Multiphase Flows: Analytical Solutions and Stability Analysis
Prof. S. Pushpavanam
Department of Chemical Engineering
Indian Institute of Technology – Madras

Lecture - 30
Capillary jet instability: Rayleigh's Work Principle

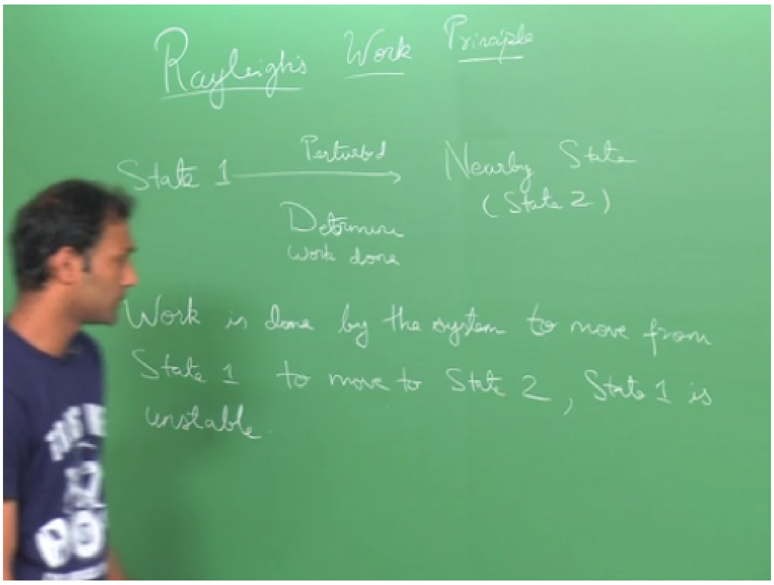
So, good morning all, welcome to the next lecture. Today, we will be revisiting the capillary instability problem through a separate analysis called the Rayleigh's Work Principle, okay. So, last class we are already looked at capillary instability and found that the configuration, the cylindrical configuration of a jet is unstable for K> l, stable for K>1 and unstable for K<1. So, you had seen that you get a dispersion curve with a maxima close to K=0.7.

So, today we will revisit the problem, something called as Rayleigh's Work Principle. It is a static argument to determine stability. When you do a linear stability analysis you do something called as a dynamic analysis where in you find out equations governing the variables, the pressure and whatever variables are there in the problem you get a dynamic equation how they are related in time and then you get the dispersion equation that we have got in the last class.

So, Rayleigh's Work Principle is a static argument wherein you look at the state that you are looking for the stability of say the cylindrical configuration of the jet and we disturbed it like all cases where we analysis the stability and then we look at how the energy of the system has changed or the amount of work has been done on the system or done by the system. So, based on that we get the criteria for stability of this.

So, I will just start with what Rayleigh's Work Principle states.

(Refer Slide Time: 01:51)



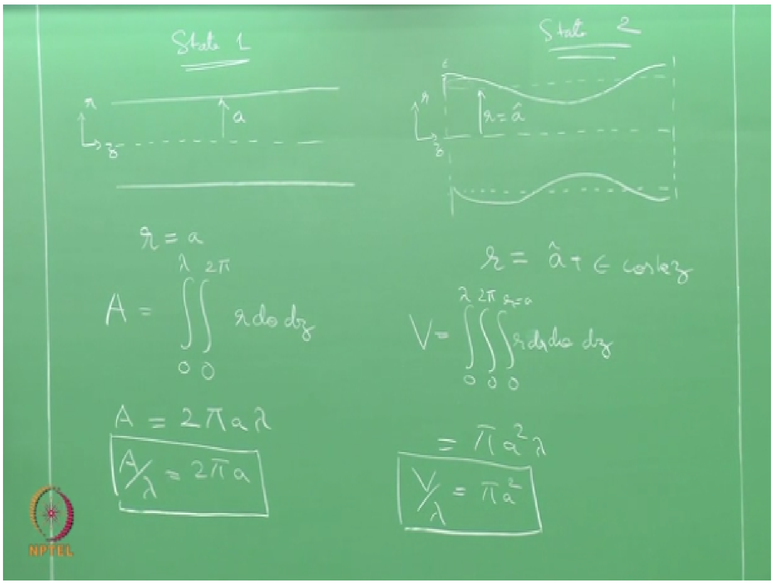
So, for Rayleigh's Work Principle first of all have a state of whose stability you are analyzing, okay and the state is perturbed to a nearby state, okay. And you find out whether work has required to be done on the system to take it to this state or whether the system thus work and goes to that state, okay. So, we determine the work done to take the system from the original state to a nearby state, okay.

Now if work has required to be done on the state to take it to a nearby state which means that it has gone to a higher energy state. So, which means this was a lower energy state and therefore this would be a stable state, okay. And if it is the other way wherein the system while in this state particularly does work and goes to a neighboring state then since it has done work and gone to the neighboring state it has lost energy and it has gone to a lower energy state.

So, this has to be a stable state compared to this. So, depending on the sign of the work done or basically the energy of the system determines whether it is stable or unstable. So, wherever it is at a lower state, energy state would be the stable state, okay. So, is that clear? So, I will just write it done. If work is done by the system to move from State 1, which I call this, to move to State 2 which is a nearby perturb state, okay.

So, work is done by the system which means it has lost energy. So, then State 1 would be unstable and similarly if the work has required to be done on the system then device will suggest to where in the system would be stable, okay. So, the problem that we are looking at today would be the problem of capillary instability wherein you have a jet of radius a.

(Refer Slide Time: 05:07)



So, this I would call its State 1, okay. And what we are saying is that the total energy of the system, right now it is an inviscid jet and it is stationary state. So, it is not even moving like the case which we had studied in the last class wherein we looked at a stationary thread of fluid and it was invisible. So, the total energy of the system is basically dependent on the surface area which is there of the jet.

Because it is when you have a jet and it tries to breakup into drops it is basically trying to minimize the surface area or basically trying to minimize the surface energy of the system. So, the total energy of the system is determined by the surface area which is available for the jet, okay. So, we have this as State 1 and we have the State 2 which would be a perturb jet. So, we will right now just look at one wave length it is infinite in the z direction.

So, this is my r in the radial and z is the axial. So, it is an infinite jet. So, I am just looking right now at one wave length of the jet, okay. So, say I perturb it. So, this perturbation is about a mean radius. So, this thing I will call it as r a cap. So, one thing you have to notice here is that this is r = a whereas, I have mentioned here to be r as a cap. That is important because of the geometry that we are looking at, it is a cylindrical jet.

So, when you started off with one wave length of jet and you imposed a perturbation which was of this pre-sign form. The volume which is added here in the crest and the volume which is reduced in the trough are not equal because of it is a surface of revolution. If it was just a planer thing wherein to a symmetric then the thing which is added and subtracted as same but since it is a volume of revolution.

The amount of volume that is getting added here is definitely more than what is being reduced by the trough of perturbation. Is that clear, no? What I want you to see is that I initially had a jet which was of radius a and now on this jet I am imposing a perturbation which is of a cosine nature, okay. So, what this means is that near the, at this crest part the radius has increased and at the trough part the radius has decreased.

But if you try and calculate the volume which goes to the power of r square basically because it is a surface of revolution. So, you see that the amount of volume which has been added in the crest is more than what gets reduced in a trough, okay. So, basically when you give a perturbation of this form you have added mass into the system.

So, to compensate and to equate the masses of the system you basically would have to have an a cap which is not same as a but $< a$, so that you can impose a perturbation as still conserve mass. So, the basic idea of choosing a cap which is not same as a is to impose conservation of mass, okay. If you keep r as a and impose a perturbation of cosine form then the mass would never be conserved.

You have to have a different a cap so that the mass remains conserve, okay. So, like I said the amount of work which would be required to take the system from State 1 to State 2 would be dependent on the surface area of the 2 States. So, essentially all I have to do is to calculate the surface area of the jet for this particular state and surface area of the jet for this particular state and compare for what conditions the area would increase or decrease.

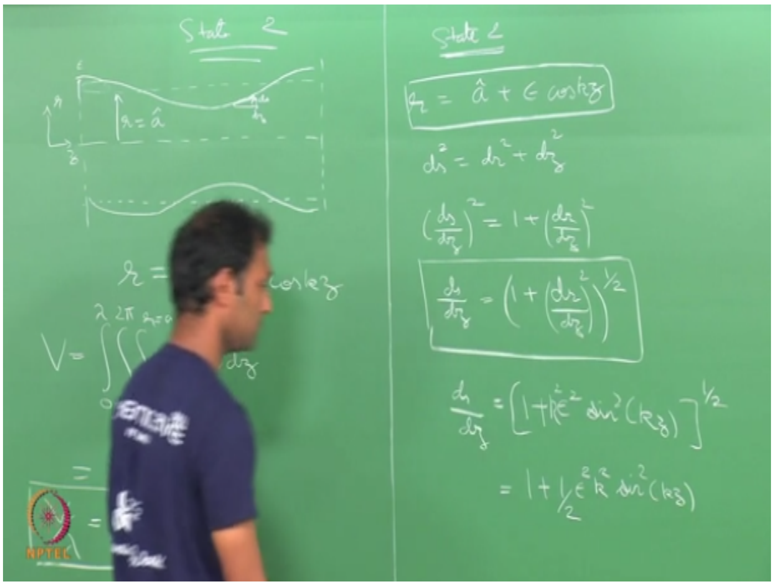
And thereby the energy would increase or decrease and therefore stability could be determined, okay. So, I will call this form of perturbation as r a cap + a small amplitude which is epsilon, okay and this its surface could be just $r = a$, okay. So, now what I would do is calculate the area which is the curved surface area, the curved surface area of the jet for this particular state and the volume of this particular state. So, let us find that.

So, areas for this particular case the State 1 would be just you know, I am looking at the curved surface area. So, it has rd theta dz which has to be integrated. Theta goes from 0 to 2 pi and dz goes from 0 to lambda. I am looking at one wave length of the jet and r is a here. So, I can just replace r by a and I would get basically 2 pi a lambda, just direct integration. So, this is the area which is there for my jet, the curved surface area.

So, I can say that the area per unit wave length would be nothing but 2 pi a, right. Similarly, I would just calculate what my volume would be. So, this is the unit volume in the spherical coordinate system. So, I integrate rdr from 0 to r which is r = a and then theta from 0 to 2 pi and z goes from to lambda, okay. There should be directly the volume of a cylinder of radius a and length lambda.

So which you could easily get to be pi a square lambda. So, V by lambda or the volume per unit wave length would be pi a square. Is this clear? So, now we do the same calculation for State 2 which is a perturb jet.

(Refer Slide Time: 13:10)



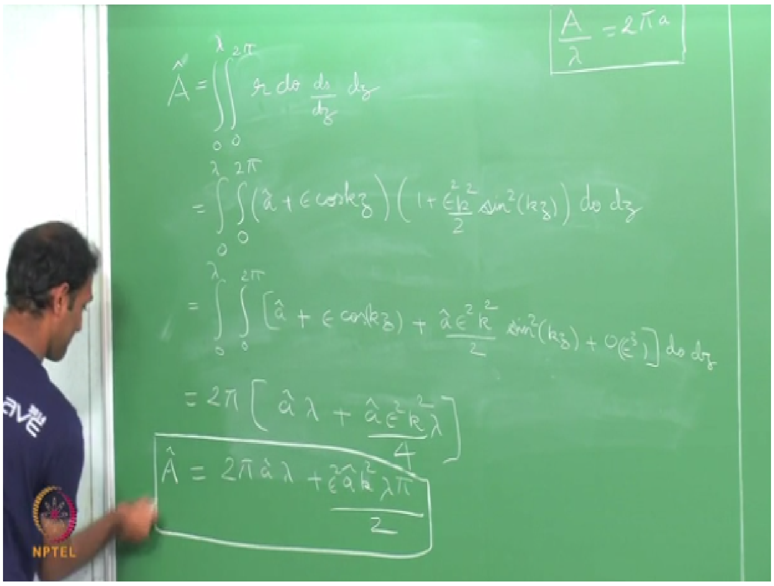
So, as we know for State 2 r is given by a cap with epsilon cos kz and if I am along the surface of the jet I can name this to be say ds which is along the arc length of the surface, okay and ds and this would be dz and dr. So, if I zoom that part, I would get ds dr and dz along the arc length, so ds is going along the arc length and z is the actual direction r is the radial direction.

So, by Pythagoras theorem I can say that ds square would be dr square + dz square. And then dividing throughout by dz, I would get ds by dz whole square 1 + dr by dz or ds by dz could be written as 1 + dr by dz to power half. So, this is important because while I am calculating the area of the perturb jet I should be going along the arc length to get the curved surface area. So, I need ds by dz, so that I can integrate directly in the z if I know ds by dz.

So, this is my r and this is my ds by dz. So, ds by dz now I can write I can get dr by dz from the equation of r here and then that would give me ds by dz to be 1 + dr by dz only this term has z in it. So, it would be epsilon square, so cos differentiation would become sin. I just substituted r in this and then got ds by dz and if I further use my binomial approximation for this I could get 1 + half epsilon square k square sin square kz. So, any doubts you here?

So, once I know ds by dz, now I can calculate my, curve surface area for the perturb jet using A.

(Refer Slide Time: 16:42)



So, theta goes from to 2 pi and dz z goes from 0 to lambda because it is again one wavelength. So, now we have to replace r by that form which is a cap + epsilon cos kz. So that gives us 0 to 2 pi 0 to lambda a cap + epsilon cos kz and ds by dz would give me 1 + epsilon square k square by 2 sin square kz.

So, I can just multiply these guys and get a cap + epsilon cos kz + a cap epsilon square k square by 2 sin square kz + an order of epsilon cube term integrated in that. Is this clear? So, I can directly integrate in theta and write it as multiplied by 2 pi because none of these terms are a function of theta. So, that would be give me directly 2 pi multiplying it and integrating the first term with respect to z.

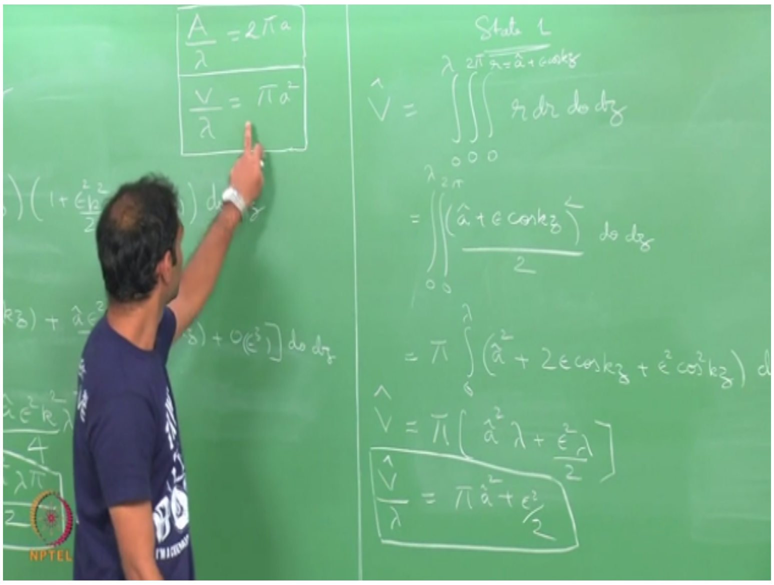
It is a constant, so I could directly write it as a cap times lambda. The first term is a constant a cap, so integrating it in z would directly give me lambda which are the limits lambda – 0. The second term has cos kz in it, so if I integrate the cos term it would become sin kz, okay and sin kz if I put the limits the bottom limit would directly give me 0 because sin 0 is 0. The top limit if I put sin kz when z becomes lambda, lambda is nothing but 2 pi by k.

So, 2 py by k gets cancelled, kk gets cancelled and I will just get sin 2 pi. So, even that becomes 0. So, essentially when you integrate you will see that and put the limits the cos term gets cancelled, okay. So, what you will be left with is integral of this with respect to z and that should give you a cap epsilon square k square by 4.

We can see that the integral of this would be just lambda by 2 and the sin square could be written as 1 – cos 2x by 2 and you will just see that if the cos part again goes to 0 because it is cos just like this term only 1 by 2 would be left and then multiplied with these constants. So you will directly get this term. So, this is the area of the perturb jet which is nothing but 2 pi a cap lambda + a cap epsilon square a cap a square lambda pi by 2.

So, since this unperturbed was called a by lambda I will just call this guy a cap or something. So, this is the perturbed area a cap which you get like this.

(Refer Slide Time: 21:49)



So, next we calculate the volume for the perturbed z which is nothing but V cap integrating so you get the volume by integrating the volume element in the 3 coordinates. So, again you have to replace r by a cap + epsilon cos kz. So, that would give me a cap + epsilon cos kz whole square by 2 replacing that here integrating this would give me r square by 2, so that is directly r out there by 2 integrating 2 pi and 0 to lambda. Is this clear?

Like integrating rdr would give me r square by 2 and r for the perturb jet was this a + epsilon cos kz. So, I just substituted instead of r a + epsilon. This r is that r, it is r here, a cap + epsilon cos kz. So, again this is none of them is a function of theta so you could directly integrate to get 2 pi. So, you will get pi integral 0 to lambda a caps square + 2 epsilon cos kz + epsilon square cos square kz.

And again the cos term should go to 0 like we had got for the area case and cos square could be again written as 1 + cos 2x by 2 and then you will see that that cos 2x part again goes to 0 and then you have left with just 1 by 2 here. So, I will get pi terms integrating a caps square would be give me lambda and this term would just give me epsilon square times lambda by 2 yes or V cap by lambda could directly be written as pi + r square by 2.

Now this is my perturbed volume per unit wave length and then what I had calculated previously for an unperturbed thing was pi a square. So, the conservation of mass basically means that I have to equate both the volume because density is the same for the liquid even when you perturb it or in the unperturbed stage. So, I could directly equate the volume and get a relation between how a cap should be dependent on a, okay.

Only once I get how a cap depends on a can I compare the areas because the area here is dependent on a and the area which was calculated for the perturbed jet depends on a cap, okay. So, if we have to compare the areas you have to get a relation between a and a cap that comes from the conservation of mass which is equating both the volumes because the density is same, okay.

So, I have V cap here, I have V here, I just equate both of them to get a relation between a cap and a and then substitute that in the equation for area and then I can compare the area and get the conditions for what k would the system be unstable and for what k would be stable. So, we will just do that.

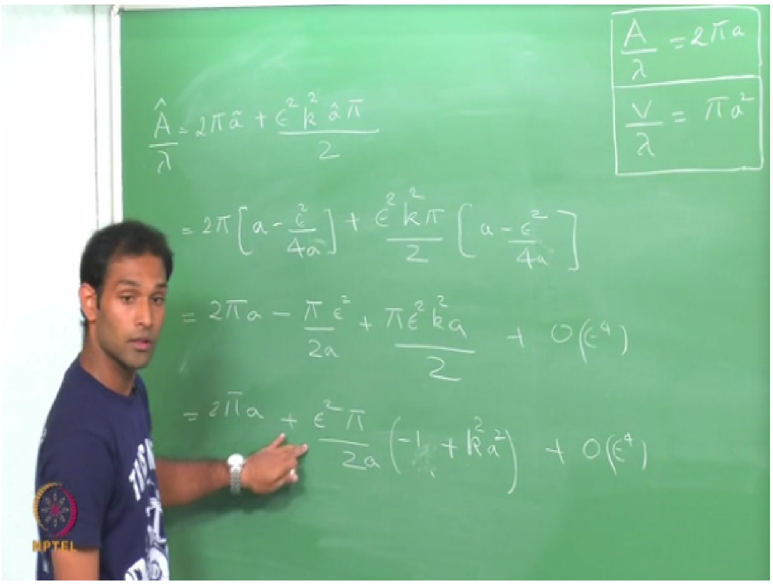
(Refer Slide Time: 26:24)

So, equating V and V cap to ensure conservation of mass we get pi a square = pi a cap square + pi. So, pi gets cancelled and then we get a cap = a square – epsilon square by 2. So, a cap would be a square – epsilon square by 2 to power half or I could just take a common out and then I could write 1 – epsilon square by 2 a square and then again using the binomial equation approximation we get 1 – epsilon square by 4 a square.

So, I will just write down the A cap by lambda that we have got was 2 pi a cap + epsilon square k square a cap pi by 2. Is it clear, still here? We just substitute the value of the expression for a cap that we have got into the expression for area of a perturb jet, so that we can calculate the area in terms of a and then compare it with the area of the unperturbed jet, okay.

So, what we will now do is just substitute a cap in this expression for the perturbed area and then compare it with what the area of the unperturbed jet was which is given by a by lambda.

(Refer Slide Time: 29:47)



So, doing that gives us a cap by lambda is 2 pi a cap + epsilon square k square a cap pi by 2. So, instead of a cap I have to replace it by a – epsilon square by 4 a + epsilon square k square pi by 2 into a cap which is a – epsilon square 4 by a. So, expanding it we get 2 pi a –, so it should be 4 a squared here and 4 a squared here, correct. So, 2 pi a and then this would be give me pi by 2 a square epsilon square – this should give me pi epsilon square k square a by 2.

Is that correct? Okay, so 2 pi a –, is this correct? Yes, this looks correct, yes. I did not carry the a square anyway, somehow. I can just write it as order of epsilon. Okay, so if I just write it as 2 pi a and take the epsilon square out and try to club this term together pi comes out let us say what else comes out, nothing else, 2 comes out. So, I am just left with 1 by a - + k square a. Is that correct? + higher order terms.

So, I could just take a common, so that I get something like this. So, for the time being we just do not worry about the higher order terms. We just compare the area for the perturb jet with the area of the unperturbed jet which is this and we see that 2 pi a was the only term which was there here and we have an additional term here which comes because of the induced perturbation that we had given to the jet, okay.

So, what we will do is we will just find what the change in area is for the 2 case that is just nothing but delta A, okay.

(Refer Slide Time: 33:45)

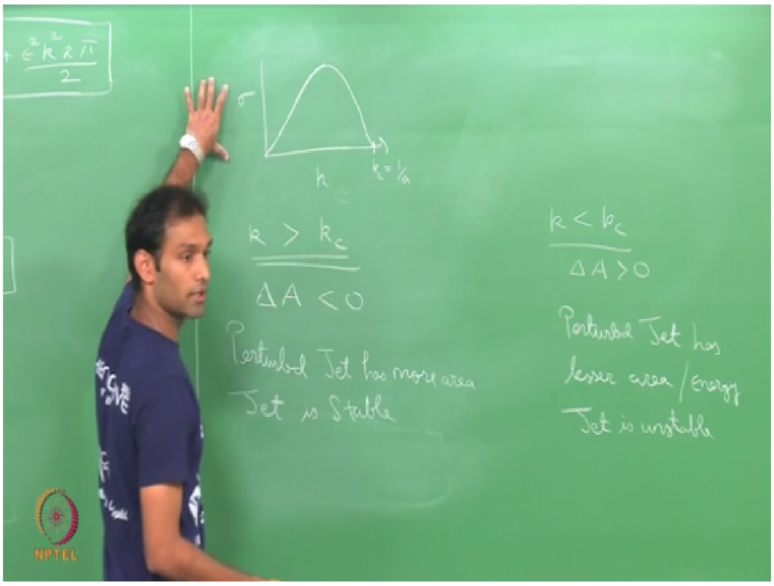
We will find to be of A lambda to be A by lambda – A cap by lambda. So, that would give us 2 pi a gets cancelled. So, this is 2 pi a A by lambda – 2 pi a + epsilon square pi by 2a into k square a square – 1, okay. Any doubts? So, this term gets cancelled and what we are left is nothing but epsilon square pi by 2 a into 1 – k square a square. I took the minus here, so 1 - k square a square. Is that correct?

I mean you could take it any way you could have subtracted this – this or this – this does not matter. So, this is my change in area per unit wavelength. So, if I try and solve for the case where delta a is 0 no which would be the condition where in the perturbation that I am giving is not changing the area. So, that is like the onset of instability where, on one side it is positive on one side it is negative and at that particular k is 0.

So, that is like the point which you had got in the linear stability analysis as k going to 1 in the non-dimensional frame. So, here if you put delta A to be 0 implies that my ka will be 1. Because this term is non-zero, so 1 – k square a square should be 0 which means ka should be 1 or I will call this a vertical k which is called the Kc or the cut off wave number or the critical wave number. So, that kc is nothing but 1 by a.

So, if you would see that the kc which is 1 by a is exactly same as what we had got through the linear stability analysis.

(Refer Slide Time: 36:19)



Remember the linear stability analysis, you had the growth rate which was sigma squared or sigma k. You had a maxima and then you had a cut off after which it was stable below which it is unstable. So, this was kc which was 1 by a. So, you get the same information by not doing any complicated linear stability analysis just by energy arguments from Rayleigh's Work Principle.

So, this is the motivation to do a Rayleigh Work Principle analysis of the system. So, what we will now do is, we will look at k on either side of kc, so k > kc which means if my k is > kc this the negative term becomes dominant or basically my delta A would be < 0. So, if delta A is < 0 which means the perturbed jet has lower area. Is that clear? If my k is > kc, so if I put kc which was 1 by a is that clear?

If k becomes > kc, kc was the thing which was making both the terms = and getting it cancelled. If my k is > kc then this term would be dominant or I will get my area to be negative which is < 0, okay and k is < kc and delta A would be > 0. So, what is this tell me about the stability of jet if the perturbed jet has lower area compared to what it was.

It means it has gone to a lower state which means the jet has gone to a lower state when I perturbed it. It has more area because it is negative, yes. So, it has more area because this term is negative, this term is negative. So, the delta is negative which means this has more area. So, if the perturbed jet has more area which means it has gone to a higher energy system energy state.

It is gone to a higher energy state by perturbation that means where it was stable. The previous state was stable because it was at a lower energy state, okay. So, the jet is stable and when k is < kc and you have delta A > 0. It is the reverse of what this would say, so my perturbed jet would be having lesser area and thereby energy, therefore jet is unstable. So, this is evident from the dispersion curve also you have positive growth rates for k < kc. So, k < kc the jet is unstable therefore it has positive growth rate.

Whereas for jet for k > kc you have negative growth rates which means that the jet is stable. **"Professor - student conversation starts"** Yeah, that is all for the class here thanks, any doubts? A = 0 should give a neutral stability right? Delta A = 0 gives a neutral stability, yes. Yes, so but so in that curve, I mean from the linear stability analysis A = 0 is also a neutral stability point but through this we do not see that?

Yes, that is true. But k = 0 is not, it is not a perturbation which would conserve mass in any case. You cannot give perturbation k = 0 and have the mas conserve. Sigma square yes, but you would get the same thing if you plot. You get the same form of perturbation. Sir, k goes to infinity then a will go to a hat anyway sir, that should …No, it would be, the jet would always be r = a cap + epsilon if k goes to infinity, k goes to 0 like it is cos kc right.

So, you would always have the r which is larger than the thing. So, you cannot actually conserve that is why k = 0 is not considered as a perturbation, I mean whatever analysis, whatever values of growth rate you get you cannot actually conserve mass by that perturbation. So, it is not than anything apart from k = 0 is considered fine. In the linear stability we took the perturbation of the interface as A + some delta right?

Yes. So there we took A and…That was because it was infinite, so it does not matter much. Here also it should be infinite right? Just considering that. No, when you are doing an analysis per unit wave length, I think it matters for the mass to be conserved. Whereas, in that I think the continuity equation and all those the equations which are there for conservation of mass would take care of the way the velocity perturbations change when you impose that perturbation.

So, that the mass remains conserve because you are solving for those mass conservation equation which is the continuity equation along even when you, yes it is a flow and then you (()) (44:10), yes. **"Professor - student conversation ends"**