Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 36
Flow in a circular curved channel: Governing equations and scaling

So, good morning and in today's lecture, we will be looking at flow in curved channels, so the whole problem of fluid flowing through a curved channel is one that is very interesting firstly for theoretical reasons, which we will be covering today and secondly because of the wide range of applications that the flow has in industry and even in biology. So for example, if you start with the Natural Sciences first, blood flow through a body happens through a lot of curve channels especially in the arteries.

And in many cases, it is creeping type flow and that is one motivation initially where people started looking at flow in curved channels. Then, practically when you look at industries, you have so many curved channels in heat exchangers and mass transfer equipment and just for that, you can enhance mass transfer and why that happens also we will see, you know in today's class. The other aspect is that very often you just do not have space to have straight channels all the time.

So, then you need to have curves again. So, to understand what goes on in curved channels, what are the new features of the flow, once again we can apply perturbation techniques to see how the curvature comes in, how the centrifugal force comes in and what effect has in the flow.
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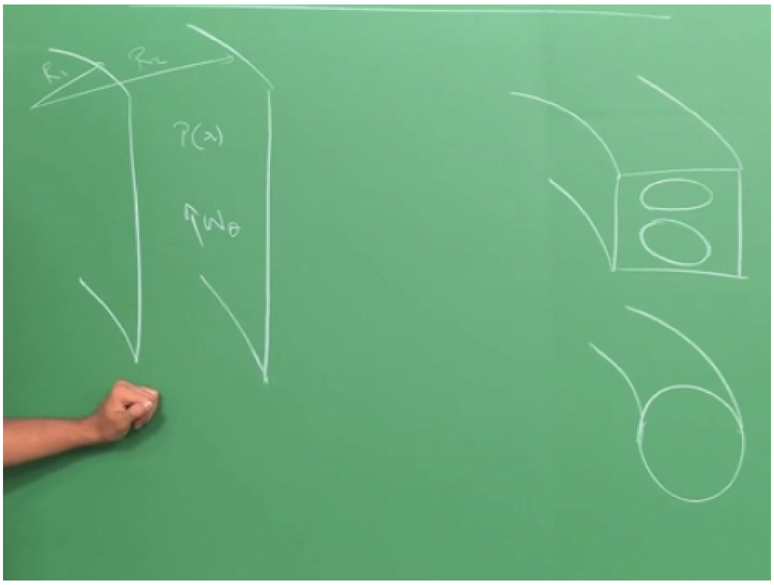

And naturally, once again we can look at stability issues, so in this class, I will give an introduction about flow in curved channels and then we will work on one of the problems. So, right off, there are 2 cases we can consider; one is this simple case of flow inside a annulus, so you have 2 cylinders of radius, let us say R1 and R2 and the fluid is flowing into the plane.

So, this is one situation where basically the walls of the channel are infinite in the z direction and fluid is going around, so that is also a type of curved channel. In fact, it is one of the first curved channels that was studied because of its simplicity, it is something like the stratified flow problem, you have this for curve channel, so that is one problem. The finite analogue of this would be actually flow in a square duct, right.

Equivalently for a slightly different cross section, we could have; I mean a curved pipe that is not; should not be too very different. So, in most of our analysis, in the past we have seen that when you look at infinite cases like the stratified flow problem, the results we got for a semi-infinite case and the finite case were very different, we just had to account for wall effects.

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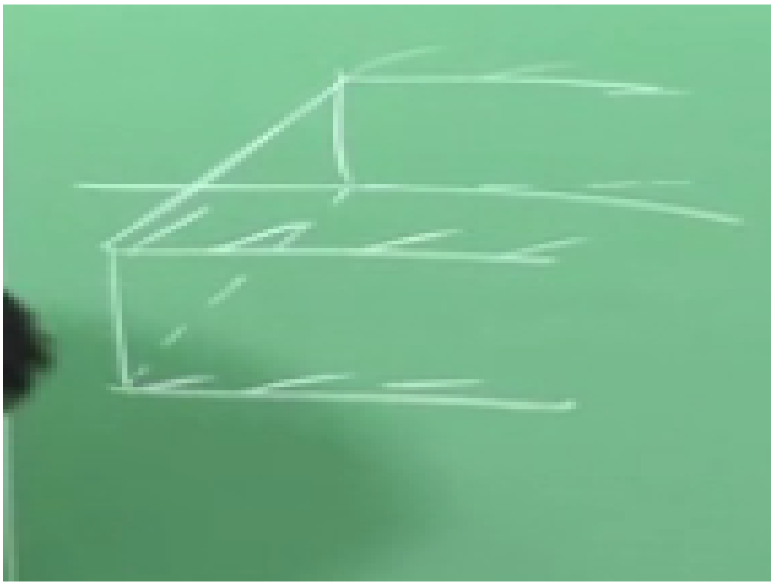

So, for example in the stratified flow problem, you have 2 flat plate, basically this was trying to mimic this situation, so when this dimension went to infinity, we have the stratified flow problem and if it is finite; it is a finite box, then we would have to solve it in a square duct but I showed you in the earlier presentation that the solutions that we got with the infinite plates and that with the square duct were quite similar.

Especially, if you were to cut through the centre, so that is the same reason why people thought of studying this and that problem but as you would know now, if we solved this problem and most of us have already done this in earlier courses in fluid mechanics, you would get a solution where there is nothing very different, you will have a velocity in the theta direction but there will be no other components of velocity in the base solution if you see.

And the pressure will vary as a function of R that is the classic Taylor cued base state, where you have purely azimuthal flow and that is it, just unidirectional azimuthal flow and you can solve that almost straightforward implicit directly, so, there is nothing very fancy about that On the other hand, if you come here to the finite wall case, you will find that and that is what we look at today that there is no base state that is unidirectional, when the duct is finite.

You see here, there is a unidirectional base state with flow only in theta but in this case, you will immediately get circulations just because the duct is finite, so there is a big qualitative change when we go from the infinite walls case, for the case where we have a finite height duct and that is not something you have come across so far. So, the story of these 2 channels just tells us that we need to be very careful, whenever we go to a limit.

So, actually what it is; you can think of a very tall duct not infinite but very tall, then this would be a kind of limiting perturbation of a very tall duct or a duct with very high aspect ratio. So, what happens is when we take that aspect ratio to infinity, you have a singular perturbation type solution, a type problem where the quality behaviour changes. So, here there is absolutely no circulations in the base state, whereas here there will be finite circulation simply because of the finite height.

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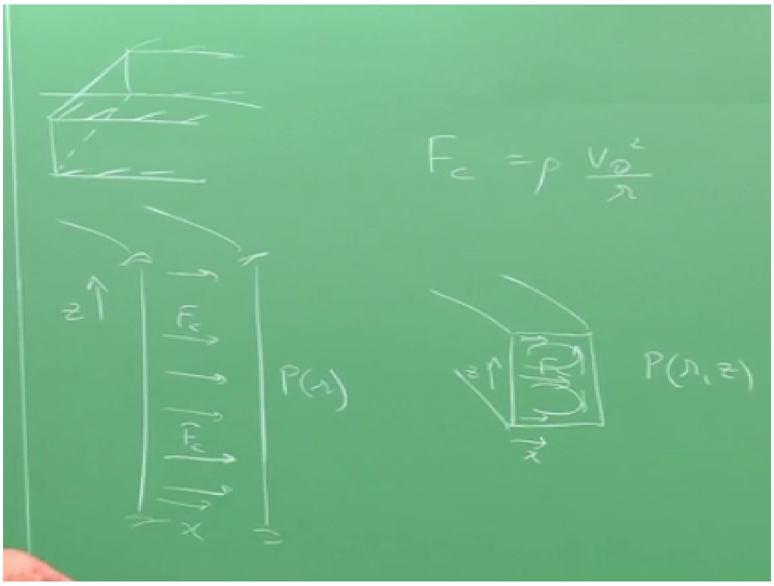

Now, why does that happen before we look at the math, we can look at the physics and that will help us in what we are going to do today. So, in this the infinite duct case and the finite duct case, so what happens is in both situations when the fluid is flowing around the curve, it experiences a centrifugal force, so there is centrifugal force pushing the fluid outside, right, so centrifugal force.

I have the same thing happening here, there is a same centrifugal force but now heuristically speaking after the centrifugal forces v theta square over r, of course you will have density, so if we have uniform v theta by some reason, then the centrifugal force the fluid experiences is goes as v theta square/r. Now, what happens is that in this situation, you can conceive of a v theta that is uniform in the vertical direction.

So, let me call that in this case z, I will do the same thing here, okay, so just take this as the x direction for now. So, now in the z direction in this case you can; because it is infinite, it is either going to be periodic in z or I is going to be independent of z, so the base state we say v theta is you know independent of z not a function of z. So, in that case the centrifugal force which goes as v theta square/ r is going to be independent of the z direction. which means that the fluid gets pushed you know uniformly throughout z.

So, if I look at how does this force get balanced, it gets balanced by the pressure and that is what you will see if you solve the problem that the centrifugal force is pushing the fluid out because there is a wall here, it gets balanced by pressure, so pressure becomes a function of r; so P is a function of r but it is not a function of z but in this case, v theta will have a value at the center of the channel but at the walls, it has to go to 0.

So, here because you have bounding walls, the v theta cannot be independent of z, v theta will be a function of z, so my centrifugal force is going to become less at the wall, while it is greater at the center. So and because of that the pressure that is trying to balance it will be a function of r and it will be a function of z also. So, because it is a function of z immediately, it will set up a vertical flow and I will have circulation but here there we know vertical flow setup.

Because P is not a function of z, so qualitatively thinking that is the basic reason and mathematically you will see it worked out that if I take this problem and try to obtain this simpler steady state and I put these assumptions in; I find that I have a unidirectional flow in the theta direction, whereas that set of assumptions will not give me a solution and immediately you will know that there is no unidirectional solution.

The base solution itself has all 3 components of velocity and so today, we will be focusing on this problem, where for the first time our base state itself is quite complicated. In this case, the base state is unidirectional but then as you can imagine that as you keep increasing the velocity, the centrifugal force increases, ultimately the flow becomes unstable and you will have circulations even in this case.

And that is the famous Taylor cued problem, where as you keep increasing the centrifugal force will get different fluid patterns and different vertices, so that is a stability problem here. Whereas, this is not a stability problem because the flow is always circulated, so we will focus on that. So, in in this class, we look at the circulations that are set up naturally inside a square duct at arbitrarily small centrifugal forces.

And in the future class, we look at the flow to the infinite duct and see how it; I mean how you get the base state, which has some circulations and how it becomes unstable. So, there are 2 parts of the same coin, in a later stage we can come back and see what we learn from it, all right.

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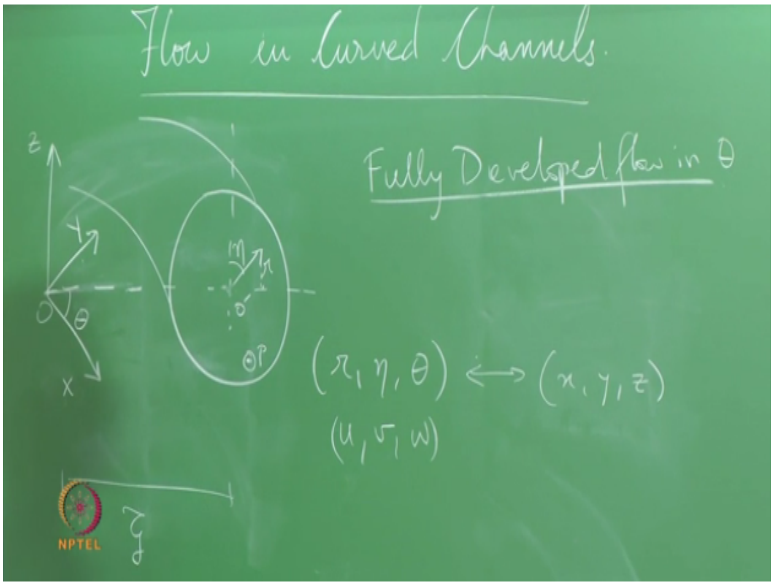

So, let us focus on; for simplicity we look at the circular problem, all right. So, first things; first we need to write down the governing equations, the full nonlinear set. So, before we can do that also, we need an appropriate coordinate system. So, what we can do is; if this is the origin and you have the x, y, z Cartesian axis, then we can set up a new system, which has an translating origin, o prime and this distance from zeta is actually the radius of curvature.

So, it is a radius from this origin to the center line of the curve channel, so within this thing, we can mount a polar coordinate system, so you have r here and I will call the angle from the vertical eta, all right. So, r, eta gives me any point let us look at this point P, so the position of P inside the cross section is given by polar coordinates r, eta and then the position of the cross section itself comes from; yeah, come from theta.

So, if I take x, I measure theta from the x axis, y does not have to be here also, this is just; yeah, this dashed line is just the line vector joining the center of the cross section of interest to the origin and the y could be here also, okay. So, theta measures the location of the cross section, so as I move through the curve channel, my theta keeps changing and inside any particular cross sectional slice, the point P is given by r and eta.

So, totally I have a coordinate system, which has r, eta, theta and this coordinate system can be related back to the x, y, z coordinate system by a mapping, okay. So, with this coordinate system in mind, we can now look at writing the Navier stokes in r eta theta. So, how do you go about doing this? We have seen in an earlier lecture, how we can write down the Navier stokes equations from a general vectorial form in any coordinate system using the vector identities and vector methods.

So, we will have to; we will do just that not in the class right now, directly begin with the equations but you can go back and see how it is done like you are done before for cylindrical and spherical. Once you find the mapping from x, y, z, r, eta, theta, you can go apply those methods and find the gradient the divergence and so on and get the Navier stokes equations.

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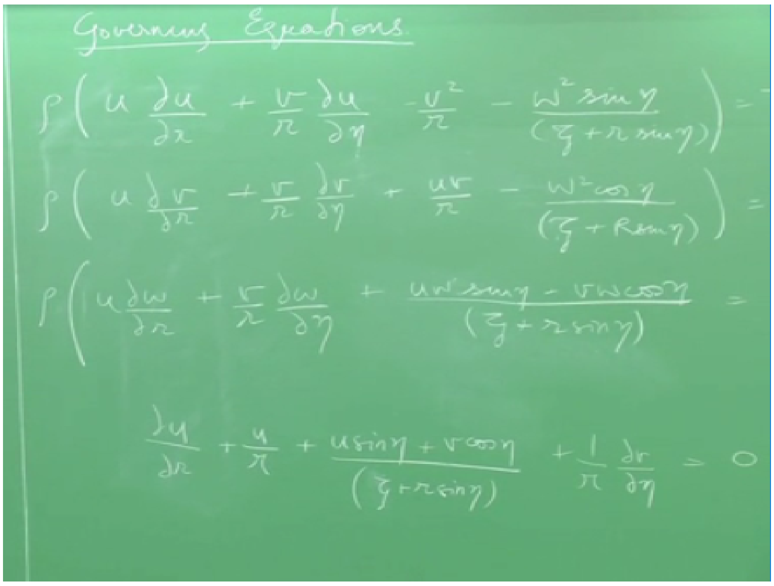

So, the velocity components are u, v and w, in the r eta theta directions and the equations as you see will be; will bear some similarity to the cylindrical coordinate equations naturally because the coordinate system is similar, so that is the equation in the new direction. See, one thing to point out immediately is, it is very similar to cylindrical coordinates, so when you look at the any component or any derivative in the eta direction, I will get the 1/ r factor that is the length factor.

When I look at derivatives in theta or a quantity to do with w, you will find the scale factor zeta + r sin eta. The reason for that is that zeta = r sin eta is actually the is this radius scale factor, this is zeta + r sin eta will give me the distance along the vertical right, r cos eta gives me the vertical, r sin eta gives me the horizontal in the plane, so zeta + r sin eta is this total projected distance to any point that I am looking at.

So, that is the denominator scale factor that will come throughout the equations. So, this part is like exactly like cylindrical, this comes from like v theta square/ r sort of thing but here v theta is just w, so it is like w square/r but then the r zeta + r sin theta, so we will go to the next, so those are 3 momentum equations, what else do I have to write; the continuity equation. Those are the equations in the toroidal coordinate system and all their glory.

Of course, these are the okay; so one thing I have to say is that today we look at only fully developed flow, so that means that I have not considered any derivatives in the theta direction except for pressure. These are equations for steady fully developed flow, if you do not look at fully develop, you have additional terms that take care of the derivatives of velocity with theta all right, so if you will have written them down, we will proceed, okay.

So, the one thing I have to point out here is that this term is the analogue of that v theta square/ r term I was talking about so that is one of the main centrifugal force terms that sort of drives momentum from the theta direction velocity, it creates the force that will drive my v and u components of velocity because this is the u component of velocity is momentum equation in the r direction; in the eta direction.

So, both these directions; the force acceleration; centrifugal acceleration is coming from here; centripetal acceleration, so that is one thing, second thing is that we see this guy gets multiplied by sine eta, whereas here you have cos eta, so it is almost like the sin eta gives me the component of that centripetal acceleration along the radial direction, so it makes sense that sin eta will come, you know in the outward r direction.

And the other component will come in the cos direction, so the first thing we should do is scaled equations and when we do that we are again looking at; all right, so these are the equations and the first thing we should do now before we proceed is to scale them. So, for that we need to select our characteristic scales. Do you have any questions, so you should be having a question right, okay?

Maybe you all do not know what the equations look like that cannot be expected but you should be; I would expect you all to know the general form. So, the right hand side seems fine, you have u dou/dou r, v dou/ dou eta, the similar sort of thing that is; these are the centrifugal Coriolis terms and so on, continuity equation looks fine. What about the viscous terms? Are they of the form that you used to see or they are not; you would have thought dou square/ dou r square will be there something, because that is just (()) (25:42), right.

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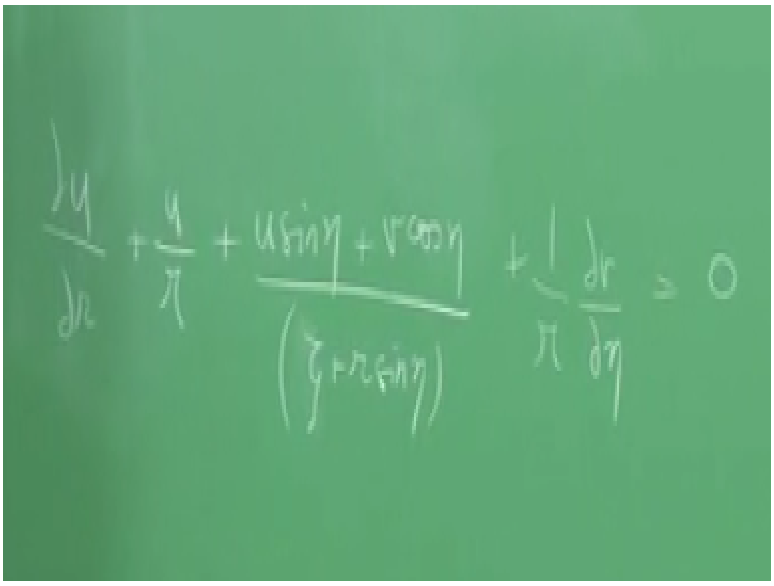

So, especially in this equation, so what I have actually done here, this is not del square v that that much is obvious, so the thing is that normally you have Mu del square v in the; v is the vector, del square v in the Navier stokes; the stokes, I mean the viscous terms. So, it turns out that of course, del square v can be written as a gradient of the divergence of the velocity - the curl of the curl of the velocity, all right.

So, what happens here is because it is incompressible, this gets knocked off, so directly del square v will be negative of the curl of the velocity, curl of the velocity and that is this whole term here is just that, alright. The reason for doing that is that it is much easier to write the equations here, if you actually go compute del square v, it would be much longer, you can do the same thing in cylindrical coordinates and see that it will give you a more compact representation.

And for the purposes of this study, it works out easier like especially when you use steam function formulations or this curl of curl of velocity is a better way of looking but if you had proceeded with del square v, you would not have gone wrong, it is just another alternative representation, all right, so that should be clear, fine. So, now let us scale the equations with the right scales, okay.

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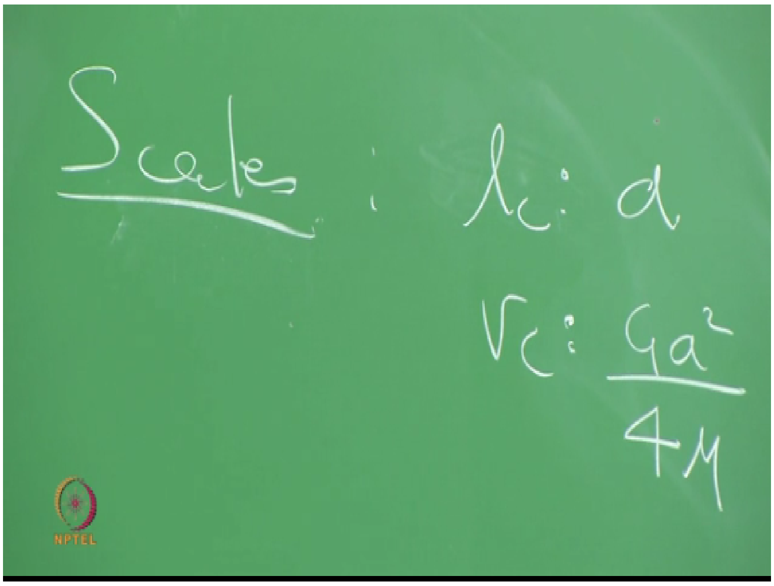

So, what would be my scale and r; length scale? So, I have not said; it should be the; let us say this is the radius, I will call the radius it a, so if the radius of this channel is small a and I have another length scale which is zeta, correct. So, which scales should I choose for the length scale, should I choose eta or should I choose a? A; why a? But even this is a proper length scale, you know in the problem, right.

So, what he has said basically is that the reason, which were interested in the length scale at which we want to see changes in the velocity and so on are happening here, see there is nothing to tell us that variations and the velocity do not happen in the zeta scale, if the; of course, here I am saying the flow is fully developed but if it is not, the flow could vary in the axial direction at you know with the scale of zeta.

So, the real reason is that we want to focus on the circulations that are happening in the cross section of the channel that is one thing, the second thing is of course, in this case we have already looked at fully developed flow, so we are not going to be interested in variations in theta and that is where really the zeta thing would come, and the third thing is; if it will allow to go to the limit of a straight channel.

Because what we are trying to do here is study the influences of curvature of a curve channel and we are going to use perturbation techniques like we always have, so if we want to perturb a curved channel about a straight one, we need to send zeta to infinity right because if zeta is infinite, then this curvature becomes very small and it ultimately reaches a straight channel or a zeta for a straight channel would be infinity.

So, if we want to send this to infinity, we cannot very well be scaling things with zeta, it also tells us that what we are really interested in is happening at the scale of the radius, so lc will take as a, so that is a length scale throughout; velocity scales will come from the equation, so once again the pressure is; I mean the flow is a pressure driven flow and at fully developed flow, there will be a constant pressure, which I will call G; pressure drop.

So, if you look at it that way this is the term that is driving the flow, this guy and it is not immediately obvious I will admit but if you do the usual scaling that we go through you will see that this G, which is the pressure gradient along the center line is what ultimately gives me the scale for the velocity and it will look very similar to that for what we do when we did a straight pipe.

So, you will actually get; suppose G is the pressure gradient here, then you will get G times a square because of two length scales from here by Mu, so the velocity scale will take as Ga square/ Mu and this 4 is a factor that simplifies calculations, so dimensionally it should be consistent, this is the same scale we use in Hagen-Poiseuille flow, so that is it, so with these scales, let us scale the problem.

So, you should find here now that if I; of course pressure will be should be G times a, because G is the scale of the pressure gradient; the pressure drop, so it will be G times the length scale. Now, the important thing is because I have; because I have taken fully developed flow, I cannot actually retain this term in a general sense because I have to realize that dou P / dou theta is a constant, otherwise it would not be consistent, so how can we see that?

Yeah, "**Professor – student conversation starts"** I miss something, yeah, w would not be the continuity because it is; why would not be there, w would not be there, think about it, so the w only come, if w changing with theta, so it is not fully developed, so you lose that derivative alright. "**Professor – student conversation ends"** Where was I? Yeah, I was talking about this term, so because you are saying it is fully developed flow, you will always be able to show that the pressure drop in that direction is a constant.

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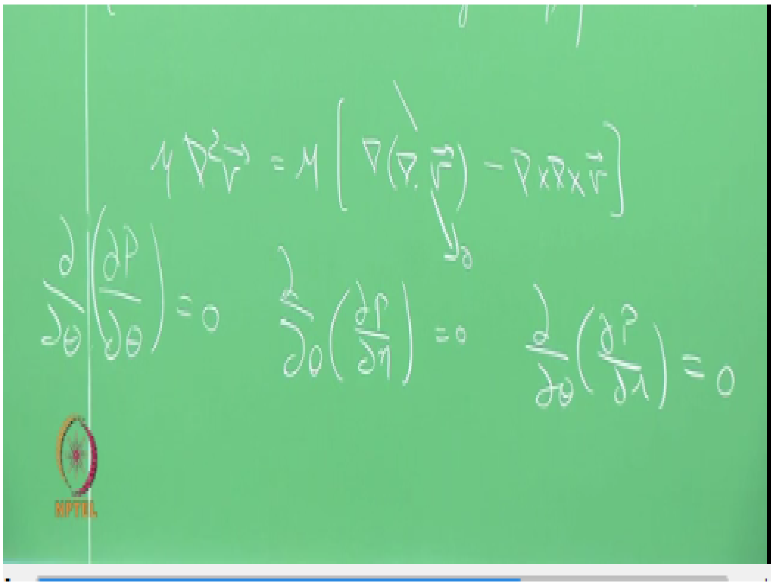

So, how can you see that? If you differentiate this equation with theta, okay you will find that everything here will drop off because all the velocities are theta at variant and the same thing will happen there, so I will get dou P/ dou theta, its second derivative will be = 0 that you will find immediately just by differentiating with theta, all because all the velocities will drop off dou u/ dou theta, dou v/ dou theta, dou w/ dou theta, all 0.

Then I can come here and do the same thing okay again, I will get, right, once again all the velocities are theta invariant. So, what will happen is now of course, I can interchange the derivatives, so in these 2 cases, so what I will get is that dou P/ dou theta taken as a function alright, is independent of theta alright, it is also independent of eta of r, so this dou P/ dou theta is constant throughout the channel not only with length also with cross section.

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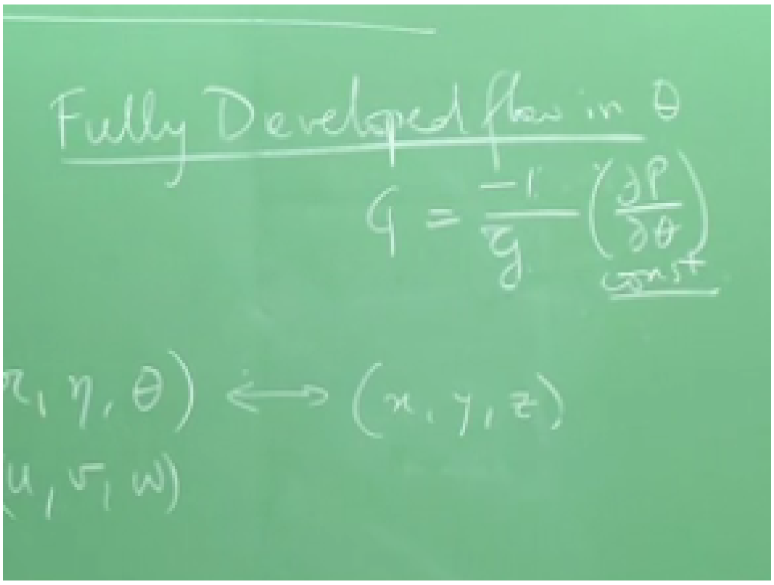

So, at any point in the channel, you go and you checked dou P/ dou theta it is the same, so that show actually related to my G, so to be very specific, G is 1/ zeta, where this dou P/dou theta is a constant, which we have now shown flows for any fully developed flow, regardless of whether there were circulations or no circulations or base state or whatever it may be, if the flow is fully developed, this is a constant and that is what my G is.

It is the pressure gradient along the center line, right but see although dou P/ dou theta is a constant, the gradient is given 1/zeta + r sin eta times dou P/ dou theta, so the driving force depends on this length scale and that will vary in the channel because parts closer to the channel have a smaller you know arc length pass further away have a larger radial distance, so the gradient; the driving force will vary but dou P/ dou theta is same.

"**Professor – student conversation starts"** See, if I scale it in that way but see what I am doing is; I am fixing the value not of dou P/ dou theta of the driving force, so if I take zeta larger and larger, this guy will just change its value but I am putting the total thing to a constant, if I just did this then what you said will be true. So, these are all subtle points that you have to be careful about especially when you read papers, some guys will give this value.

And then they will have this 1/zeta coming everywhere in the base state and it will look funny like you put that to zero some things will happen, so you can always you should always go back and you know understand what he has done at the values, all right. "**Professor – student conversation ends".** So, that is settled, so now dou P/ dou theta is a constant.

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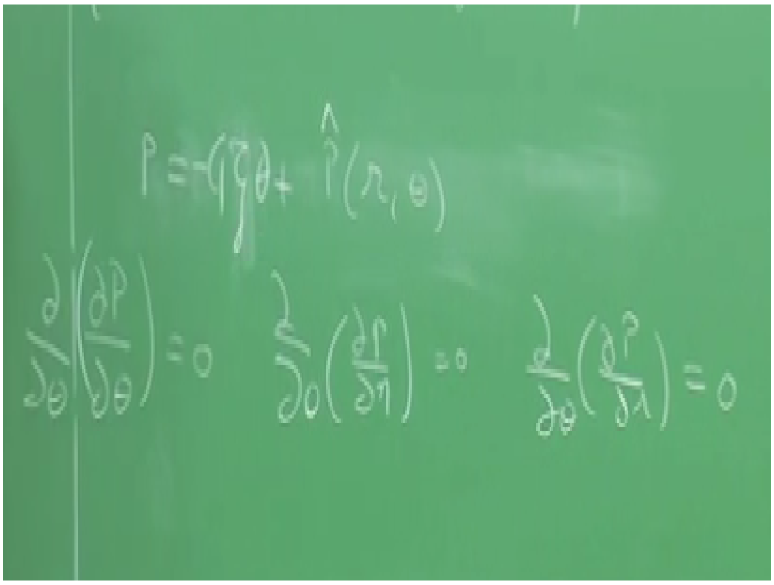

So, from this I can say that P; from these 3 cases, I can write down P is z + P hat of r, theta, so that is a general functional form for P, I should use theta times G, right because dou P/ dou theta is a constant. So, this is important to realize in any fully developed flow so, because of this, I can take my pressure scale as G times a, if I did not have that because the driving force is coming from here, I would actually there would be some problem taking G times small a.

Because I will be scaling the w direction also with small a and that lead to some issues in the scaling but it is only because it is fully developed flow, I can do that otherwise, I should look at the zeta scale for this part of the equations. So, that means I can replace this, I do not need this guy is nothing but - G zeta, so that is important to realize. If we did not do that and we treated dou P/ dou theta as a general variable, you will not get the scaling rate.

So, this is actually a constant term and this is the driving force, so now that we know this is the driving force, we can go and scale our equations right and do it properly. So, let us do that now, so I will work on these equations because they are too big to keep writing again and again. So, if you look at this term first, what I can do is; I want; I know r/zeta is my scale, so a is my scale for all the length scales.

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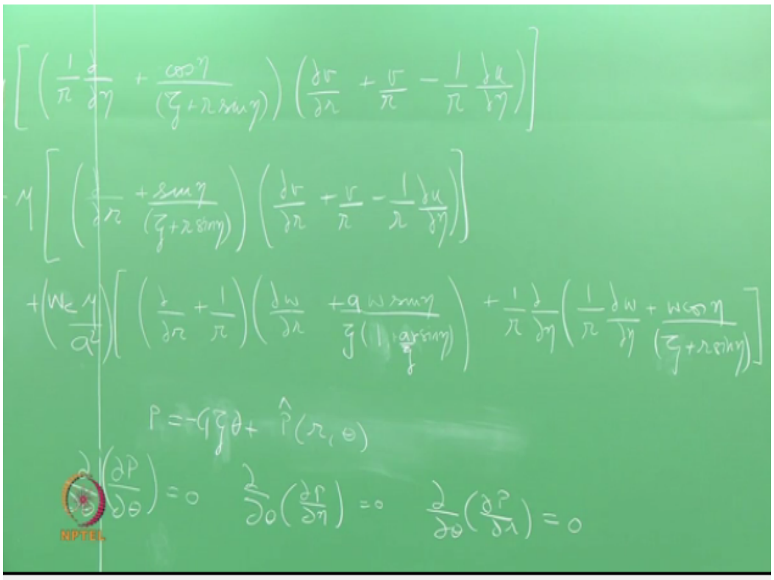

So, if I do that here and I replace r by r star times a, where r star is the dimensionless variable, a square will come out from the bottom, right, I have one w on top, so that will give me the w characteristic, so I have got my usual group but what will happen here is; if I pull out a, I should take out this zeta guy and I will get 1 + ar/ zeta. What I did was I have removed the zeta, so I will get 1 + r/ zeta sin eta, then I replace r/ar star.

So, I am just not putting the r star in, so I get a/ zeta here okay and zeta comes out but then yeah and then because I pull out the a square here, right, so I will actually have a in the numerator, so I will do this something similar, one a came from here and one a came from there, so if I am removing any a, I just multiplied and divided by that a.

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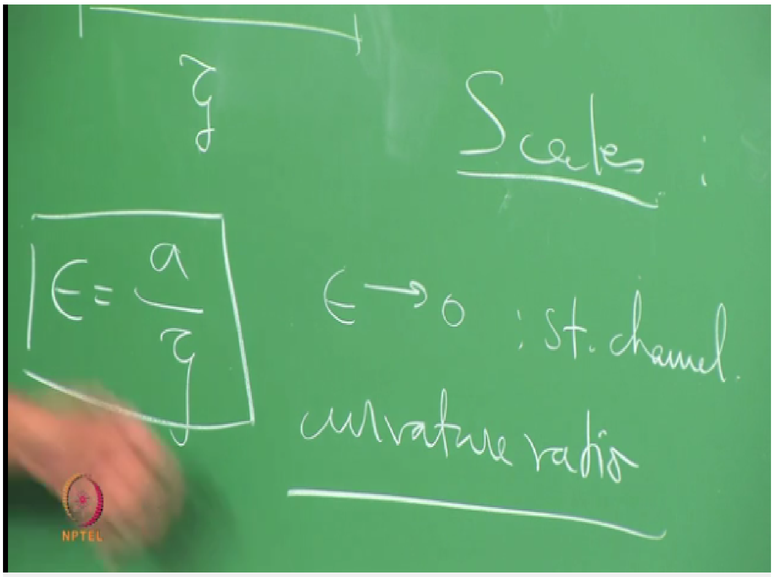

So, if now this a/ zeta is the; is what I am going to take to be my epsilon parameter and if we want to look at a state channel as epsilon goes to 0, I will get a straight channel right and as; yeah, and larger the epsilon the more curved the channel is, so this is called the curvature ratio, so that is how that comes out naturally, so wherever you have theta + r sin eta, I am going to get right, epsilon upon 1 + epsilon sin eta.

The same thing will happen here and that makes sense because all those zeta + r sin eta, I remember what the scale and that direction, so if it goes to infinity that has to become order epsilon. So, not only you have epsilon here the entire term gets multiplied by an epsilon, so all these terms that represent the curvature of the axial direction are going to be of order epsilon alright, that is clear that is the first thing.

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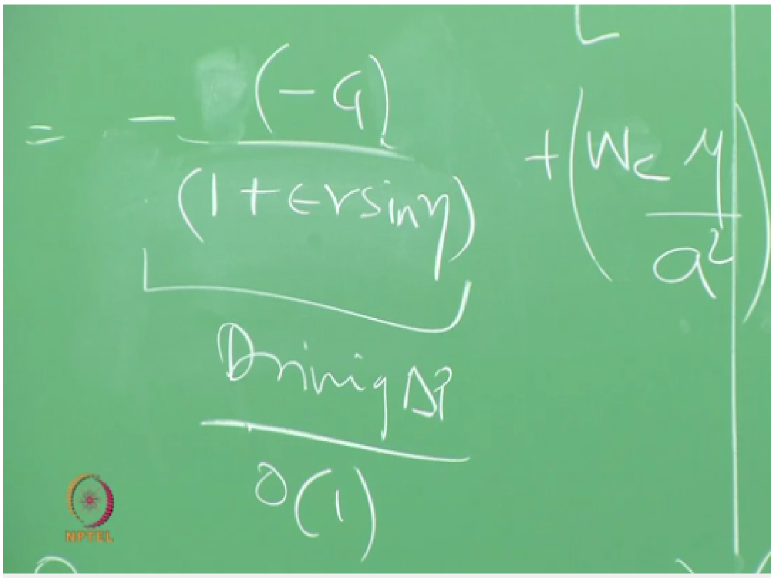

So, then we get this guy out of course here, yeah I should have been a little careful, so I will get zeta here, so then the zeta and this zeta will get knocked off, so I will just get, right; so this term is the driving delta P and it is order 0, the largest part of it, it is not an order epsilon even though there is epsilon here, I put that off I will just get +G and that is my driving force.

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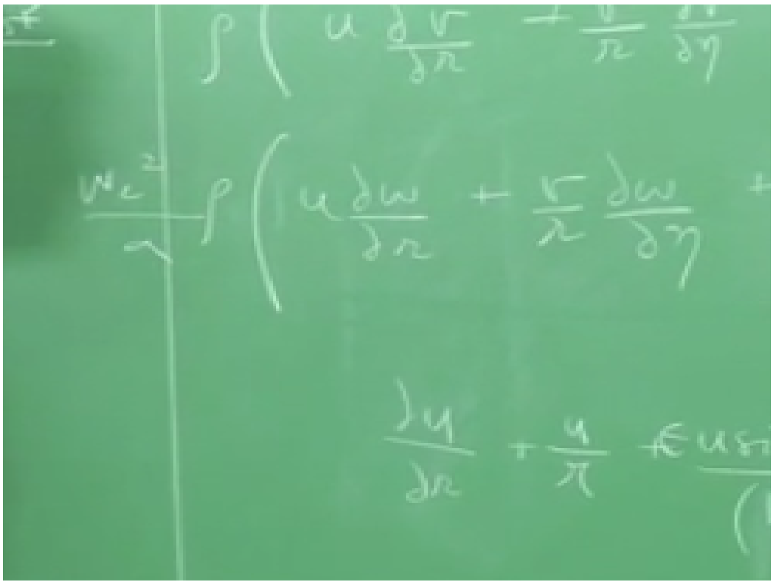

So, the zeroth order term in whole thing has to be the driving force of the problem and that is retained here, right. The other pressure; if all the other derivatives in theta order epsilon but that cannot be order epsilon because that is what driving the flow. Then on this side, if I again I can do the same thing, so I will get wc squared over here, right by a and finally what I can do is; take this stuff to that side or rather bring that here.

Because I want to retain the viscous terms, okay. So, now we have a choice, we always have this choice when we reach this point either I make these terms order 1 and take this to that side that means I will retain all the inertial terms or I can retain all my viscous terms, I have no choice about the pressure drop because the pressure drop is the driving force, so because I want to look at; I mean I have decided to look at viscous forces in this class and that has applications to micro channels and stuff like that.

So, if you want the viscous forces to be there and also we will see later on that the whole thing I am doing has a lot to do with the Reynolds number, so far we have not spoken about Reynolds number at all but since we are discussing centrifugal forces, they will be proportional to the density as you can see so somewhere the Reynolds number has to come. So, far we have not looked at it, we have just talking about small epsilon.

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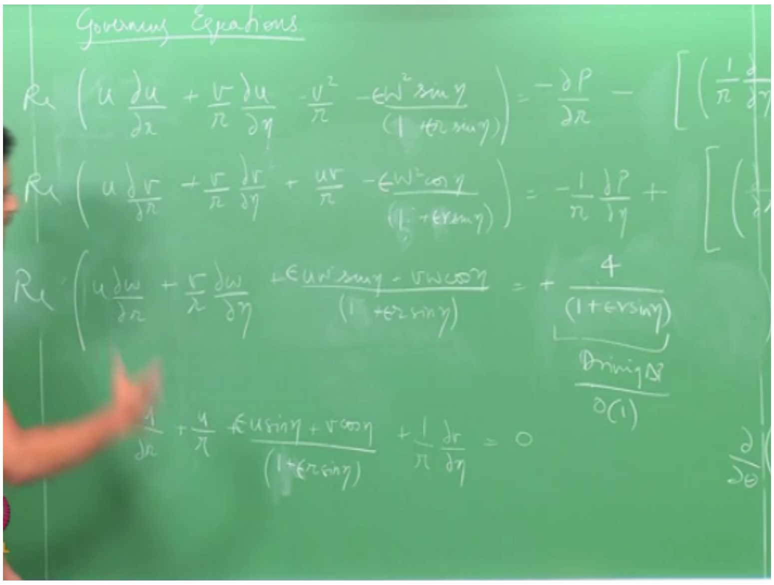

But you will see that I need to scale this in the right way, so that we get the right result later. So, what I will do is; I will take this to the other side because I want my viscous terms to be there, so I will get a square here, wc Mu and this guy becomes 1, right. Here, I will get a squared by wc, fine and now this whole group is just the Reynolds number, right, I will get rho a wc/ Mu, so that is what you will usually get.

If you scale by viscosity, you will get the; where the viscous terms will get the Reynolds number multiplying the inertial terms right and here we see now that you can find out what the velocity scale is. So, I have made this order one, this guy gives me the Reynolds number, so they are as important as Re and this I said I do not know wc but I know that this is the driving force, so I will make this whole group = 1.

So, that will happen if wc is Ga square/ Mu, so that is what I written here and I threw in a factor of 1/4 that is purely algebraic, if you do not need to take it, if you do not want but what you will see I think later on is this 1/4 is also closer to the average velocity but since that is, what I am following Gary Leal, so I just follow that, so this whole guy will go off and I will get just 4, right all right, so these are the scale equations, we stopped for today.

Tomorrow, I will start from the scalar equations and then we will solve the problem by perturbation methods. So, one important thing to realize is that these are the equations in the same equations, we have not done anything to them, the fully nonlinear coupled equations for flow in a curved channel but we have scale it in such a way that these epsilon terms are now very clear, so we know that all the terms that are coming in because of curvature are going to get multiplied by epsilon.

So, directly on inspection you should see that if you put epsilon to 0, we should get back the equations of the cylindrical coordinate system because this is basically the cylindrical coordinate system if zeta theta = z, zeta goes to infinity. So, if you put epsilon to 0 here, you will directly get back the cylindrical coordinates that is the continuity equation you can see that immediately and this guy gets knocked off.

So, when you looked at the zeroth order problem, it will be a cylindrical coordinate Hagen-Poiseuille flow because it just flow in a straight pipe with viscous terms and then when we look at the first order, we will start seeing the centrifugal force coming into play.