Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 28
Capillary jet instability: Problem formulation

So, we will start today's lecture with this wrapping up what we did in the last class, which was the problem of the Rayleigh Taylor instability, right. So, the Rayleigh Taylor instability, we went through the math in algebra and found that when we impose the condition that we want a nonzero solution, the relationship between the growth rate sigma and the wave number in terms of all the other parameters is given by this relationship, okay.

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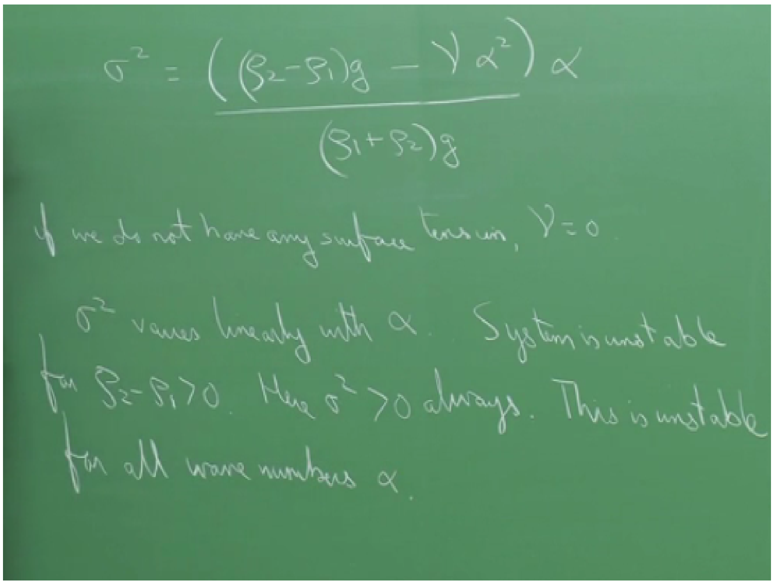

And what we were trying to basically summarize is that if we do not have any surface tension gamma is 0, then sigma squared varies linearly with alpha, okay that is what we see because this goes of, gamma is 0 and this is a constant and then system is unstable for rho 2 – rho 1 being > 0 because here sigma square is positive always, it was unstable that means, when you have the heavier fluid on top, lighter fluid in the bottom is unstable system that is what this says.

But this is unstable for all wave numbers okay, this is unstable for all wave numbers, alpha because sigma squared is always positive and the more the alpha, the more this thing, so there is no selection of a particular wave number that is going to happen, okay the higher; the higher the wave number, the more is a growth rate.

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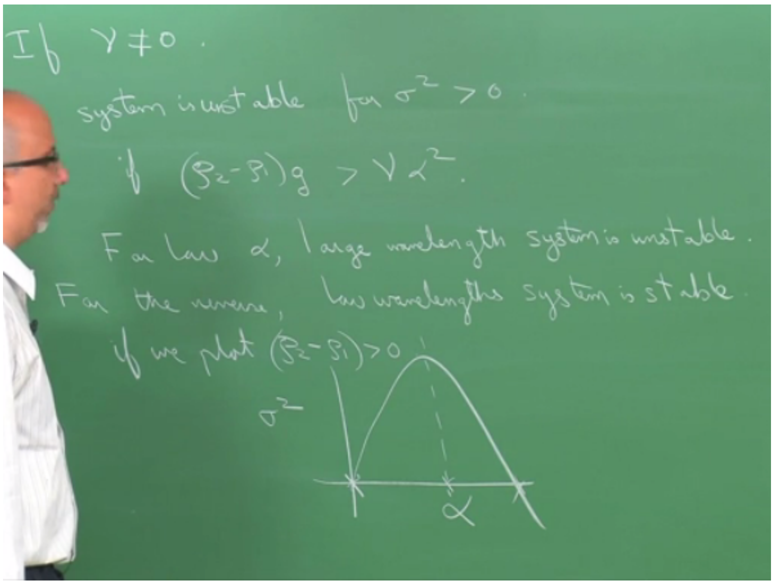

Whereas, if gamma is != 0, then system is stable for sigma squared, sorry; it is unstable when sigma square > 0, okay the system is unstable, when the sigma squared is positive because if sigma square is positive, sigma is going to have a plus the square root of that or minus the square root of that, so there is one component, which is going to be growing okay and even one component grows that means it is unstable.

So, see it was unstable for sigma squared > 0 and when does that happen? This happens if rho 2 – rho 1 G is > gamma alpha squared okay, so it means for large or sorry; for low alpha or large wavelength, system is unstable, when alpha is low, this condition is going to be satisfied, okay and for the reverse that is low wavelength system is stable okay. So, what that means is the surface tension actually is going to have a stabilizing influence, okay.

Because it is associated with this negative sign here okay that is something, which I want you to keep in mind and if you want to now plot for rho 2 - rho 1 > o, if we plot for rho 2 – rho 1 > 0, maybe I just plot sigma squared versus alpha clearly, for alpha = 0, sigma squared is 0 and that is some value of alpha depending upon the gamma for which again is 0, so there is an interval here in which there is going to be going up and coming down okay.

For alpha large; for alpha large, system is stable that means sigma squared is negative okay, for in between for low alpha, system is unstable, sigma squared is positive okay, so there is a change of the stability and what this means is there is some kind of a maxima, which you are going to see in this dispersion curve, okay. What is this maxima correspond to? This tells you the wave number, which is going to be the one which is fastest growing.

The one which is going to actually grow fastest, so these wave numbers are also unstable but what we are going to actually see the natural experiment is going to be a pattern, which is going to be dominated by this wave number alpha, okay. So, this is similar to what we saw in the Rayleigh Benard convection problem, where we found the alpha by the point where the growth rate was having a maximum or the Rayleigh number was having a minimum, okay.

So, I just wanted to point out this analogy here, yeah, Jason, is there any problem? **"Professor – student conversation starts"** Yeah, negative value of alpha, yeah we are going to look at; Yeah, so if you are talking about; you are saying alpha is negative, I am just wondering if we have actually made an assumption of alpha being positive anywhere in the derivation alpha x, e power alpha xx is what we have.

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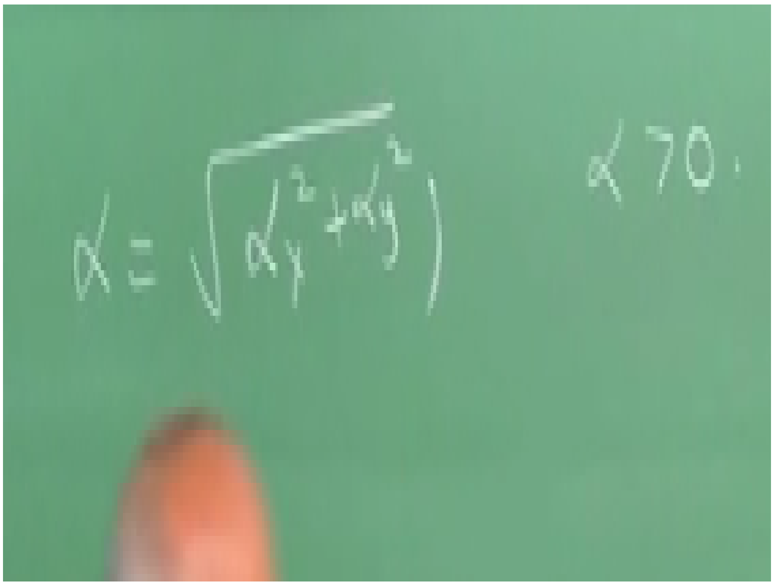

No, it was exponential, and the alpha that we have is actually square root of alpha x square + alpha y square, so alpha is positive, so this is 2-dimensional problem at alpha is nothing but square root of alpha x square + alpha y squared, okay and this is a positive quantity, yeah, exactly, so alpha is positive. So, we are looking only at the positive half of the plane. **"Professor – student conversation ends".**

So, the point I am trying to make here is when you include surface tension, there is something like a selection of a particular pattern, particular wave number okay, which is what you are going to see and that is occurring naturally in the system because of the physics, okay. So, I think we will just stop the Rayleigh Taylor discussion with this, what I want to do is go on to the next problem, which is the problem of the capillary jet instability, which we have mentioned a couple of times in the class before, okay.

Yeah, **"Professor – student conversation starts".** Yes, basically, for large alpha; the question is for the large alpha sigma squared is negative but remember, yeah, yes indeed sigma square is negative so as far as sigma is concerned, you will get a purely imaginary number that means there is no real part, the real part is 0, you will get plus or minus, i multiplied by some number okay.

So, that means the real part is 0, so you are on the boundary of stability, the threshold that is the reason I did not discuss this case, where you have the; so you really cannot tell if it is stable or unstable okay. If you know for sure is positive, you know for sure is negative then we can make this conclusion, so I am focusing only here well I know for sure is unstable, okay. Here it is purely imaginary, so it is marginally stable or neutrally stable.

If you give a disturbance is neither going to grow nor it is going to decay okay, the real part is 0, so whatever disturbance you are going to give is just going to stay as it is, okay. So, basically this portion of the curve you really cannot, so what you need to do is go for higher order terms to understand what happens. So, you can actually conclude about stability or instability only if the real part is either negative or positive.

If it is 0, you really cannot tell anything, what you need to do is go for higher order terms, okay so that is the limitation of this linear stability analysis, in a sense you can only tell you for sure a stable unstable depending on whether it is negative or positive, the real part. System is stable yeah, yeah, for no wavelength here; no wavelength is; o! you are objecting to this statement here, oh, yeah, maybe it is right, it is marginally stable, maybe, I will make this analysis, this is what I say, may be it is neutrally stable is what I am going to write; is neutrally stable.

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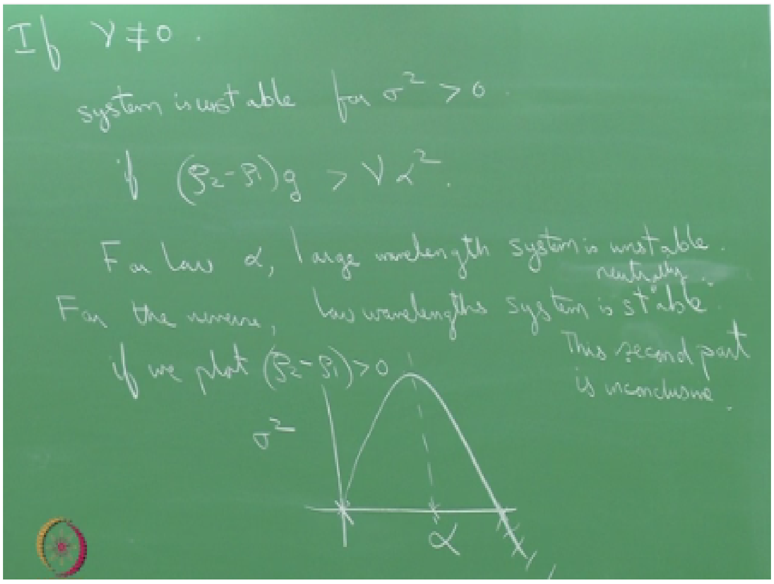

I feel if we cannot really make any conclusion, you know is inconclusive, this (()) (11:43) has a good point, okay. So, what is mean is the lesser you write the less chance of you make a mistake, so more you write, the more chance of you making a mistake, right okay, I think that is an important lesson more than either stable or unstable. **"Professor – student conversation ends".**

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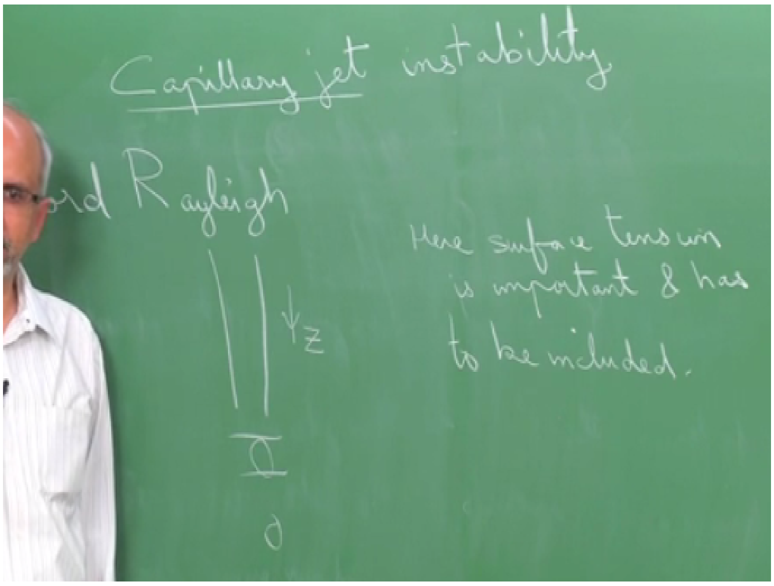

We will talk about this capillary jet instability okay and this particular problem we have analysed the; I mean, we mentioned in the past but was analysed long time ago by Rayleigh. I think we need to give them enough respect, so we will call Lord Rayleigh, okay and we are going to basically do this analysis, again is going to be a very simplistic analysis with a lot of assumptions but then there is a lot of risk information which comes out of it.

So, you can use that as a basis for doing more complicated analysis by relaxing some of the assumptions okay that is the idea. So, what is this capillary jet instability problem; is one supposing, you have a jet of liquid falling vertically down and let us that is the vertical direction, what we expect is because of the gravitational force, the jet is going to accelerate okay and because the accelerate, it is going to construct.

Eventually what you see is; this guy is going to pinch off and you are going to get drops, okay. There is an experiment which you see every day in the morning, when you open the tap in your bathroom okay, so the idea is; is it possible for us to make a prediction of the size of these drops. How does this happen, what is it that is causing this thing to break because theoretically, it can keep on shrinking and then keep on thinning down and it can go on forever.

The velocity will keep on increasing, so what is it does causing it to break into drops, it is clearly the surface station effect okay and that is the reason this is called the capillary jet instability because capillary is basically associated with surface tension okay. So, again here surface tension is important and has to be included in the model, right, otherwise you would not be able to actually get this breakup, okay.

The other thing we are going to do is we are going to do something similar to what we did for the Rayleigh Taylor problem, we are going to assume that the jet is inviscid because viscosity is not really the one, which is causing the jet break up, viscosity is only going to possibly change the rate of growth, it is not going to decide whether it is positive or negative, it is going to make the growth rate small, the viscosity is high.

So, we have a very less, very viscous fluid like honey and you just drag a drop honey from a height, the honey is not going to break okay because it is going to break after a very, very long time, so the time for breakup, the growth of the disturbance is going to be very large, okay, all the growth rate is very low okay. So, viscosity is not really going to decide this, it is only going to make it the break further away, okay.

What we are interested in is; when it breaks what is the size of the drop, supposing that is the question you are asking then you can actually neglect the effect of viscosity, okay. Anyway, we will neglect it and if you think it is important, we can always include it later.

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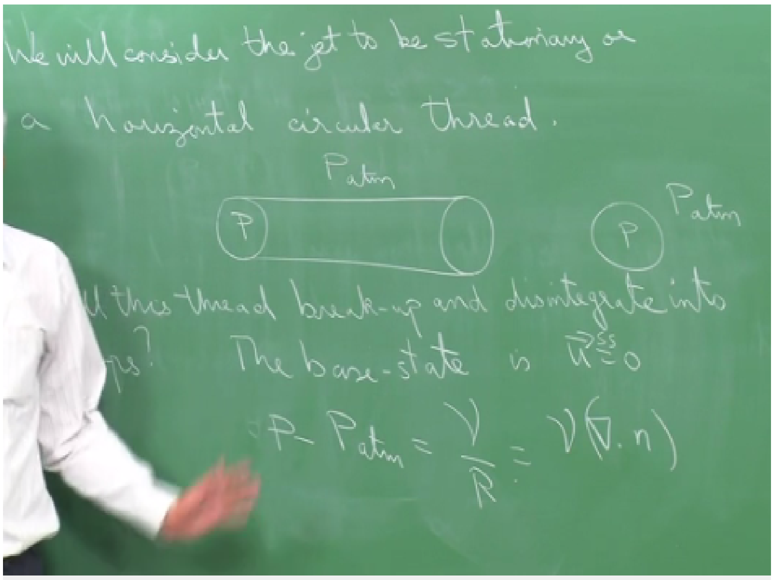

So, we will follow Lord Rayleigh and we will make another simplification, which is; we will consider the jet to be stationary or a horizontal circular, I just call it a thread, so this is very hypothetical situation again, which means you just imagine something like a circular jet with no velocity, it is just a blob of liquid, which is in the form of a circular cylinder and is stationary okay, that is your geometry.

So, the question now is; is this, if you had such a liquid thread would it break up into drops because of surface tension okay and there is no acceleration, like we had here, so that would clearly depend upon the size, where the surface tension force is going to depend upon the size of this thread. So, the smaller the thread the more is going to be the effect of the surface tension force because the surface area to volume ratio is higher, okay.

So, the question we are asking is; will this thread break up and disintegrate into drops, okay and the next question would be what would be the size of the drops that we get? So, what are the advantages of making these assumptions that I have, so whatever done I have made a lot of assumptions, one inviscid, no viscosity, I am also saying that this jet is not moving, it just sitting there, okay, which means I go back to this problem which was similar to my Rayleigh Benard problem where my base state was 0 velocity.

My Rayleigh Taylor problem, where the base state was 0 velocity, here again my base state is 0 velocity that just helps me do the analysis okay and get some insight, so we are not happy with any of these assumptions then you have to go and that become to the homework problem, right, so that is what we will do. We will assume that this is the base state, we do not assume the base state, the base state is the trivial solution u = 0, all three components assume, okay.

Liquid is not moving and clearly corresponding to this base state that is going to be a pressure distribution okay and what is going to be the pressure distribution when you have a flat interface, oh not a flat interface, a circular interface there is going to be a difference in the pressures, the pressure here is not going to be the same as the pressure in the atmosphere, okay. Let us say the ambient fluid is atmospheric and you have P atmosphere is here and this is P.

Because only one pressure which is that of the liquid that is the pressure of the liquid but clearly, P - P atmosphere equals gamma/ R, okay. This is 1/ R turns out to be del dot n, we will see that when we do this analysis. So, this is actually gamma del dot n; del dot n turns out to be 1/R okay, so that is my base state, there is a pressure difference, there is a pressure jump across this interface and that is because the liquid is actually curved.

So, if you look at the cross section, the cross section is actually circular okay and in order for it to maintain this circular shape, you need to have P > P atmosphere, it is different. So, in order to analyse this problem, use the same approach as what we did earlier, write down the governing equations, we have the base state and we need to do the linearization and go ahead with the solution.

Again, same business of trying to decompose into wave numbers, what we will do is; we are going to assume that this thread is infinitely long okay, the thread is infinitely long and so then, I can actually decompose it in terms of some kind of a Fourier mode okay. The other thing that we can do is; make a further assumption of that being axis symmetry in the problem okay that is there is no variation in the theta direction, okay that is just to make algebra easy.

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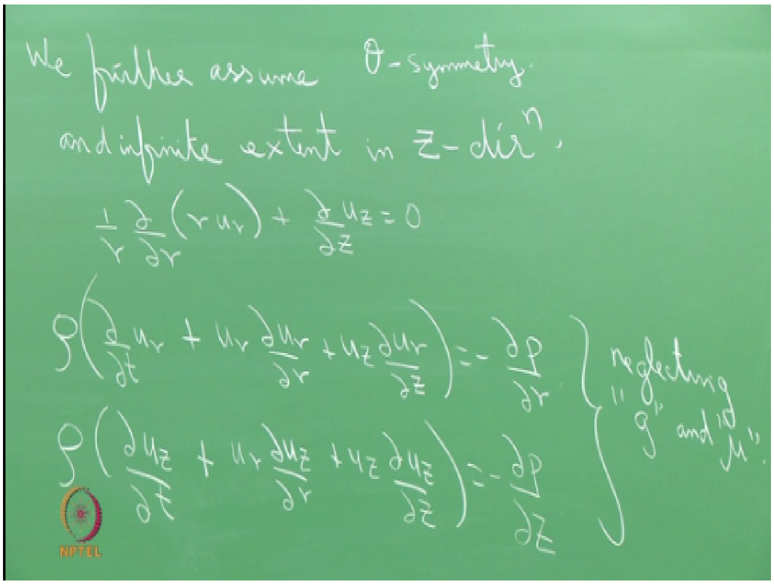

In fact, the people have worked with by including the theta direction also and there is a result which you know, you can get again analytically. So, we further assume; what do we assume? Theta symmetry, okay and infinite extent in the z direction, so it is going, it is a very infinitely long thread because if I do not assume infinite in the z direction, I need to put boundary conditions and I will get a loss, how to answer that question, okay.

So, this basically gives you some insight, so we write the equation of continuity, which is 1/r under these assumptions, this becomes this, +d/dz of uz equals 0, that is the question of continuity and then I have rho d/dt of ur + ur dur/dr + uz dur/dz equals – dp/dr, okay and then similarly I have; the viscous term does not show up because I am assuming as inviscid okay and let us say I am just doing this thing without any gravitational effect.

Because I do not think gravity is a one, which I am interested in study, if I want to study the effect of gravity, I do the vertical jet problem well, that is the one which is actually going to cause the break up. What I am focusing on is; I am focusing on how the surface tension is going to actually cause a breakup, okay, the gravity is neglected and then I have; okay, remember what we have done is neglecting gravity and viscosity.

I mean we keep surface tension but that is going to come in the boundary condition, so you can already see a little bit of what is going to happen, so those are my governing equations, subject to the boundary conditions, which are my kinematic boundary condition and my normal size boundary condition, okay those are the 2 conditions, which I need to invoke just like we did for the Rayleigh Taylor problem and the interface here, okay, so we will do that.

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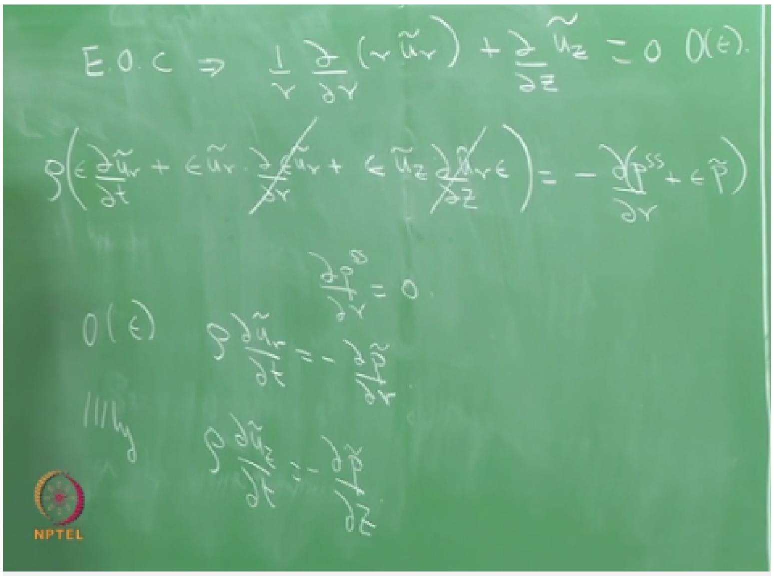

But first let me just deal with the equations here, assume urs of the form, ur ss + epsilon ur tilde okay that is my steady state that is my perturbation, which is of order epsilon. What I want to do is; ur ss is of course is 0 because nothing is moving, the jet is stationary okay and similarly, you can put for the user also. So, if you put that the equation of continuity for the perturbations, this becomes tilde + this is of at order epsilon, okay.

Do the same thing for the Navier Stokes equation, what do you get? d/dt of ur tilde + epsilon times z + epsilon times ur tilde times d/dr of epsilon ur tilde + epsilon uz tilde d/dz of tilde times epsilon again equals –dP/dr of steady state + epsilon P del tilde, okay. These terms again the convective terms are of order epsilon squared and therefore, these guys drop off an order epsilon.

The other thing which I want to mention here is; dpss/dr is 0, what we are talking about is the pressure inside the liquid in the radial direction is going to be 0 because if it is not 0, there is going to be some kind of a convection, okay. I assume there is no convection, no velocity in the radial direction therefore, dpss/dr has to be 0. What that means is the pressure is uniform in the cross section but there is a pressure jump across the interface.

Because P1 – P2, P atmosphere is gamma/ R so, in the cross section, the pressure is uniform okay. So, what this means is pss is independent of R and so at order epsilon, my linearized equation is this, you can do the same thing for the other direction. Similarly, you get rho dau uz tilde/ dau T equals – dP tilde/ dz, okay. Again the steady state, the gradient is 0 for the pressure in the axial direction.

So, as far as the equations are concerned, these are the equations okay but then I need to solve this subject to some boundary conditions and what we do is; we do not worry about boundary conditions in the axial direction, why? Because it is infinitely long and so we are going to give periodic perturbations in that direction. What we need to do is worry about the boundary conditions in the radial direction and that involves the normal stress boundary condition and the kinematic boundary condition.

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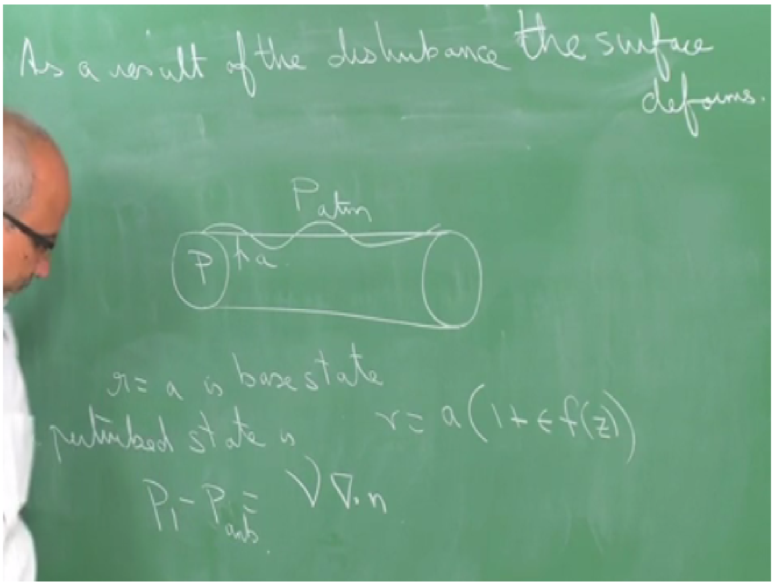

What I want to do is; I want to talk about; so, when I am doing this linearized analysis, I am going to give a perturbation and what this perturbation is going to do is; it is going to be of some arbitrary perturbation, I am drawing it periodically here but I mean is some arbitrary perturbation here, okay. As a result of which, the surface then deflects, so when we are going to give a disturbance, I am not restricting my surface to be circular and conserve volume, okay.

So, as a result of the disturbance, the surface deforms; the interface is going to deform okay that is the general situation and that is all very interesting, I have to include this in my model otherwise the guy are going to break, right, I mean we keep my interface flat, it is going to remain a circle for ever, so I need to include this thing and see how this guy is going to behave and if your radius of the unperturbed surface is a, then r equals a is the base state, okay.

And what we can do is; in a perturb state, this r equals a multiplied by 1 + epsilon times f of z, so I am giving a perturbation, the perturbation is in the form of fz, some arbitrary function of z, okay and it is a very small perturbation and that I am indicating my epsilon here, same thing as what we did for the Rayleigh Taylor problem, okay. Only thing is the Rayleigh Taylor problem, I had h, which was function of x and y okay.

But now to make my life simple, I am just saying that things are not changing in the theta direction that is the reason I am not included my theta dependence here okay because it is axisymmetric, theta does not show up, only it varies only in the z direction just keeps the algebra a bit simple but at the end of the day, you saw when we included 2 directions x and y, the 2 wave numbers is I can actually combine most of the time, right.

I got 1/2 x squared + 1/2 of squared and I just said that was equal to some alpha squared, okay. So, mathematically only it becomes slightly different otherwise, the analysis is the same okay. What do we want to do is; we want to get the normal stress boundary condition, in the normal stress boundary condition what I wrote earlier, which is P1 – P2 = gamma * del dot n that is my normal stress boundary condition.

And this is going to be valid always, this is P the actual pressure in fact, I should not write P2, this is actually P ambient, right plus constant and what I want to do now is; calculate this del dot n, which is my curvature and but I need to find del dot n for this deform surface, so that is the general case.

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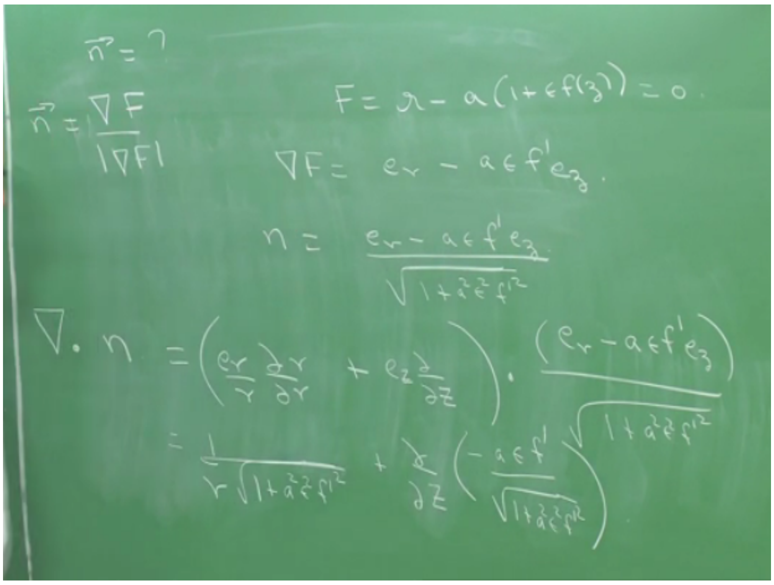

So, how do you find then, before finding del dot then I need to find n and remember this gradient of F/ absolute value of the gradient of F that is n, okay and F is r - a times 1 + epsilon f of z, okay, this f = 0. So, what is the gradient of F? er - a epsilon f prime; f prime is df/dz/ 1 +; No, this is a gradient of F, I am just differentiating this with respect to r that is associated with 1 er, differentiate with this to by z, I get a epsilon F prime and that is associated with ez, okay.

Now, n is going to be this one; so that is my unit normal vector okay and what I need to do is calculate del n, I am not sure if we did this problem already, maybe we did, remember, somebody was talking about 2 curvatures and that is what I want to show today, okay. We need to calculate del dot n. I have to calculate del dot n, it is er/ r d/dr of r + ez d/dz, okay operating on n, which is; okay.

So, what I need to do is; operate I am doing the dot product; er and ez are perpendicular to each other, is there a problem, these are fine? Taking the dot product, so er dotted to er is 1, er dotted to ez is 0 and I need to cancel a this term with that term and this with this, okay er and ez are perpendicular. What I also want you to recognize is that this particular term is a function only of z, it does contain r this entire thing is independent of r, okay.

So, for all practical purposes, when I am differentiating with respect to r, this guy is going to give me 0 but then I will have to use the product rule; r multiplied by that right, I have to use the product rule, so what I will get is; 1/r times this with that 1 + a square epsilon square F prime squared that is what this is going to give me; 1/r times this and er dotted to er is 1 and there is d/dr of r, which is 1, okay, this is just mathematics.

So, you do not have to worry too much about it but and now we are going to do this with that but now you remember that this is a function of z, so you have to use some quotient rule, like we did the last time and maybe what I will do is; just write this, +d/dz of - a epsilon F prime/ square root of 1 + a square epsilon square F prime square, okay. So, use the quotient rule now and it is going to be similar to what we did.

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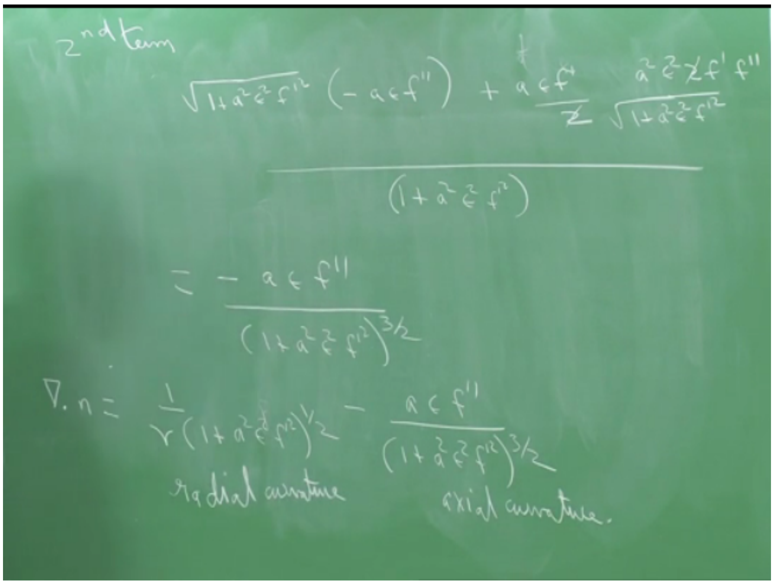

I would like to actually write the answer but I am not sure of the sign minus or plus, so I just have to do it again okay. I am looking at the second term over there, it is d/dz, we can just write the second term denominator times the derivative of the numerator, which is; I think it is minus; + epsilon F prime numerator times the derivative of the denominator, well which is 1/2 times double prime. So, you bring that to this side and you can do the simplification, okay.

Since I have done this problem before I think this is what you will get that is what we get, so minus sign, this is exactly what we got last time, you take the denominator here, multiply it and this will cancel off with this and you have this multiply by 1 contributing, so that is the second term and therefore, del dot n is – 1/r times 1 + a square epsilon square f prime square to the power 1/2 - a epsilon f double prime/; that is what we get.

Now, I want you to focus on the fact that the curvature term is actually made up of 2 terms okay and I want to give you some physical significance to these 2 terms, the significance is that this particular term is what I would call a radial curvature and this is an axial curvature okay.

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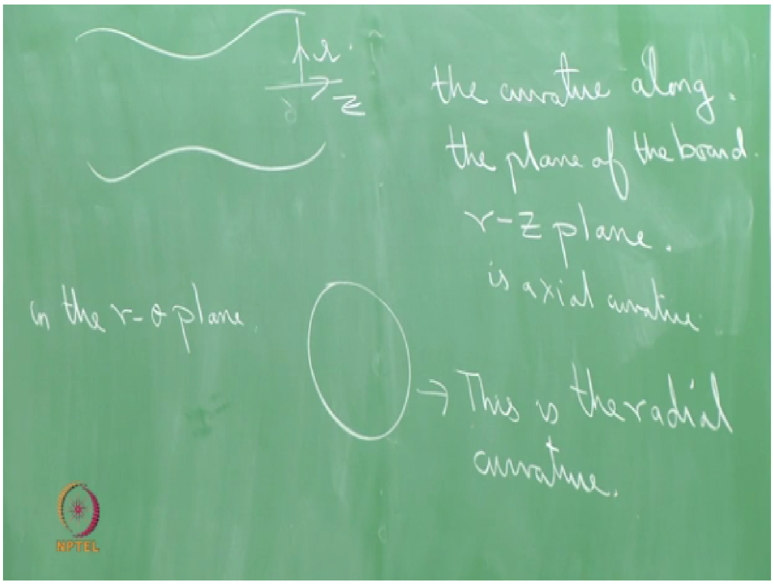

And just to explain what these things mean, look at this deformed thread, so you have a curvature along the plane of the board okay, the curvature along the plane of the board and this is the rz plane, this is z and this is r, rz plane is my axial curvature and this is my f of z remember, this is f of z, so this is associated with my f double prime, okay. In the r theta plane, the r theta plane is actually perpendicular to this, okay, it is circular.

Why do I say circular? because I assume theta symmetry. If you want we can assume non circular but basically now in 1/2 theta plane, this is the shape, so there is a curvature of the cross section and this curvature is radial curvature. So, essentially 2 effects; one is a radial curvature, one is the axial curvature and both of them together give you the actual curvature, which you have to include in the problem, okay.

So, well, we have got this particular thing, del dot n, what we have to do is; we have to use this in our boundary condition; the normal stress boundary condition and the normal stress boundary condition we have to again do a perturbation series analysis, get the term of order epsilon to the power 0, get the term of order epsilon to the power 1, okay and because the equations of the order of epsilon, the boundary conditions also have to be of the order of epsilon.

The way I am going to convert this to order epsilon is just by doing a binomial series expansion and take this to the numerator, take it to the power - 1/2 and do a power series expansion, do the same thing there and find out what is the term of order epsilon, what is the term of order epsilon to the power 0, okay. So, once we do that I mean the normal stress boundary condition is clear okay.

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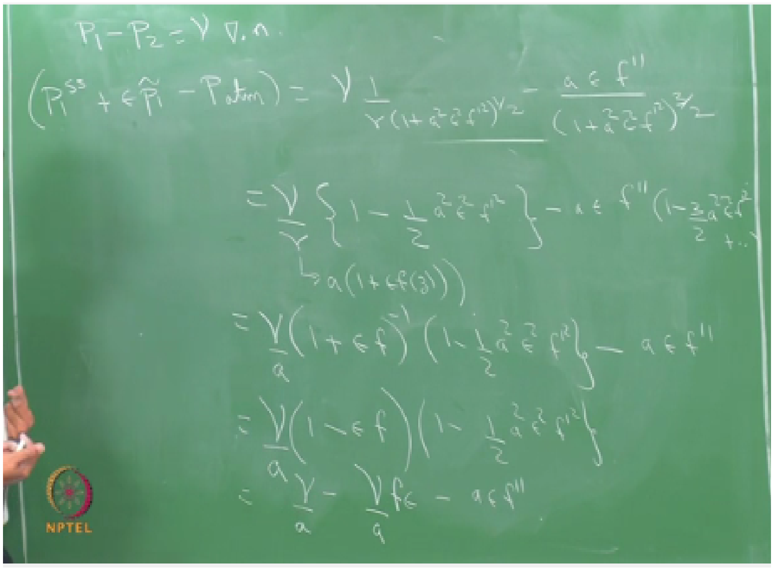

The normal stress boundary condition, remember it is P1 – P2 equals gamma del dot n and I am going to write this as P1 ss - ; oh sorry; epsilon P tilde - P atmosphere, P1 is P1 ss + epsilon P1 tilde, P2 is P atmosphere, outside liquid okay, equals gamma my surface tension time is this guy okay, so do your binomial series expansion 1 – 1/2 of a square epsilon square F prime squared etc etc, right, is that right,

Yeah, **"Professor – student conversation starts"** But the r can change with z, right yeah, that is correct, that is what we are going to get, let us do this, then you will know yeah, it means that is what we are going to find out, at order epsilon, we will find what it is; what it is; is r epsilon to the power 0 and what it is at order epsilon to the power 1, okay. We will find that out when we do this, so you are talking about this term; right you talking about this term.

You are telling me whether it is going to be always of order, whether it is going to be always equal to 1/r, yeah but the r can change with z yeah, No, I do not; It does not mean that extra term is = 1, the radial curvature you are saying is =1/ r and no, that this is the radial curvature okay and let us do look like this; let me finish my; I will answer your question but let me finish this analysis and then we will come to your question. **"Professor – student conversation ends."**

So, how does this binomial series thing work; 1 / 1 + nx - a epsilon times F double prime times 1 – 3/2 times a square epsilon square F prime square + etc., okay, this is right, yeah, so now I have forgotten something here, this r itself is varying with z, I need to include the fact, r is actually 1 + a epsilon f of z because I am going to be evaluating this along the boundary, at the boundary r is != a, for a perturb surface, r is actually 1 + a epsilon f of z, okay.

So, this has to be; I need to do a binomial series expansion of this as well, take you to the top, so this is now give me; a * 1/2 you are right, yeah, yeah, thanks. So, this is going to be gamma times by a times 1 - epsilon f to the power -1 times 1 - 1/2 times a square epsilon square F prime squared, I do not worry about this term because this is a higher order term, okay when I multiply this and this is going to give me - a epsilon F double prime, the rest of the terms are higher order terms.

Here, I can come up by a 1 + epsilon F times 1 - this thing, I am not sure is this is a signed problem. Where is that 1 +? Yeah, that is good, it should be plus because this would be minus yeah, now everything is fine. I am wondering, if I made a mistake again, so this gives me gamma/ a – gamma/ af epsilon – a epsilon f double prime. So, I do not know if this answer your question that the radial curvature is actually gamma/a I mean at order epsilon.

It can be written as gamma/a – gamma/ a f epsilon, so this is your radial curvature when you do the binomial series expansion, okay. What I want to do is; I want to look at the left hand side and the right hand side, these terms r of order epsilon to the power 0, this is going to balance this term, which is of order epsilon to the power 0, this P1 tilde is going to balance the order epsilon terms.

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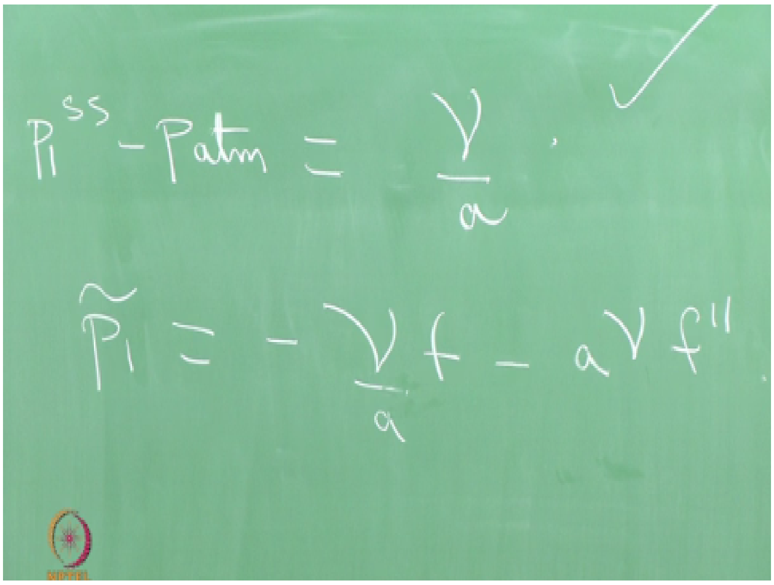

So, what this means is the boundary conditions; the normal stress boundary condition gives me T1ss - P atmosphere equals gamma/a and P1 tilde is = - gamma/ af -; I think the gamma multiplying this also right, gamma multiplying this; gamma af double prime yeah, that is basically what your normal stress boundary condition is at order epsilon, so this is how your perturbation pressure is going to vary, okay.

So, all I have done is written the normal stress boundary condition to order epsilon, I mean whenever you have any term like something having an epsilon; a function of epsilon in the denominator or a sin or a cosine term, you are going to do a Taylor series expansion and then reduce it to order epsilon, a power series epsilon that is what we have done okay and this is remember fine, because the base state that is what I expect and for the perturbation, this is what expect, okay.