifugal forces.

So once again if we take the asymptotic expansion and go to order epsilon 1, we can derive the equations. So I will directly write those down and I leave it as an exercise for you to derive them, which straightforward the usual procedure that we have done so far.

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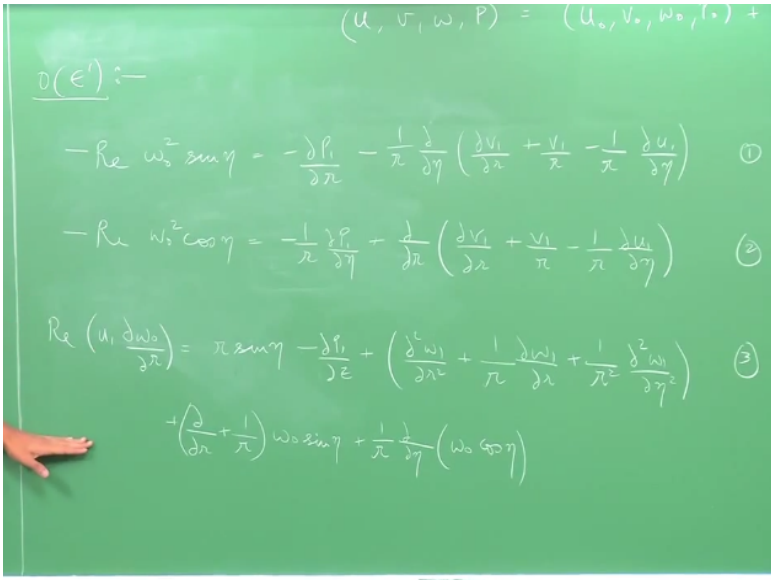

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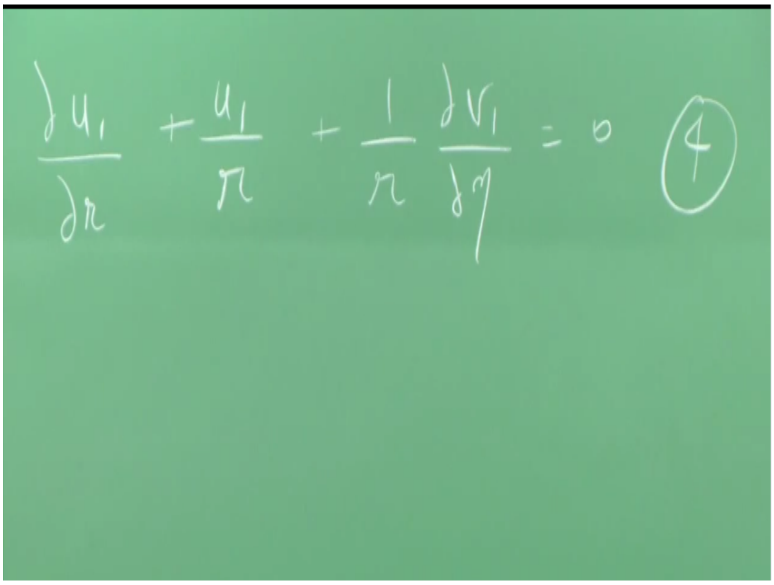

So at order epsilon 1, so while I am writing this down you should look at the equations yesterday and see if you can identify where each term is coming from, from the original nonlinear equations. So for example this term is coming from the centrifugal force term in the governing equations and these are the viscous terms, which are order epsilon. So just look and see if you can identify where it is coming from.

So these are the 4 equations that we get at order epsilon 1, so you have the momentum equations in u in the radial direction in the eta direction in the axial direction and the continuity equation, which is equation number 4. So there are some interesting things that we need to see in these equations. The first point is that centrifugal force is an inertial term, it is basically a nonlinear term in the Navier-Stokes equations.

So when you go back you will see that you had w square you know in the governing equations. So that term actually is nonlinear and if you want to solve it you would have to solve the problem numerically but what we have done here is by doing the perturbation approximation, we find that that term which was nonlinear before has come into the problem now as a linear term.

Because although we have w0 square, w0 is already known from the base problem as 1-r square. So if you look at this problem, which is in terms of the variables u1, v1, w1 and p1 you will find that it is completely linear set of equations firstly. Secondly, it is nonhomogeneous with the nonhomogeneity being the centrifugal force term, which has w0 square.

So this thing is typical of perturbation calculations in general and you can think of it as a weakly nonlinear calculation. So the purely linear calculation would have never given us circulations so that would be equivalent to putting Reynolds number=0 right. That is the creeping solutions or Stokes solutions just puts Re=0 so you neglect all the inertia. If we did that even at first order, we would have got 0 and 0 here.

And the whole set of equations would be completely homogenous, which means that even u1, v1, w1, would have been 0. So I would have just persisted with you know flow going round a bend but without any circulations, if you take Re actually to be 0, but if you allow for nonzero Re but say that wait even though there is some Reynolds numbers effects that those effects are very weak.

We have said that by saying small epsilon. So when we did that you do retain Re and you retain its contribution but you are able to do that in a way that allows you to calculate step by step analytically. So in that sense you are accounting for the first effects of the nonlinear centrifugal force terms. So these are those 2 terms are basically inhomogeneities now in this set. This is the equation for the correction to the axial velocity.

So w0 is what it is in the straight channel when it curves, it should be corrected by w1. So we actually find this is an equation for w1 and that of course is the continuity equation. So this third important thing to notice that if you consider equation 1, 2 and 4, you will see that nowhere do they contain w1 right. So these 3 equations the momentum equations for u and v, u1, v1 and the continuity equations are a self-consistent set for you know the 3 variables u1, v1, p1 and you have 3 governing equations.

So we can actually solve 1, 2, and 4 first and then come here and solve equation 3 for w1. **"Professor - student conversation starts."** Oh sorry yeah, yeah, it is a good question.

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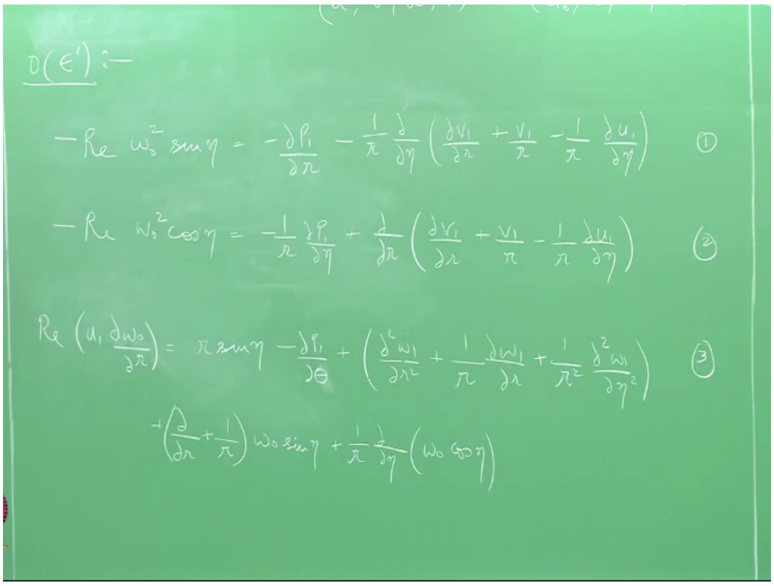

I will come to this term actually, you will actually end up retaining yeah that is it, but I will come to that later. **"Professor - student conversation ends."** So this is the change in the pressure gradient in the theta direction. Alright so whatever is saying previously is that we have 3 equations for 3 unknowns that we can solve first for u1, v1, p1 and then we can come and solve for w1.

Is that clear? So this will always happen when you have fully developed flow because in fully developed flow you would not have changes in w1, changes in w with theta in the continuity equation. So the continuity equation will only involve u and v, so that along with these 2 equations allows me to solve for u, v first and then I can solve for w1, which is the correction to the axial velocity.

So once again we come back having the whole methodology of perturbation series so what we have done here if you look at it physically is to say that I have a flow the primarily flow is axial just going through the channel. So that you first calculate that is w0 then using w0 you get the centrifugal force that is w0 square and use that to calculate the circulations which are u1 and v1 that is coming from the centrifugal force w0 square.

See you will get u1, v1 which will be some circulations and some pressure gradients p1 and that you use to calculate now the change on w itself which is w1 and if we keep going forward we will keep doing this. So first calculating the circulation then coming back and seeing its effect on axial velocity. Then again going back and calculating the circulations and so on.

So this iterative procedure I mean makes physical sense because you get the base flow, find the centrifugal force, find the circulations, come back to the base flow and by doing that we will increasingly improve our approximation. So today we will just look at order epsilon which means you just solve these equations up to w1. Alright so now that we have got the equations and understood what we were about.

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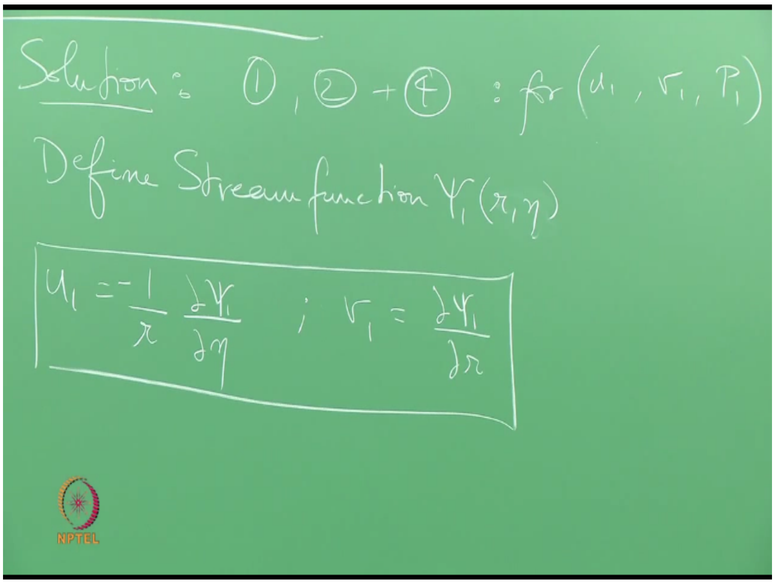

We will get down to the mathematical solution. So as I said we can solve 1, 2 and 4 for u1, v1 and p1. So how do we go about this? You can see that the continuity equation actually has the form of a 2-dimensional system because there is no w1. So we can use the stream function formulation that we have done previously for 2-D flows. So stream function formulation works not only for 2-D flows but also when you have fully developed flow.

Because the form of the continuity equation allows me to do that. So if I define I will call it psi1 becomes it is at order 1, oh sorry write r, eta and this will be identical to the stream function formulation we have in cylindrical coordinates. So in fact okay so the idea is we have replaced two of the variables in terms of one variable psi1. So now that we have this what is the next step that we do when we are looking at stream functions?

Right we have to eliminate pressure from the equations which amounts to taking the curl of the vectorial form so what we need to do is differentiate, we want to eliminate these 2 terms right. So we will differentiate this with respect to r and differentiate this with respect to eta*1/r and subtract right. So that will knock off these 2 terms.

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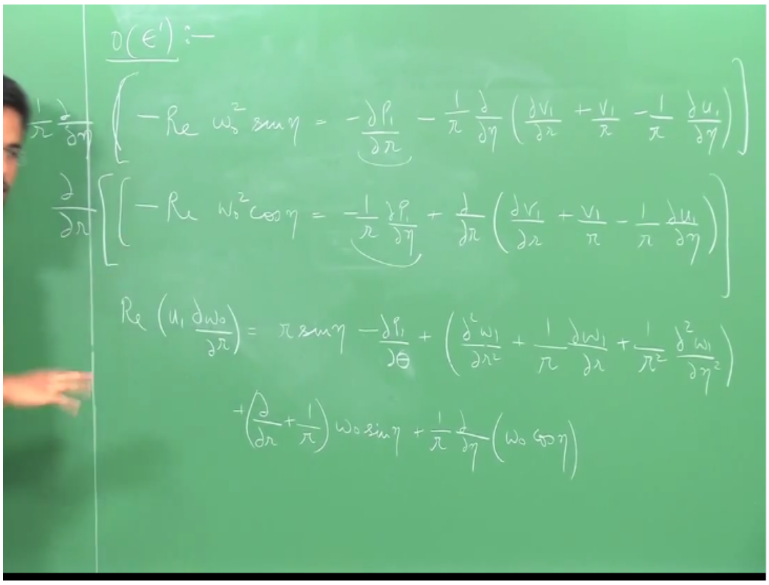

The other important thing is remembered this equation we will operate on the whole equation right and on this guy we will do the derivative with r. The important thing to realize is that when we take the derivative with eta will get cos eta here whereas derivative with r will leave cos eta free. So when we differentiate and add you will get cos eta together. So you will not have sin and cos, both the forms will be cos only. So that will tell us ultimately a stream function psi1 will be of this form.

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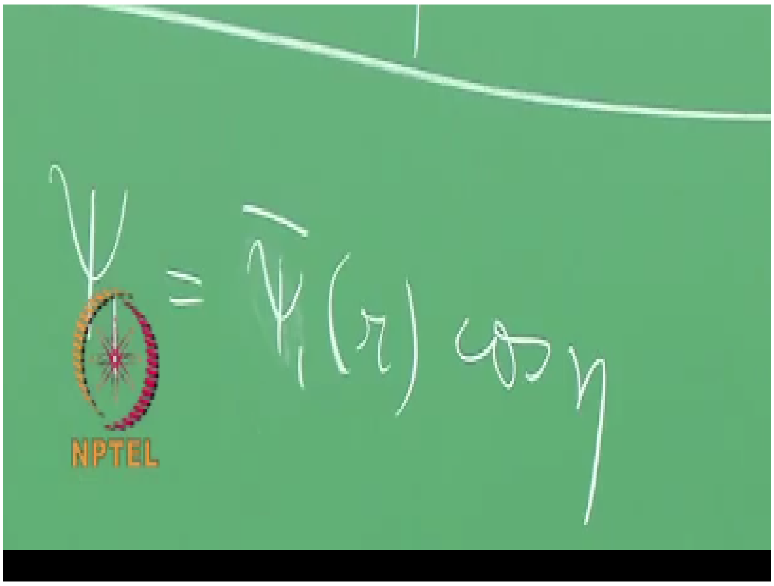

You can split it up into its r dependence and the cos eta dependence because both the inhomogenous terms will end up having cos eta right and cos is an even function about eta. So that means that my stream lines will be symmetric about the left half and the right half right. See eta=0 is here, so if you have cos eta, -eta and +eta will be equal for cos. So the stream functions or whatever the circulations we have will be symmetric about the vertical plane.

So that is something to keep in mind but just proceed with the calculation. Differentiate this with eta and this with respect to r and then subtract the 2 equations. **"Professor - student conversation starts."** No, I am differentiating this with 1/r, okay fine yeah, yeah, that will be better, correct.

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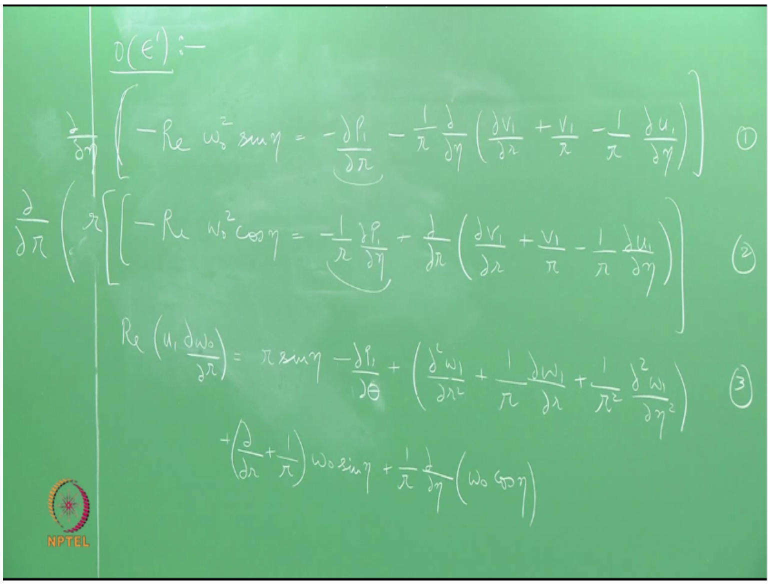

So you first multiply by r and then differentiate with r. **"Professor - student conversation ends."**

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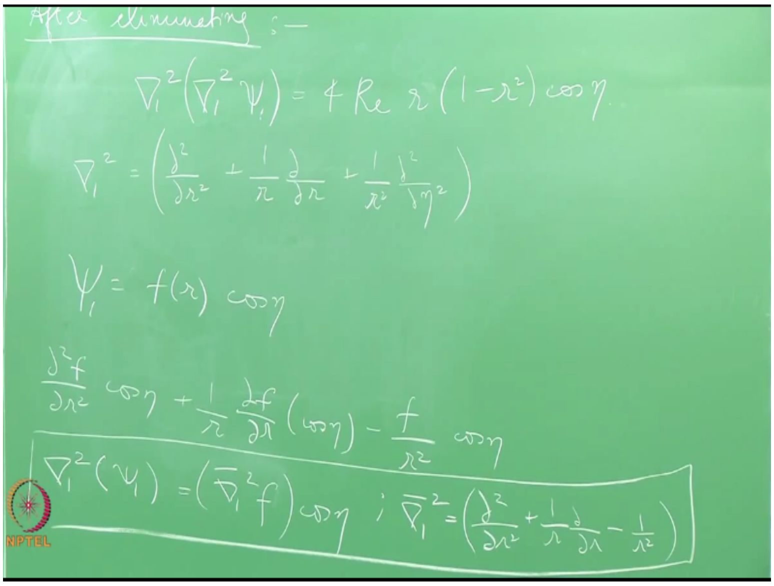

Alright so after you eliminate these 2 equations if you have not been able to get it so far you can go back and work it out, but it will work itself out. You will land up with an equation for psi1, so the most important thing to check is since these equations were second order, naturally the equation for psi should be fourth order because we have combined 2 equations into 1, so you can see that it is in fact the fourth order problem.

So you have del square operating on del square psi where del square is given by this operator. It is just the del square and cylindrical coordinates and the right hand side contains of inhomogeneity, which will have Reynolds number because that gives me the amount of centrifugal force and you have this cos eta dependency like I pointed out earlier because sin eta gets differentiated to cos eta.

So now whenever you have these types of PDEs you need to see whether you can separate the r dependency and the eta dependency. So in this case if the right hand side the inhomogenous term was 0 then psi would be simply 0 because you just have no-slip homogenous conditions on the wall, but v is not 0 so that is why you will have some circulation but the eta part is everything in cos eta.

So it makes sense to propose a solution where you have psi1=let me say some function of r which we do not know times cos eta because if we consider this form here the cos eta dependency would get canceled out so we just check that now whether if we can put this solution back here whether the cos eta parts go off then we will get an ODE for r. So we can just try that off.

Easy ways to consider what happens to del1 square because whatever happens here will get repeated again right in the other operator. So if you put this back here in del 1 square what will you get? We operate this on psi1 right, you are going to operate this on psi1 that is in the operation. So f of r will be retained here and cos eta will just pop up right+1/r dou/dou r operated on this term will retain the f inside the operator and again cos eta will come up.

And the final term; however, you are differentiating with respect to eta twice right. So first derivative with cos eta will give me sin eta with –sign, second derivative will give me back cos. So I simply get –f/r square cos eta right. So this whole thing del1 square of psi1 can be written as del1 square bar which is a different operator acting on f*cos eta okay where del 1 bar operator is dou square/dou r square+1/r dou/dou r-1/r square.

So I hope that is clear, you will need also about this, so the cos eta just pops out.

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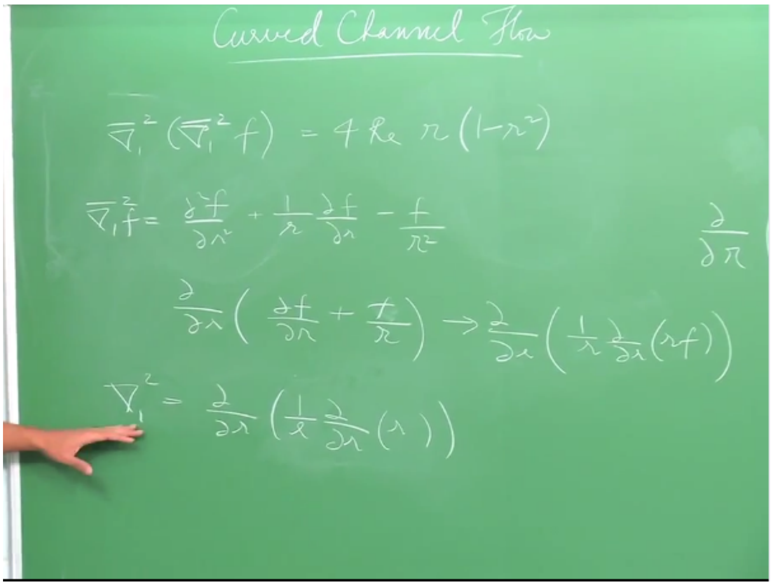

So if we substitute this solution here what will end up happening is you will get del 1 bar square because the cos terms will get canceled off from either side alright. So what we have ended up with is a fourth order ODE for r. So at this point, this ODE may still look a little intimidating but actually it is very easy to solve. So that will come by simply realizing that we can rewrite the operator in a more compact way that allows us to integrate it four times and get the solution.

So how can we do that? If you look at this expression, it can be written as dou/dou r of something. So what is that of? This will give us a first term right+u/r right. So this will give me the first term and if I differentiate taking r out I will get this term, when if I differentiate r I will get -1/r square times f. Now this itself can be written as another nested derivative I guess that is right. So now we have written del 1 square in a nested form.

Yeah **"Professor - student conversation starts."** Okay, okay, fine, correct. **"Professor - student conversation ends."** So finally the operator is written in the nested form. So now you can see that if you want to solve del 1 square=an inhomogenous term. All we need to do is integrate with respect to r right, multiply by r integrate again and then divide by r. So now it is really easy to solve this problem. Is it clear now?

I mean once you got this you have this here so to remove the derivative you just integrate both sides. If you integrate both sides with respect to r this guy will go off, then multiply with r, then integrate again, then divide by r that will leave you with del 1 square=all those operations on this side.

Then you again repeat that 4 more times you will get f1. So f1 will be obtained finally after 4 integrals with respect to r and then multiplying dividing by r as appropriate. So now the problem has been simplified I mean it is such an easy solution that you can get immediately. So you can work that out.

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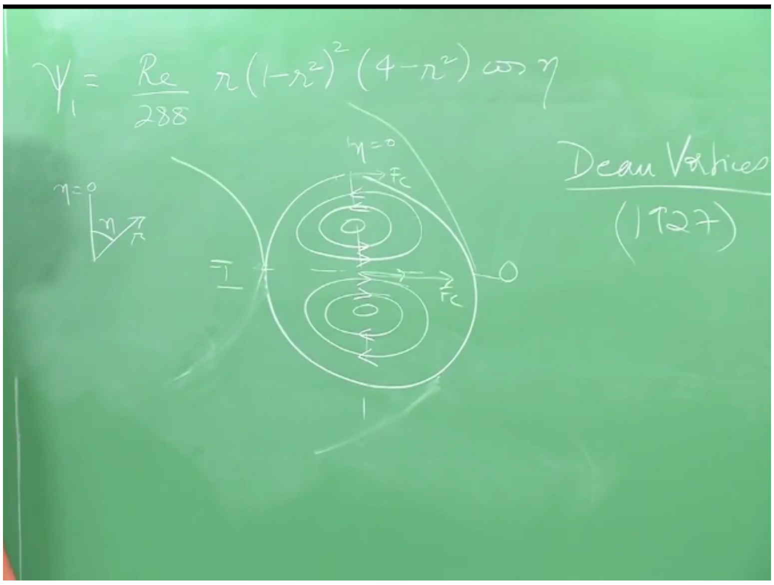

So finally after integrating and then applying the boundary conditions we will get the following solution. So this is what f1 looks like you all should go back and do the calculations for practice. Alright so now let us pay attention to what f1 is? So f1 if you remember sorry this is not f1, this is psi1 the stream function. So now if you remember fluid mechanics theory and stream function, the contours of the stream function will give us the stream lines in the steady state situation.

So like I said again the eta dependency is cos eta that means it to be symmetric about the center line. So if we look at just the cross section forgetting about w we just have the cross section with the radial coordinate r and eta. So if you remember eta goes like this, this is eta=0 right and the radial distance is r and if you plot these contours in Matlab or Mathematica, you will get something that looks like this.

And they will repeat at the bottom. So because of cos eta it is symmetric about the mid plane while the value of function is not symmetric on this side, it is however the contours have the same shape and from here you can get the velocity feel by applying the stream function formulation. So u is -1/r dou psi1/dou eta. So what is this telling us now? If you look at the u, you will find that the stream functions give us the circulatory flow.

So what we have seen now to order epsilon 1 is that the flow through this curve channel, the axial flow creates the centrifugal force that results in secondary flow imposed on that which has these circulations. So this is how the channel curves. So right so this is the inside of the channel and this is the outside. So fluid is pushed along the center right and once it hits this wall, it recirculates back.

And the whole reason why we have these circulations at all is remember that the centrifugal force here Fc and this centrifugal force is much larger because here w0 square was much higher, here w0 square is small because it is near the wall so the push in this direction is much stronger than the push so it is basically like as the fluid gets pushed here and gets pulled back so you get circulations that is because it is finite here.

If you had the infinite case, you would not get the circulations at all because the force everywhere would be the same. So that we will look at a later class but this is essentially the problem and if you combine this u1, v1 and added with w0, you will get a fluid particle moving through the channel and also drifting along these stream lines.

So you will get helical vortices. So as it moves to the channel it will do a helix and these 2 vortices are called Dean Vortices and is because of these guys that you have enhanced mixing in curve channels. So his first paper was in 1927 when he applied the same procedure same perturbation calculation for flow and he calculated these symmetric stream lines and that is the first theoretical work that was done on the problem.

So I will not calculate to w1 you can do that as an exercise and the entire thing is given in Gary Leal. So you can work it out and maybe hunt down his paper, he has got some interesting results that he has and this started off a huge analysis into single phase flow in curved channels.

Some of you all must have come for my seminar so that is when I did the same calculation for the 2 phase flow where instead of just having a single phase in the whole domain you have a core fluid, which is phase 1 and surrounded by the annulus of phase 2 and then you do the perturbation calculation. So now essential to the governing equations I mean in addition to the governing equations you have the normal stress condition, tangential stress condition and all of those being perturb.

So the similar idea can be used in so many different problems and so many variations of this are there. In fact, on (()) (38:52) in fact how far you can take perturbation. So there is a paper by Peterson a recent one I think in the maybe 2007 or 2010.

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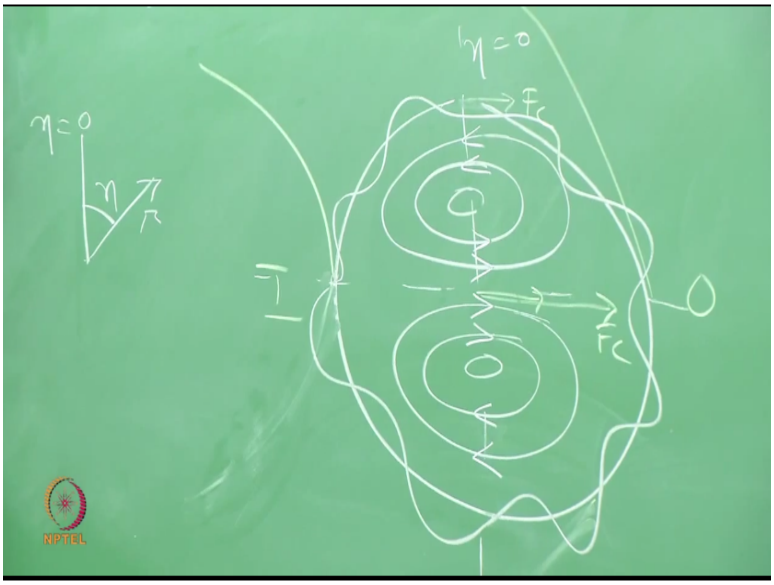

What he does is he looks at against single phase flow in a curved channel but now he makes these walls wavy which will remind you of your quiz 2. So in the second quiz we gave state flow through a channel which had waviness. Here he took the waviness and the curve, so he has 2 perturbations parameters, one in the curvature and one in the wavy wall and that is the physics of fluids Peterson, you can check it off.

So then he does it is a double perturbation. So first you perturb to order epsilon square he went in epsilon which in the curvature and then each of those guys have perturbed in an 2-dimension perturbation series.

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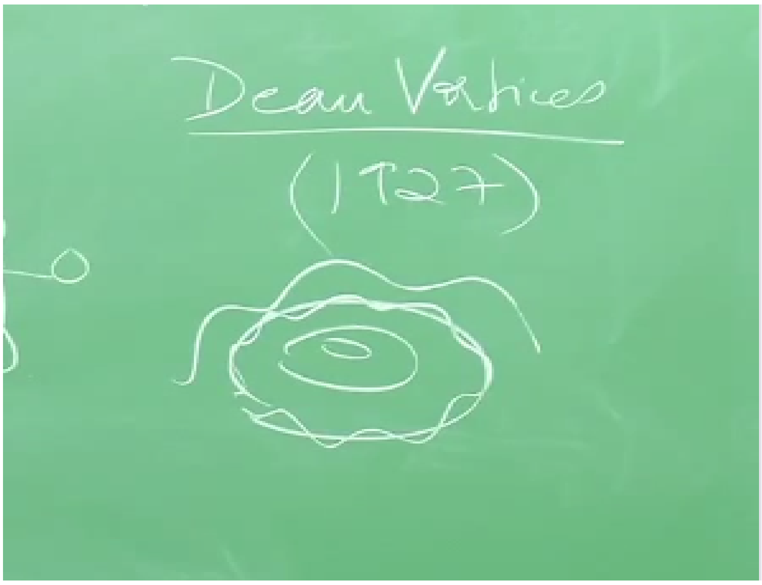

So then he has a solution of these vortices then become if you look at the stream lines the stream lines also become wavy and it is quite interesting. So in this sense you can keep going on adding different effects. Alright, so I think with that we will wrap up our class on Dean Vortices.

And in the next class we will look at the instability problem which is also studied by Dean in 1928 where there he looks at you know 2 infinite walls where circulations do not come immediately but circulations will set in as an instability so there is a subtle difference there and there we can apply our linear stability analysis