**Multiphase Flows: Analytical Solutions and Stability Analysis**
**Prof. S. Pushpavanam**
**Department of Chemical Engineering**
**Indian Institute of Technology - Madras**

**Lecture – 33**
**Turing patterns: Results**

So what we will do in today's lecture is continue the discussion on Turing patterns, which we started off yesterday and towards the end of the last class, this is what we had done.

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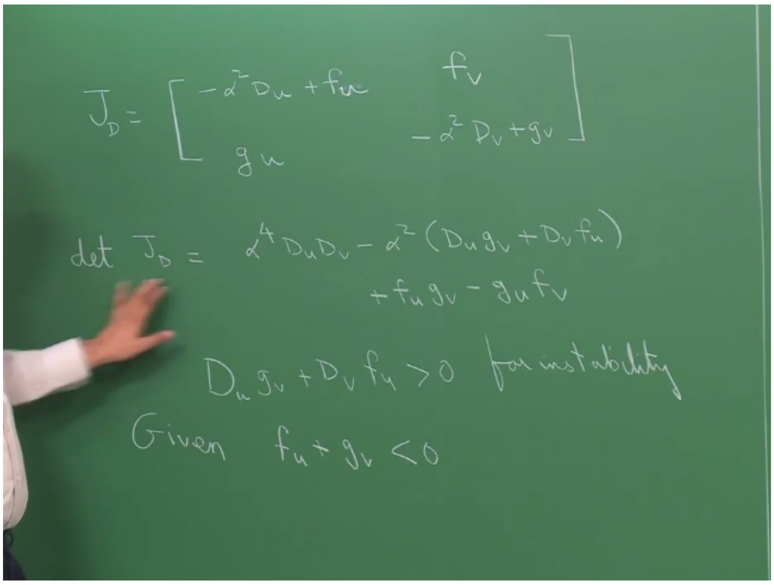


We had found that the Jacobian matrix for the reaction diffusion system was given by this okay. This is when finite diffusion coefficient is actually added to the system okay and the determinant of this particular matrix is given by this expression. It is fourth order in alpha and what we have established was that the trace of this particular matrix would always be negative.

Because it is already known to us that fu+gv is negative okay. So there were 2 conditions, which could possibly be violated for getting an instability. So what we established was that the trace condition cannot be violated. The only condition which could possibly be violated was the determinant condition and in order for the determinant condition to be violated, we know that this is positive.

Because the system without diffusion is stable and this is positive and the only way that the determinant can be negative is if this particular quantity is positive okay. So in order for it to be unstable and this is a necessary condition for instability.

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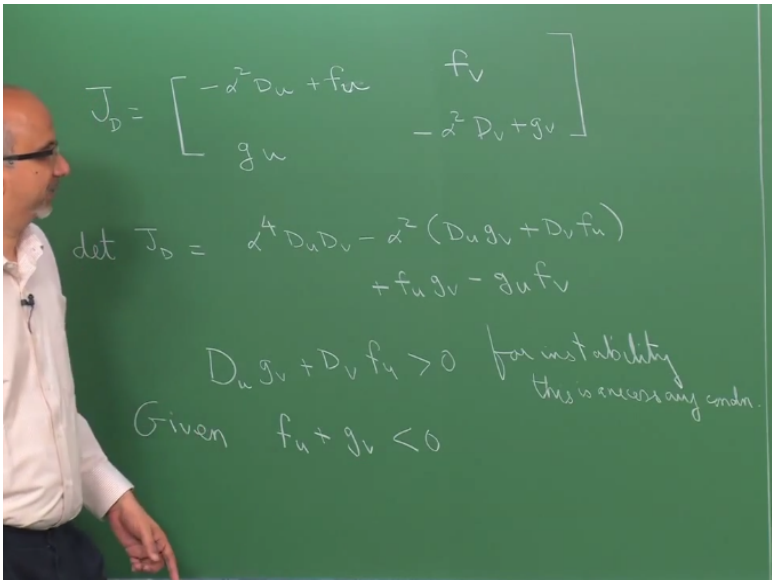


So for instability this is a necessary condition. What we already know also is that fu+gv is<0. So in order for you to possibly have an instability, there are 2 options, one is that both of them are negative. If both fu and gv are negative, then clearly this is always going to be negative and you will not have a violation. So the only option available is one of them is positive and the other is negative okay.

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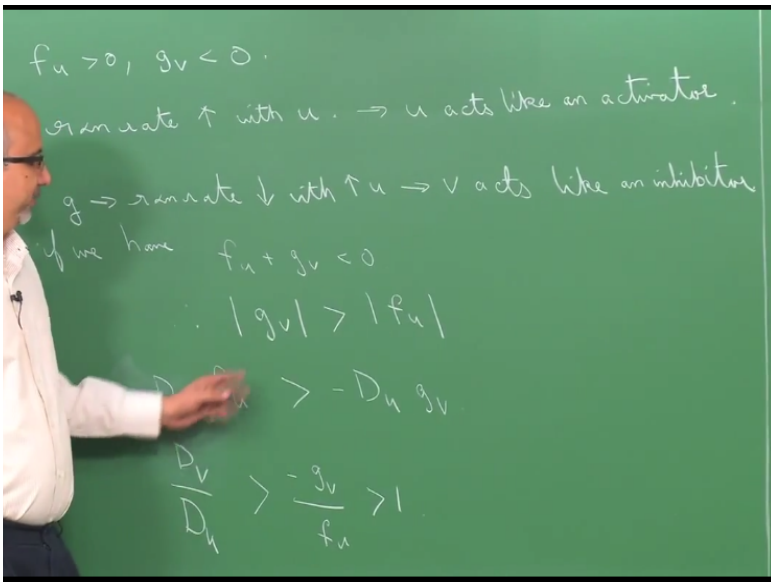


And the negative guy is dominating the positive guy okay and so what we have done towards the end of the last class was we have assumed that fu was > 0 and gv was negative and because of the nature of this dependency what we said was look the reaction rate increases with u that is what it means, f is a reaction rate so as you increase u the reaction rate increases and so this is u acts like an activator okay.

And as far as the other reaction rate g, the reaction rate decreases with an increase in u. So that means v acts like an inhibitor okay. So basically what you need to have is in your chemical kinetic expression, chemical reaction rate expression you need to have this kind of a dependency that one species is actually going to slow down the reaction as you increase the concentration.

And if you have your regular first order kinetics then you do not have this action of inhibition okay and you will not be able to get these Turing patterns for example okay. So this is what we have concluded but these are all qualitative arguments and what we need to do is see if we can get some more quantitative information. So if we want fu+gv to be negative, if we have fu+gv to be negative and if gv is negative, you are going to assume the other way but we are just assuming this okay and we proceed.

That means what? The absolute value of gv must be > the absolute of fu right. Therefore, the absolute value of gv must be > the absolute value of fu. Only then this is going to be negative. If the absolute value of gv is less, then this is going to be positive and now I come back to this expression Dv fu must be > - of Du gv, gv is negative so –gv is positive okay. So basically what this means is Dv/Du must be < - of gv/fu.

And we already know that the absolute value of gv is > the absolute value of fu okay. That means Dv must be this must be > 1.

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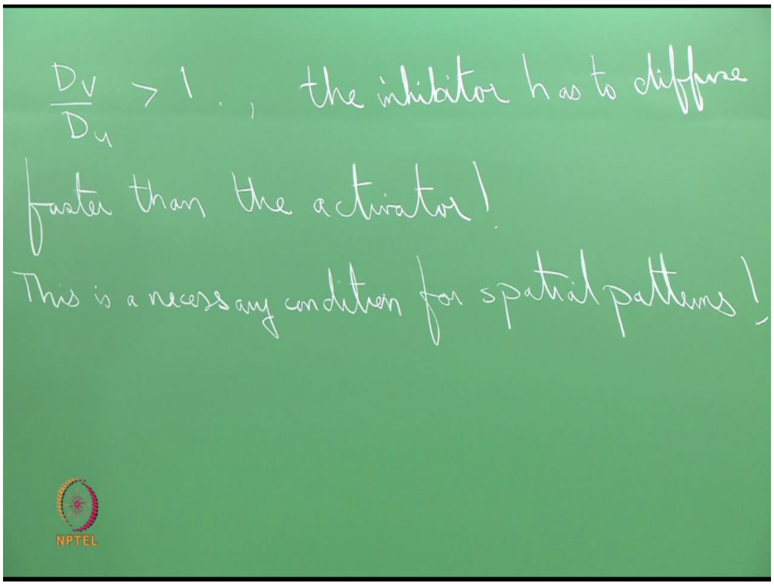


In other words, Dv/Du must be > 1 okay. So all I am doing is I am using the fact that the mod of gv is>mod of fu and I go back and I am trying to find conditions on the diffusion coefficients okay because clearly the diffusion coefficients are you can think of as parameters which I can do possibly control my system okay and if I can do that. So I am just manipulating the second thing.

I am saying that this is going to be positive and that means I can possibly have instability and if I have an instability, the resulting solution is going to be a periodic pattern because that is what we assume it to be at the form of sin alpha x okay and this is > - Du gv. So now I am just dividing by Du taking fu as positive taking down here and this is positive by positive okay and from this, this is >1.

In other words, what it means is that the inhibitor has to diffuse faster than the activator that is the conclusion I have. That is this is a necessary condition remember, this is only a necessary condition for instability, just because it is diffusing faster does not mean that you will have a spatial pattern okay because this is only guaranteeing that the middle term is positive.

But which is going to dominate the first end of third term in the determinant that we do not know. See I want this to dominate these 2 terms. I am only looking at the sign of this term okay. In order for you to actually have an instability, you have to make sure that this is dominating over these 2 terms in which case the determinant will actually be negative okay.

Otherwise, this guy can be positive but small value and these guys can dominate and still determinant will be positive okay.

So again I want to emphasize that this is a necessary condition for spatial patterns okay. So now how can I make sure when you indeed have spatial patterns? In order for you to have spatial patterns, you need to have the determinant negative right.

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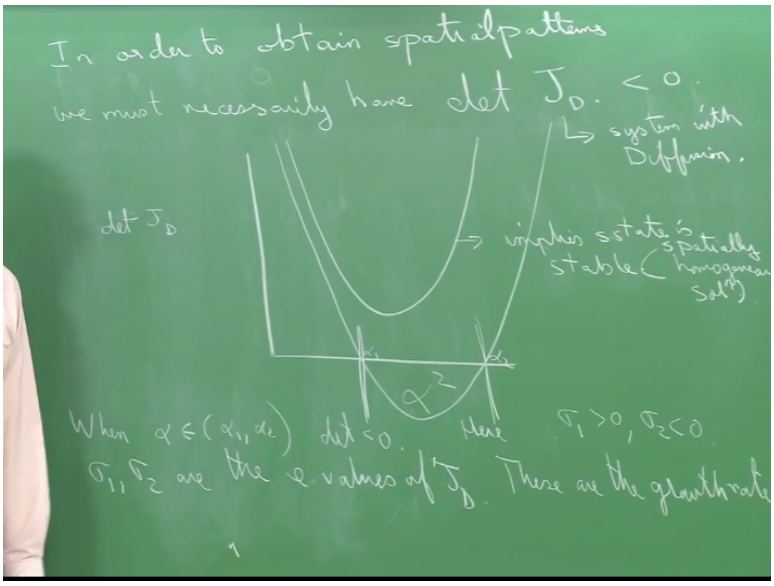


So in order to obtain spatial patterns, we must necessarily have the determinant, as added a subscript D here just to tell you that this is a Jacobian of the system with diffusion. When I do not have D that means there is no diffusion okay. Determinant of JD must be negative that is this is system with diffusion okay. So basically what I am going to do now is make a plot of this determinant of JD versus alpha squared okay.

Clearly, this is a quadratic. For lower values, this is going to dominate as alpha goes to – infinity this is going to dominate; it is going to be positive. For large values, this is going to dominate; this is going to be positive. So what I know is this is going to be of shape of a parabola right. So the question is this is a general shape that I have if this is the shape of the parabola that means my determinant is always positive okay.

And what decides the shape, this shape is going to be decided by the values of the elements in my Jacobian evaluated at the steady state and the diffusion coefficients okay. The diffusion coefficients and gv fu evaluated at the steady state decides the shape. It is possible that this guy may start dominating, the middle term maybe large in a small interval.

So that happens then this implies steady state is stable. Which steady state? The one which is having a uniform concentration because we have done the linearization about that steady state okay. That is the spatially homogenous solution. This means no matter what the alpha squared is. If you give arbitrary disturbance, you can decompose it into different wave numbers.

And for all wave numbers this alpha the determinant is positive and remember trace is negative. So what is trace represent for a 2-dimensional system? The sum of the Eigen values. The determinant represents the product of the Eigen values. So now this basically means that the real parts are going to be negative okay and because of the product of the 2 Eigen values are negative, so both are negative and trace is negative okay.

And so if Eigen values are negative remember the Eigen value was my growth rate e power sigma t. When I did the linearization, I had sigma*u star v star=the Jacobian*u star v star. So the Eigen values basically are my growth rate okay. However, if you have a situation of this kind if your determinant can be like this and that is going to be decided by the diffusion coefficients.

Then you are necessarily guaranteed that that is an interval of alpha over which the determinant is negative okay. So this is stable here, this is stable here and this is exactly similar to what you have done earlier your neutral stability curves etc when you had your Rayleigh-Benard problem okay. So the point I am trying to make here is that in this interval let us say alpha 1 to alpha 2.

When alpha belongs to alpha 1 and alpha 2, the determinant is negative. If the determinant is negative means one Eigen value has to be positive, one Eigen value has to be negative okay. That means the stability is going to be decided by the guy, which is positive and the Eigen value remember is the growth rate that means one Eigen value is going to make it unstable the one which is positive okay.

So here sigma 1 let us say is positive and sigma 2 is negative and sigma 1 and sigma 2 I used sigma 1 in the last class right. So sigma 1 and sigma 2 are the Eigen values of JD and these are the growth rates. So what is it that is going to guarantee you that this fellow is going to be negative? So far we have just made sure that we had a necessary condition but what is that that is going to guarantee you that the determinant can be negative?

You have to look at the discriminant of this particular thing and if the discriminant is positive I mean you treat this as a quadratic in alpha squared. If the discriminant is positive that means you are guaranteed to real roots and these will be the 2 real roots. So in addition to Dv/Du being>1 you need to make sure that the discriminant is of this equation is positive okay.

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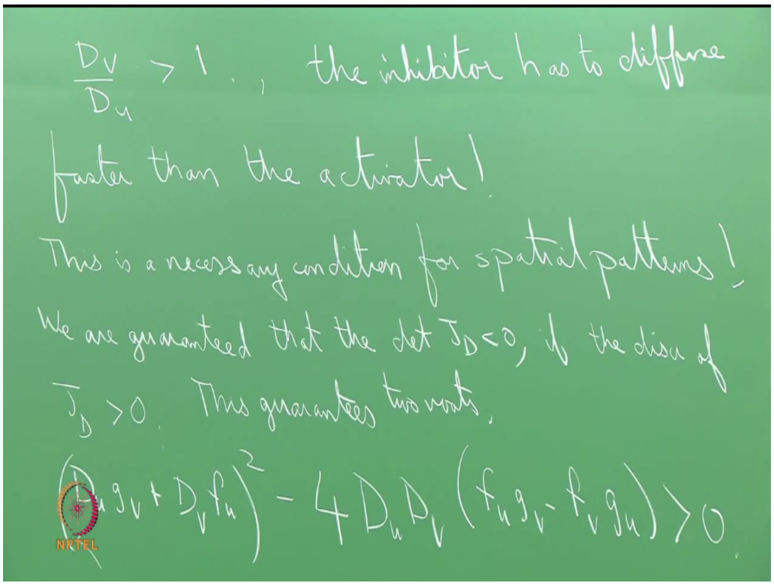


So the discriminant of this equation being positive implies so we are guaranteed that the determinant of JD is<0 if the discriminant of JD is>0 because this guarantees me 2 roots okay. So that is my condition which I need B squared-4ac, I need to make sure that Du gv+Dv fu the whole squared=4*Du dv okay. If the discriminant of that determinant expression is positive, I am guaranteed 2 real roots and this will be the 2 real roots.

If the discriminant is negative then it is going to be of this kind, no real roots okay. So that is what I have written here. So now if you have a steady state, if you have a steady state for a particular system, your reaction rate expression f and g, you know what your uss is vss is, you can find out by just solving those algebraic equations right and using Newton-Raphson or something.

Then you can also calculate the partial derivatives of the reaction rate expressions and if we need to satisfy this okay, there is a necessary condition but if we satisfy this then you know for sure that the determinant is going to be having a negative value and one of the growth rate is going to be positive okay. If one of the growth rate is positive then from our form of the expression that we have assumed for the disturbance, the steady state solution which we are going to get is going to be a periodic solution.

Because you assumed sin alpha x okay, so what I am saying is these are quantities they are known to you from your reaction rate kinetics. Du and Dv you can possibly think of as experimentally controlled parameters, actually they are not, they depend upon the system but if you actually have this condition satisfied then you can actually have a spatial pattern okay. So this is the necessary and sufficient condition in some sense.

Because this is going to give you the thing. Now the spatial pattern, I just want to spend a little bit of time on that expression. There was a neat explanation given about this Dv/Du so I think I will mention that.

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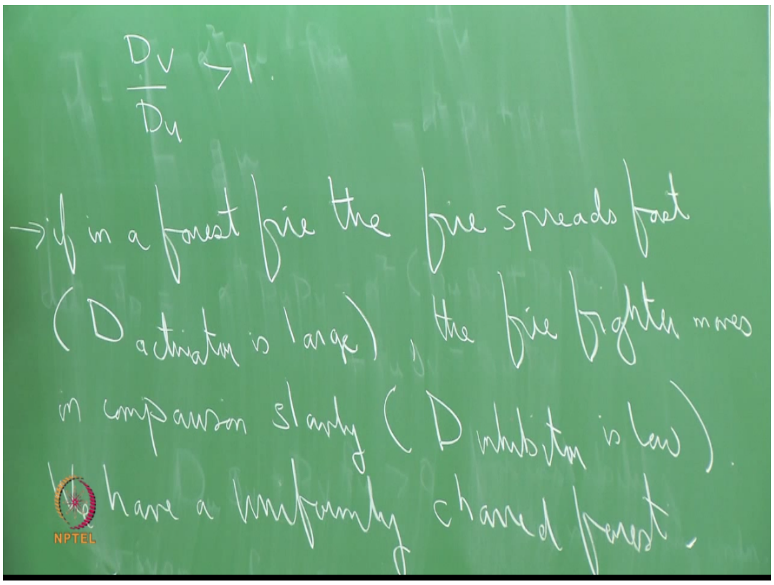


Dv/Du being>1 okay, so we were looking at these 2 species, one as an activator and one as an inhibitor okay talk in terms of chemical reactions, so rather than talk in terms of chemical reactions we will have to give a slightly broader and more general argument. Now in many places where you would have seen, you would have read in the newspapers about forest fires okay spreading.

And you know people specially in the US and Australia, the weather becomes very dry and at one place the wood catches fire and then it spreads and it causes a lot of devastation right and of course how do you control it? You control it by having some kind of a firefighter who goes spraying some chemical from the top or you know all these fire engines are going to go and douse the fire right.

So you can look at this particular thing as if there is one of them is activator and one is an inhibitor. The fire by itself has a rate by which it is spreading okay. So the fire is an activator because it just spreads whereas the fire engine or the however in the aeroplane with which you are spraying chemicals to control the fire. There is an inhibitor that is actually trying to quench the spread.

Now supposing the fire engine is moving very slowly, it does not move fast enough, the fire engine remember is the inhibitor right. Supposing the guy does not spread fast enough, the fire is going to spread everywhere and the entire forest is going to be burnt. You will have spatially uniform homogenous steady state, no forest anymore, lot of ash right but if this guy is able to move faster, if the fire engine can move faster, then you will have pockets which are burnt, pockets which are not burnt okay.

And then we are going to get a spatial pattern. So that is just one way for you to possibly think of this particular thing. So I mean I just thought this is a nice way to explain this. So if in a forest fire, the fire spreads fast that means the diffusion coefficient that is the rate of spreading, diffusion is basically a way of spreading. Diffusion of the activator is large, then the firefighter moves in comparison slowly that means D of the inhibitor is low okay.

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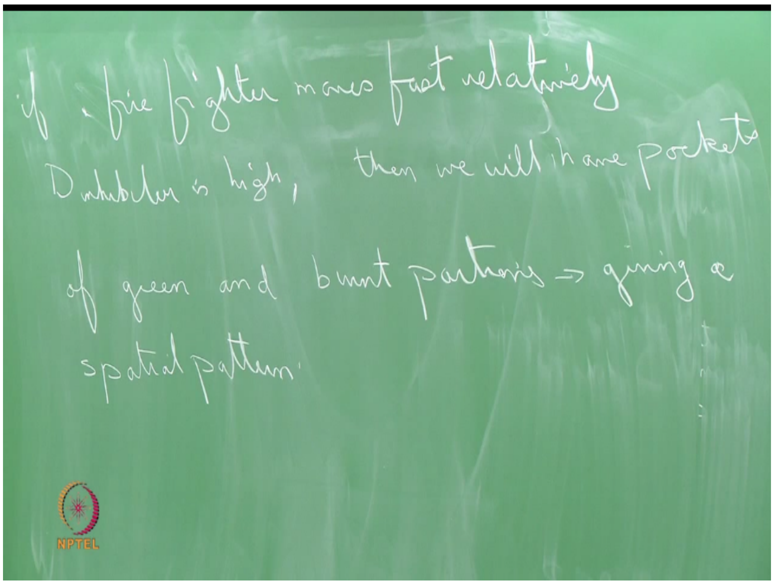

Then, we have a uniformly charred forest okay but if D is the firefighter moves fast relatively that means these are all relative things right and D inhibitor is high then we will have pockets of green and you know burnt portions giving a spatial pattern. Of course, the theory for this particular Turing pattern is whatever I have discussed here okay but then to experimentally establish because you know diffusion coefficient is not something which you can actually control.

It is not a flow rate that you can come open your valve or it is not a temperature that you can actually increase. So to actually experimentally establish the existence of these Turing patterns that took a lot of time okay and in finally it was established and then people started actually believing Turing patterns actually exist. So I think that is something which you people can read.

Because there was a huge gap between the theoretical prediction of how diffusion can actually cause an instability, cause a spatial pattern and by the time this particular theoretical prediction was actually verified, we took experimentally, it took time because you need to have the right system with the activator, with the inhibitor, so that the kinetics was having this kind of mechanism.

Then you need to make sure that the diffusion coefficients were also right okay. So that would depend upon the concentrations. So I mean with lot of persistent people we finally managed to get this.

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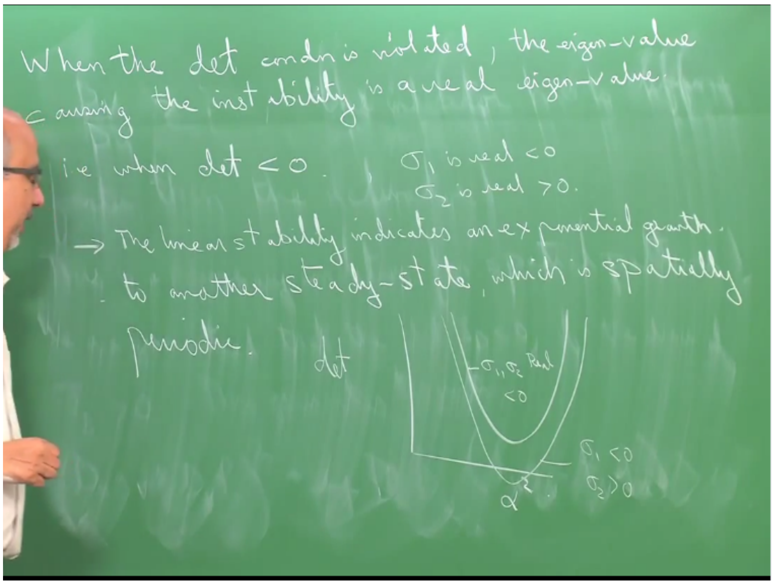

I want to just make one more point here and that is when the determinant condition is violated, the Eigen value which is actually causing the instability is a real Eigen value okay. That is the Eigen value causing the instability is a real Eigen value because now you can have a complex Eigen value with the real part. What I am trying to tell you here is that see there are 2 ways in which you can have instability.

One is the real part of a complex Eigen value is positive or the real Eigen value is positive okay. So the point I am trying to make here is when the determinant condition is violated that means what i.e. when determinant is negative, one of the Eigen values is real and negative, the other Eigen value is real and positive okay. That is sigma 1 is real and negative say and sigma 2 is real and positive for instance.

Just before the violation just before the guy cuts the x axis, both of them are going to be negative okay. This means what? The linear stability indicates an exponential growth to another steady state, which is spatially periodic okay. This is like same as what we had for our Rayleigh-Benard problem. You had a stationary solution, which was not moving at all and then when you had instability, you got a periodic pattern which was steady okay.

You have a periodic pattern spatially varying but that was also a steady state, so this is what exactly the same thing happening here okay. So I mean the new solution that you are going to get when the homogenous solution becomes unstable by changing the diffusion coefficient is going to be a steady state pattern but this is going to be periodic. The earlier solution was a steady state pattern but it was spatially uniform okay.

So what is the other situation possible? So let me just explain what I mean by this? I am just going to draw the determinant versus alpha squared okay. It is like this, so this is the situation where sigma 1 and sigma 2 are both real and negative. For this case, sigma 1, sigma 2 is real and negative whereas here sigma 1 I wrote as negative and sigma 2 is positive. What is the difference between these two?

I will change something like a diffusion coefficient okay. That is let us say I am changing only diffusion coefficient of u, so for some value of Du I have this business. I keep changing the u, this curve is going to change and it is going to come below okay. When it comes below just at that point here you had both Eigen values real and negative whereas here you have one

Eigen value negative, one Eigen value positive which means a real Eigen value has caused the imaginary axis.

A real Eigen value has changed from negative to positive okay and that is what I want say sigma 1 remains negative but sigma 2 has changed from negative to positive making it unstable. There is another scenario which is possible I just want to mention that. Suppose you have we proved in this particular problem that the trace was always going to be negative okay but suppose you have a situation where the trace can possibly get violated.

That is not applicable to our problem here because we prove the trace is always negative with the diffusion coefficient.

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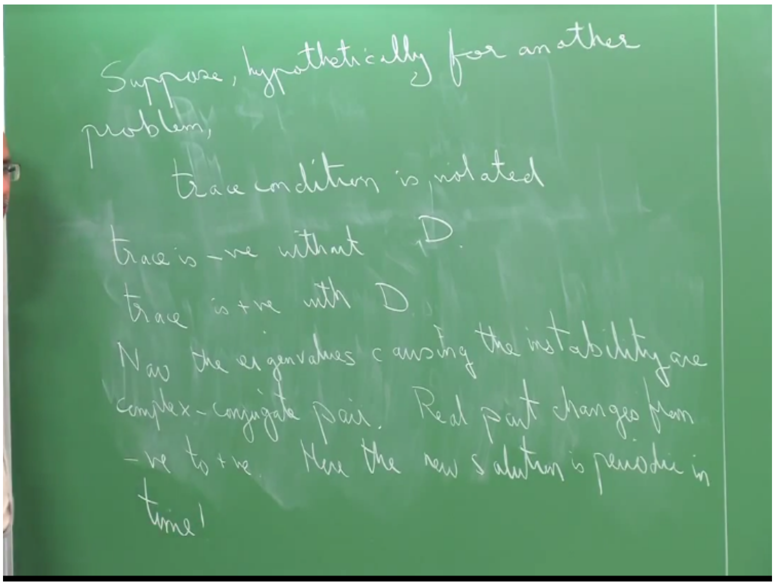


Suppose hypothetically for another problem the trace condition is violated, which means what? Trace is negative without diffusion coefficient without the D okay, trace is positive with the D suppose. Then, if you actually calculate the Eigen values, it will turn out that now the Eigen value is causing the instability are a complex conjugate pair. So the real part will change from negative to positive okay.

So supposing this happens what can you say about the nature of the new solution of the new state. The new state is not going to be a steady state; the new state is going to be a time dependent time periodic solution because in a complex conjugate pair you will have e power a+ib okay. You will have e power ibt, the e power ibt is going to be periodic in time and so what you will actually see is a periodic solution in time okay.

So here the new solution is periodic in time that is earlier there is no imaginary part.

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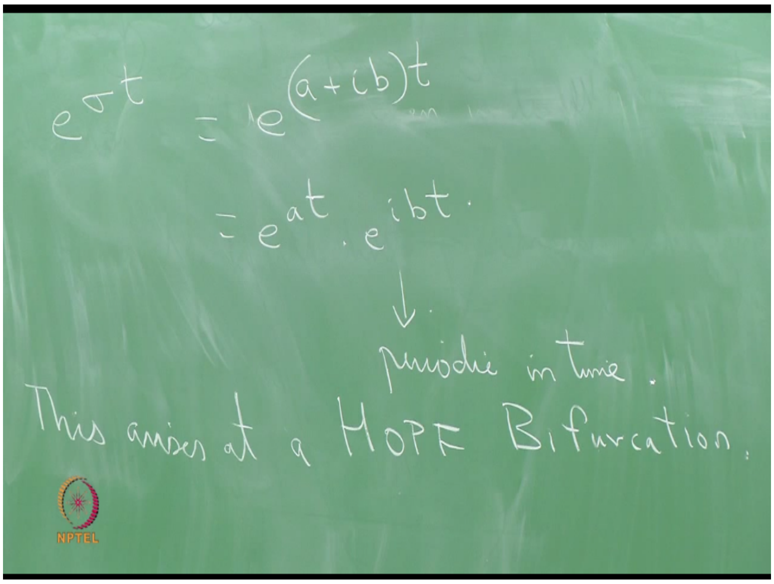


Now I have an imaginary part that means my growth rate which is in the form e power sigma t is going to of the form e power a+ib t okay and this is e power at times e power ibt and e power ibt tells you this is periodic in time. So actually put a probe and you have to measure I do not know concentration or velocity or temperature, you will actually find if this situation arises, a time periodic behavior okay.

Whereas in the other case, but in our Turing pattern problem that does not happen because the trace condition is not violated, trace condition we proved is always going to negative okay. So I mean I just wanted to mention this that this is also a possibility. So that you should not always think that whenever there is an inability you always will get a steady state. You can also get a time dependent solution okay.

And when this happens when you have a time dependent solution that means the new solution is periodic in time, this arises at a what is called a Hopf Bifurcation. So I am just dropping a couple of keywords for those of who are interested to actually read a bit more on this okay. So what we have seen here is in order for you to actually have a Turing pattern or a spatially periodic solution when you have a system with the reaction and diffusion added.

Kinetics should be such that that should be something which is some chemical which is activating, some chemical which is inhibiting and the diffusion coefficient of the inhibitor has to be faster than that of the of activator okay. So that is the prediction and then there are some necessary conditions and sufficient conditions which you have derived. So I think the very fact that you have taken a very simple 2-dimensional example with 2 variables has actually given us a lot of insight into this particular problem.

If you have taken actually more complicated system, then you possibly would not have been able to get this kind of an understanding okay. The other important point which I want to emphasize here is that look what we have been doing so far is using this framework of linear stability analysis, which is a very general framework, which can be used for understanding any system is not necessarily in fluid mechanics.

It could be in reaction diffusion, it could be in some other system, so as long as you write your governing equations okay and that is some kind of a non-linearity, which is embedded in the system which is going to be describing the system behavior. The procedure is very, very clear okay.

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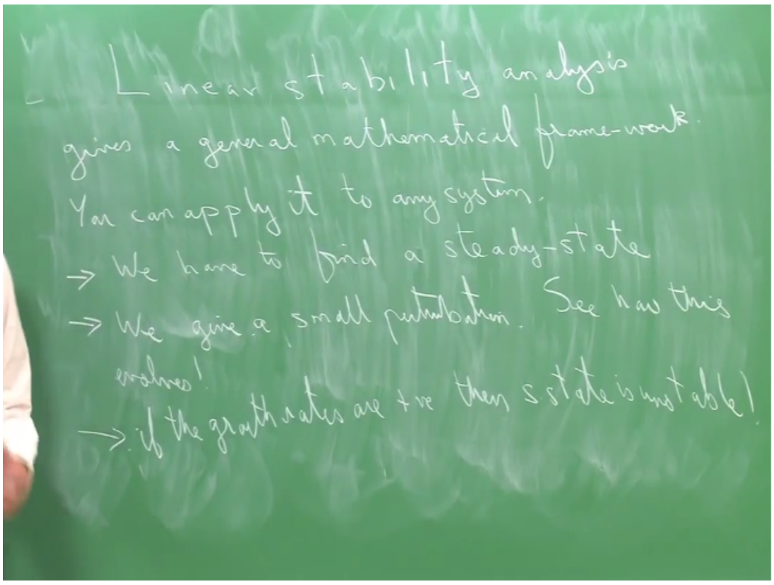


So I think the important point what I want to say here is linear stability analysis gives a general mathematical framework. You can apply it to any system and I think that was one of things which I wanted to do that is not necessarily hydrodynamic problems fluid mechanics, it can be reaction diffusion, it can be something else just only chemical reaction okay and the idea is we have to find a steady state okay.

Always trying to find the stability of the particular steady state, so need to find what the steady state is only then you can find out whether it is stable or not and that is what we did. We had a spatially uniform solution. In the fluid mechanic problems, velocity was 0, there was my steady state. In this problem, uss, vss is some constant value okay. Then, we give a small perturbation okay.

And see how this evolves, how this actually is evolving in time and sometimes if you have diffusion coefficient, you will have evolution in space as well and if the growth rates are positive then steady state is unstable and one of the things is the nature of these curves is such that automatically the intrinsic features of the system it decides what the periodicity is spatially.

It is not something which you know depending on how you do experiment. So for example in the Rayleigh-Benard problem the wavelength of the periodic solutions came was decided by the properties of the fluid okay, the space between the thing. Here the alpha where it is just going to cross the x axis is going to be the one which decides the periodic spacing right.

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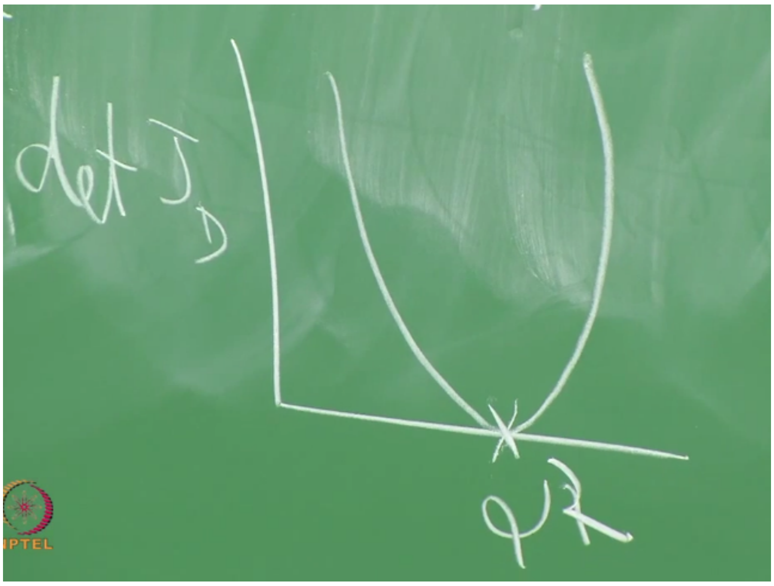


In this the determinant whatever I plotted versus alpha squared, the point where it is just tangential when I actually had the determinant of JD versus alpha squared, at some value it is just going to be tangential. The periodicity that you are going to see is that one okay and that is going to be decided by the kinetics g, v, f, u is going to be decided by the diffusion coefficients.

So whatever is there in the system that is going to decide what the spatial pattern is. So what you can of course do is in the very simple system one dimensional equation in x, you have reaction kinetics, you can just do a finite difference method and you can possibly solve the system and see when the determinant condition is violated, when it is not violated, what kind of a solution you get?

You should get a spatially uniform solution when you have the determinant is positive. When the determinant is negative then depending on the choice of the parameters you might be able to get a spatially periodic solution. This is a very simple experiment you can do on the computer. It is more difficult to do an experiment in the lab with the test tube okay. So I think that is something which you guys can do.

I do not want to start something new at this stage so we will stop as far as this particular class is concerned and in the next class what we will do is we will derive the boundary conditions for the tangential stress wherein the gradient of the surface tension will show up okay and we will have the formal derivation then will do a problem on Marangoni convection and see how Marangoni convection sometimes is the cause for natural convection and not the buoyancy term okay.

Then we will work on 2 or 3 more problems in fluid mechanics, but just to take you slightly away from fluid mechanics and kinematic boundary conditions we just threw in this example. We will go back to kinematic boundary condition and fluid mechanics next week okay. Thanks.