Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture - 3A
Stratified flow in a micro channel: Effects of physical parameters

So welcome to the third lecture in Multiphase Flows. Today what we will be doing is we will carry on from where we left off in the previous class, where we were studying Corcoran Stratified Flow between 2 infinite plates. So let me begin with a recap of what we did.

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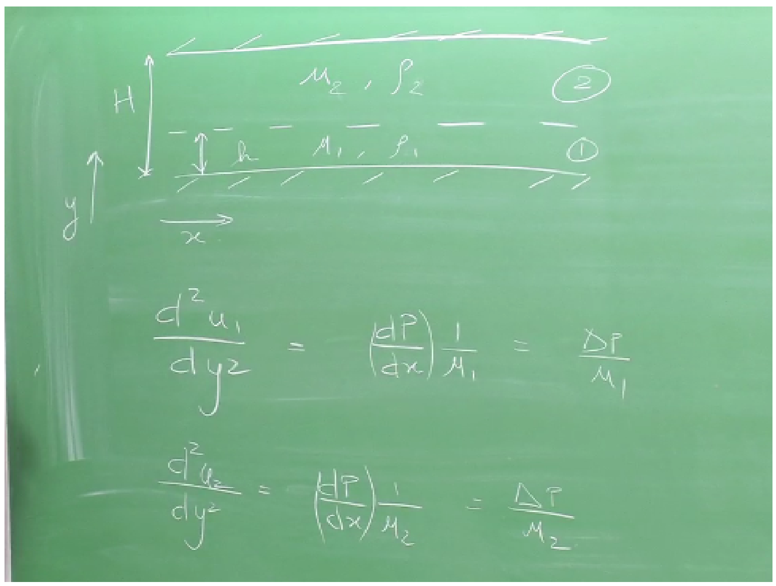

So in that system we had basically Corcoran Flow and micro channel which we decided to model as flow between 2 infinite flat plates. The width of the bottom fluid which I will call here fluid 1 have a width h, the channel of course was capital H then I had fluid 2 here means this was the x direction and this was the y direction. And for generality both fluids have different physical properties. What we proceeded to do was to take the Navier-Stokes equations granted.

And then we started simplifying the equations using the assumptions that we believed were appropriate for the situation. And how appropriate they were we will know when we reach the final stage of modeling which was validation with some experiments. But for now we proceed. And as you would have done in the previous class ultimately we will land up with a set of ODEs for the velocity field.

Further as we showed the gradient in the x direction was a constant which is an outcome of a unidirectional flow which was all also fully developed as we saw. So therefore for convenience I shall write this as-- believe that do not have the negative sign here. So what we will do in this class is I want to show you how we can get information from the system even without solving these equations. So essentially there are 3 important steps when we look at modeling a physical system and trying to use applied math to understand the physical world.

The first step is to understand the system of interest and to write down a model where made a number of assumptions and that is what we have just completed. The second step is to obtain a solution and that we also did in the previous class. The third and most important possibly of all these is to analyze the equations and the result that we get. So by that what I mean is essentially 3 aspects.

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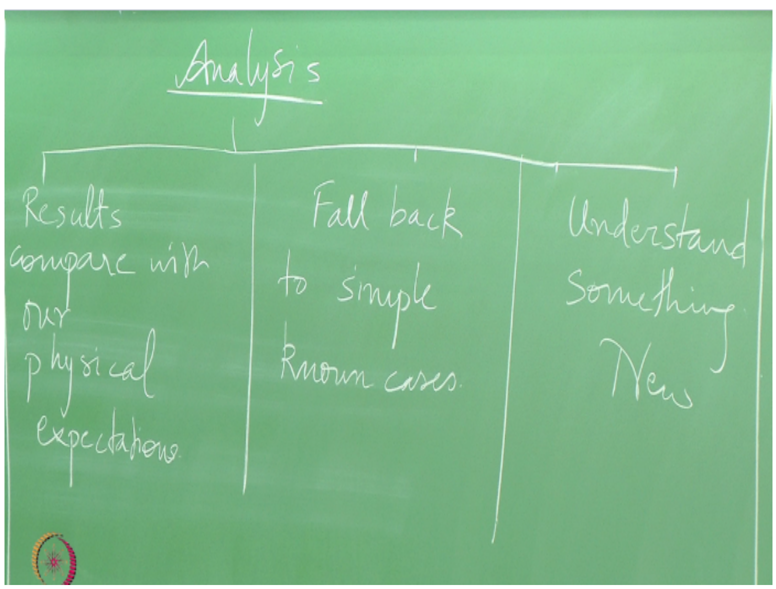

So the first point is to make sure that our results should be seen in comparison with our physical expectations. So our physical intuition about the problem and an understanding of the physics should tell us something-- I mean we should have some expectation of what we are going to see and then we should compare that with the result and see whether they match, and if they do not then either the model is wrong somewhere or we got a wrong solution or our understanding of the physics has to be devised.

The second important problem is to fall back to some simple known cases. For example, we may know the behavior of this problem in the case of single phase flow, and in the 2 phase problem we will get some new variations on the physics and what we would want to see is whether this 2 phase flow falls back to the single phase, so these sorts of things work as checks on the result that we have.

And the third and the whole objective of the exercise is to, understand something new. So these are the 3 aspects that I will be looking at in this lecture. And the important thing to realize is the process of analysis does not start only after we get a solution. In fact it should start the moment we write down the mathematical model which-- by which I mean the equations with their boundary conditions.

And that is what we will do precisely today without actually doing any computations and without using the solution that we got in the previous class. We will try and see how much we can flesh out of the equations. How much of knowledge we can obtain by just looking at these 2 equations. Okay so that is what we would be doing. So I before progress we need some boundary conditions on both these equations, right.

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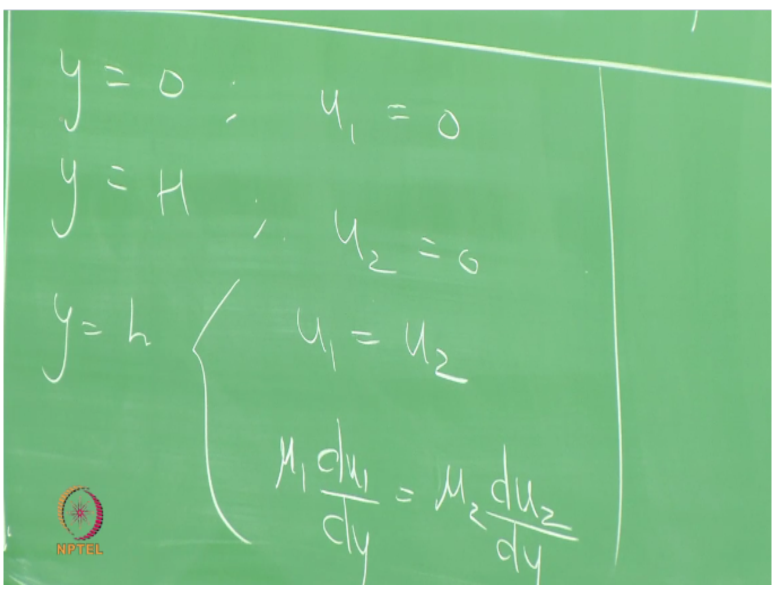

So at-- the bottom of all of course we had the No-slip condition at the top wall we had again No-slip on the top fluid. At the interface we had a pair of conditions that came from the continuity of velocity where both tangential velocities were equated. And finally we had the tangential stress balance. So let us begin. So the first thing to do is when we have an equation of this form, some second derivative equal to a constant term essentially.

To try to understand what is this equation actually trying to tell us. So of course we know a message of solutions which, where we can plug-in a formula that we have studied and obtained the result. But very often the form of the equation is considerably more complicated. So then it helps to try and understand what each term of the equation is actually saying and what you think will happen if you increase the term or if you drop another. So that is what we will try to do.

And since we have to couple equations here we will start with the simpler case of the single phase flow, just to illustrate what I mean. So let us look at the stratified version of the problem now.

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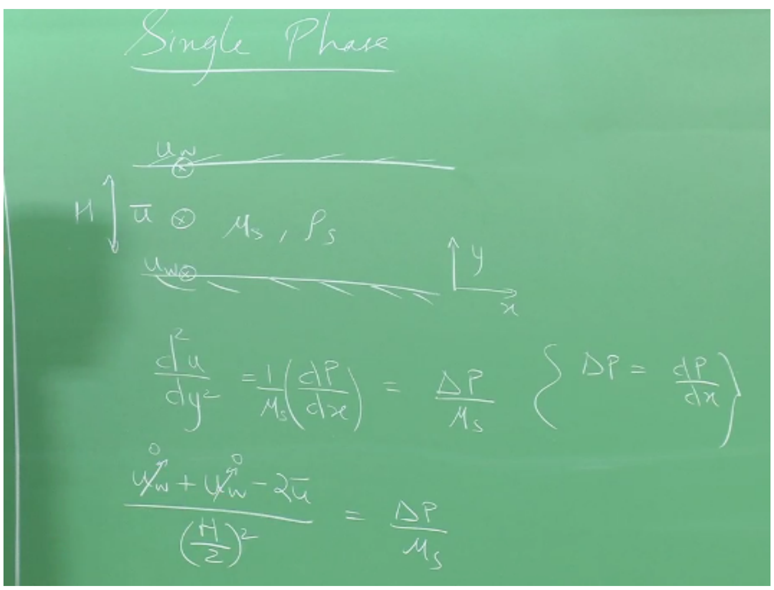

So for the single phase case we just have a single fluid flowing between 2 flat plates and now we just have viscosity and density of the single fluid, and as before the height is capital H. So we write down the same creeping flow equation to describe the unidirectional flow. And to follow my previous notations, so here you should note that this is the Delta symbol and Delta P is simply dP/dX which is constant for this case of unidirectional fully developed flow.

So now our objective of looking at this problem is to try and get some feel some understanding of how this viscosity and the pressure drop actually impacts the velocity field between the 2 plates before and then once we do this we can take this understanding forward to the 2 phase problem. So to get some insight here what I am going to do is replace the second derivative with a simple 3-point finite different scheme.

So consider now 3 points within the domain, 2 at the bottom plate and the top plate and one along with centerline. And we label the velocity is here as u wall and u bar for the velocity at the centerline. And now we can use a central difference second order scheme to write down the-- to replace the second derivative. So in that case I will simply get velocity at one wall at the top + velocity at the bottom wall - minus twice the velocity of the centerline divided by the separation between those nodes which in this case is just capital H/2 the whole square.

So I have simply replaced the second derivative with its central difference formulation and this is equal Delta P/Mu S. So now from this I can realize here that the velocity of the wall is of course 0 because of the No-slip boundary condition. So these 2 terms fall off to 0. And I can solve this now to get an estimate of the velocity at the center U bar.

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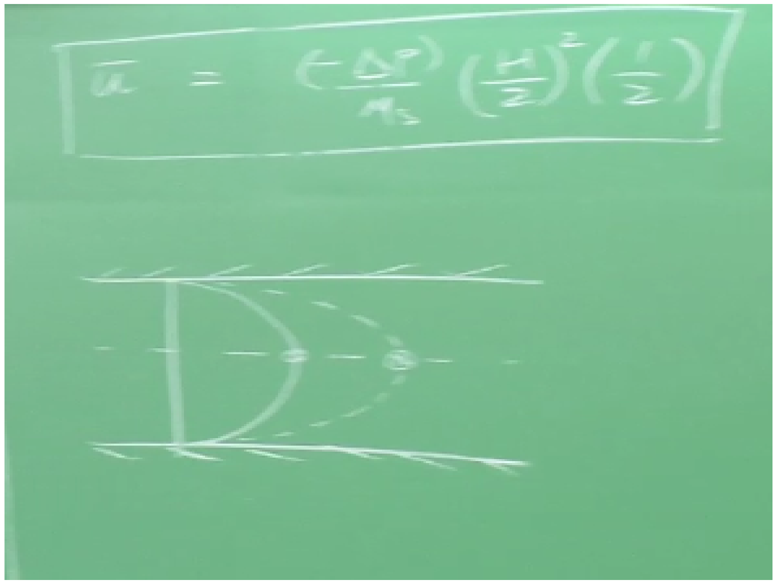

Right, so that is by expression for the velocity at the centerline. And now you can begin to see how the parameters in the problem are affecting the velocity field. So you can directly see from here that if I have a greater pressure drop driving the flow of course you should remember that in the forward x direction the pressure is falling so actually Delta P is negative and so - of Delta P is a positive term and the velocity is positive in the x direction. So the greater this driving pressure drop the more will be my velocity at the centerline.

The greater the viscosity the low will be the velocity as we would expect. And here we can see that for plates that have a larger gap again the velocities higher. So if we plot this and we look at the centerline, so I would have 0 on both ends and a velocity at the center. And I would get a parabolic type of velocity profile and as we were recognized as the Hagen–Poiseuille flow. And so if my Delta P is higher all my gap width is greater or my viscosity is lower I would have a higher value of Mu bar and in even stronger parabolic profile.

So of course you can ask the question where should the maximum of velocity lie? So that answer is that it will lie at the center line. And the reason for this is not from any calculation but simply a question of symmetry. So you see that the problem is completely symmetric about the centerline and below and so it follows naturally that the velocity profile itself should be symmetric about the mid plate which is why we have the maxima exactly at the centerline. So U bar will represent the maximum velocity.

And so now we see how these different factors that are affecting the flow profile. And the important thing to realize here is that we have been able now to understand these effects without solving any actual equations. So after getting an understanding about how these parameters affecting the problem we can of course come back and solve the original ordinary differential equations and then get some quantitative results to match with our qualitative understanding.

So bearing that in mind let us move on to the 2 phase problem at hand which is considerably more interesting.

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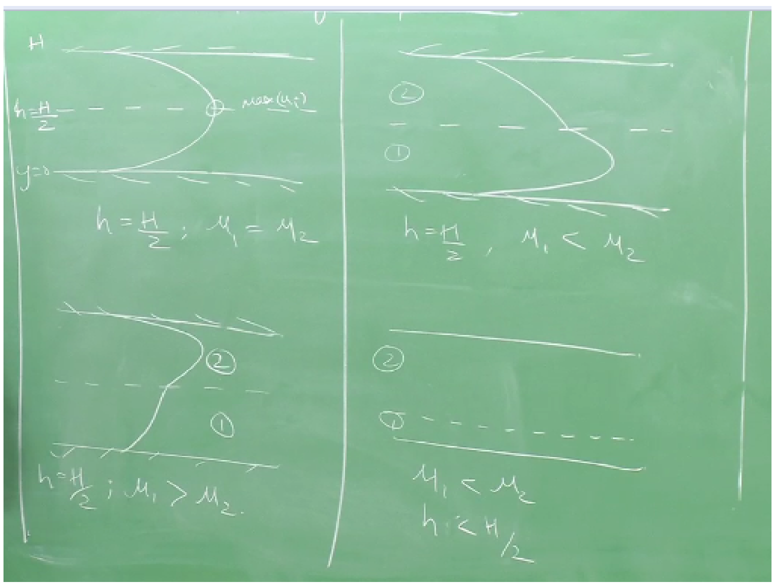

Now what we want to do is we want to anticipate the flow profile. And to do this we will use that understanding that we have just got from looking at the single phase problem. Now here this is the interface which I will to begin with look at exactly at the center. So this is y=0; H=H/2 and this is H. So in other words this is the case of and now I still have my viscosity is n and density is 2 consider. Of course we know that in this simplified version of the problem the density does not come in.

So we are really just looking at the pressure drop viscosity of fluid 1; viscosity fluid 2; location of the interface and the height of the channel. All right, so what do you think is the first gas we can make about the profile? The most immediately obvious thing that we can say once we have understood the profile over there, so that simply what I mentioned as a second point in the analysis procedure we can fall back to the case that we already know which is the single phase case and that will simply we are saying if we have equal viscosities and also put the interface right at the center.

And so what you will find in this case is just a single face profile where here we have the max velocity. So that was pretty straightforward. Now let us move on to interesting case that we have established this. I will leave the interface at the center again. But now let me make the viscosity of the bottom fluid < the viscosity of the top fluid. And what we need to ask ourselves is how will the characteristic features of this profile change which basically means where will be the maximum move, for example.

So what do you think would happen if I have the bottom fluid < viscous than this one? Maybe half as viscous, going back to a single space problem we realize that naturally the velocity we know now is inversely dependent on the viscosity, so smaller viscosity fluid would flow faster. And in this case because the low fluid is less viscous we would expect it to have a higher average velocity than the fluid above it.

And then if we note the boundary conditions that we had of continuity of velocity and the quality of the stresses you would expect the maxima simply to shift down. It gives us something like that, this fluid 1 and this is fluid 2. So now that, that we have understood this problem, the inverse case of mu2 being > mu1 again follows immediately by symmetric because there is nothing that tells me that the bottom fluid should be fluid 1.

If I just did the whole problem with the bottom fluid-- top fluid as fluid 1 I would immediately get the other case. So it would follow obviously that the—they might not look exactly some symmetric but that was the intention that I have fluid 2 bearing the maximum of the Velocity field. So now we have seen 3 interesting cases. However, in all these cases I have kept the interface at the center.

So now I want to ask a question where things get a little interesting, is what do you think will happen now. If they look at this case let us say and keep the viscosity of the fluid below it low as low as you feel like maybe 10 times lower and now when I have the interface at the center, we all agree that the maxima will remain here. But what if I were to move the interface somewhere closer to this wall? Or if I arrange a flow such that the interface is closer to the bottom wall or in other words, fluid 1 has a shallow width.

Then do you think the maxima will lie in fluid 2 or in fluid 1 given that mu1 < mu2 and H is also < H/2. **"Professor – student conversation starts"** (()) (19:45) which is H and Mu S. So Mu will definitely < mu1 but H is higher than mu2. Good, so as you pointed out that-- if you just look at the bottom fluid for a moment, you will see that it has low viscosity which will push up its velocity but it also has a low height if you just look at the height of the bottom fluid. **"Professor – student conversation ends"**

So then the question is whether the viscosity always dominates H or not? And maybe from here we can make a guess because you can see that the H goes as squared whereas the 1/Mu goes linearly. But assuming we did not have that information, a simple way we can arrive at the result which in fact happens to be that; you have the maximum moving into the top fluid even though the top fluid is more viscous.

Is because as I move the interface down I should ultimately fall back to the single phase case that has to happen, no matter what viscosities you have ultimately I have to go back to single phase if I make the layer of fluid 1 smaller and smaller. And in the single phase problem I know that the maxima lie in the center of the channel so therefore as I make the height of fluid1 smaller ultimately if I were to draw this as a centerline ultimately the maxima of the entire flow profile have to approach the centerline which will now align fluid 2.

So even if we do not know anything about this problem and even did not even know the model we should still be able to say this.

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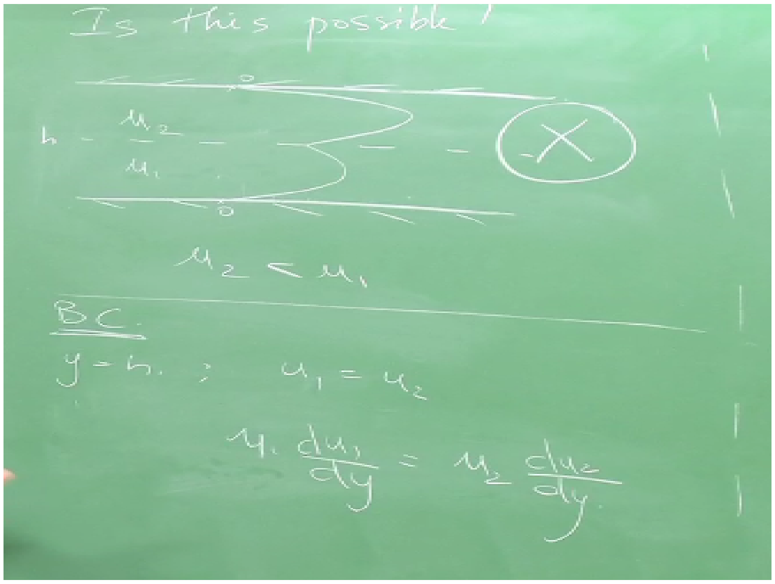

So before we try and collate what we have understood let us pose a question. Is this flow possible? Can I have maxima in both fluids? Can the velocity-- so I know the velocity is going to be 0 here and 0 here. Now the question is can to have 2 maxima maybe it is > other fluids. So this would still satisfy the case of mu2 to being < mu1 so I have the maxima there, maybe even the H in mu2 is larger.

"Professor – student conversation starts" So the question is can we have such a profile? Maybe we can do a show of hands, how many people think we can have such a profile, can raise your hands? So all of the rest of you --how many people think you cannot have this profile? This invariably happens. So we have a 50% 'No' takers, but that is all right. So what we will do know is the answer I will let you know is that this is impossible. As the question is why it is impossible. **"Professor - student conversation ends"**

So the simple reason is if you go back to the boundary conditions at the interface at y=h which is the interface, I will have u1=u2 where you will say that there is no problem there so the velocities are equal. However, I need to have a balance of the shear stress. In fact, you can think of this balance as nothing but Newton's third law that the stress exerted by 1 on 2 is reciprocated by 2 on 1.

And what you will see mathematically in this is that although the slopes are multiplied by viscosity and therefore their magnitudes can be different they both have to have the same sign otherwise there will be residual stresses on the interface that is basically what the stress balance means you cannot have residual stresses on an infinitely thin layer of interface fluid. And here you can actually see that the derivative is positive in the top fluid and negative in the bottom fluid.

So therefore we can rule this out immediately by again just looking at the equations and the boundary conditions of the opposing. Okay, after all of this you can ask me maybe a boundary conditions are wrong like suppose it slips at the wall or maybe even this is wrong maybe I have some of magnetic nanoparticles along the interface and then there is some electric fields somewhere and it just gets pulled along and how do I know that this works.

And of course here I have arranged the problem in such a way that what I am saying is going to happen. But then when you are studying natural systems in biological systems the same idea is hold you will try to apply the same conservation laws and there you may not be so sure. So that is very once again you try to see whether your intuition matches the theory and whether its theory matches the experiment.

So dismissing this velocity profile we will now look at the possible cases that we have anticipated.

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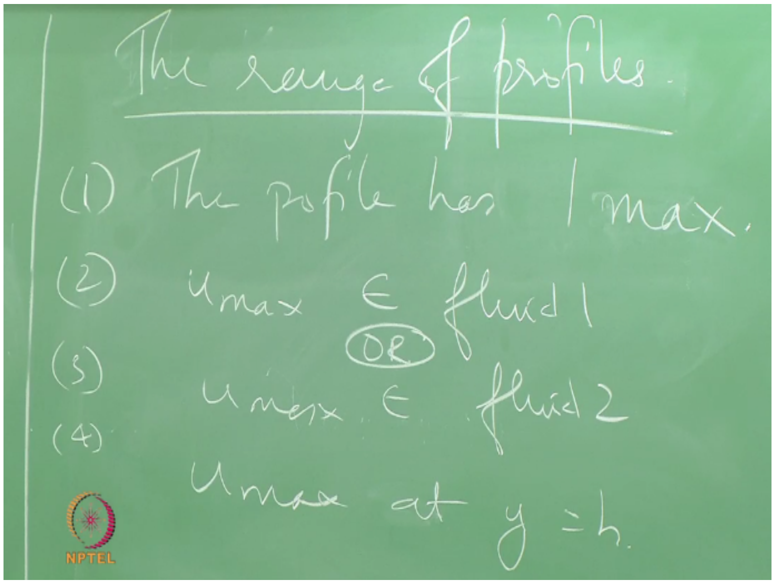

So the first point is that the profile has 1 Maximum. Second point is that this umax can belong to fluid 1; the umax could also belong to fluid 2 and there is a 4th possibility which I have not spoken about it and that simply that umax should belong or should be at y=h. This follows naturally. From these 2 cases somewhere at some point it will be at y=h. So having identified with the combination of analysis and intuition the 4 different flow profiles that we can get.

The next question that we need to answer is where will I find these in the world of the parameters or in other words a parameter space investigation.

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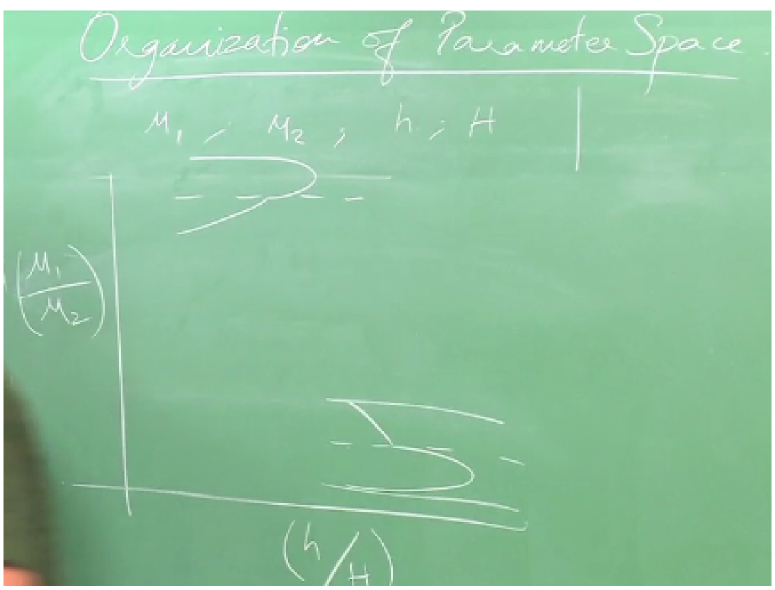

So let us call it Organization. So which are my parameters, I have viscosity of fluid 1 as a viscosity 2, I have the height h and capital H. I also have of course Delta P but that is somewhat like the magnitude parameter because it affects both fluids equally so it is unlikely to change any qualitative to make any qualitative differences in the profile. So therefore there are only some parameters which will affect the qualitative nature.

The statement I just made about Delta P not affecting the problem qualitatively is we will show that more rigorously in the next 2 lectures where we look at scaling analysis. But for now we proceed with this, simplified version of the statement. So the question is for which viscosity of course we have some (()) (27:41) of what is happening here but as we know all these cases were for H= I mean at the at the midpoint interface at the midline.

Now what would happen if I start moving the interface up, you know at some point the maxima will shift. So therefore there is an interesting relationship between the viscosity relation of the 2 fluids and their thickness ratios. And somewhere as I vary these parameters I am going to get different profiles. So the question we want to answer here and which will be the third aspect of what I wrote down for analysis, learning something new.

And that answer that we are trying to give to the question is where for which parameters or for which fluids I will get the desired profiles. And this is an important question because there are many applications where maybe fluid 1 is a stream containing is impure water containing some sort of toxic chemical and fluid 2 is solvent. In such a situation you will want the water to be completely purified.

So ideally you will want the water to spend the maximum amount of time in the channel. And that would happen assuming the interface is at the center; that would happen if the water flowed with a low velocity. So you have a lot of the solvent you do not mind sending the solvent through replenishing it and recycling it several times, but what you really want is the water to come up here. So then you will want to adjust you cannot of course change the viscosities.

But knowing the viscosity I would want to adjust the interface location so that my water flow is slowly. So then what you are going to draw right here is going to be important. So what I am looking at is some kind of a flow regime map where I have parameter perimeter along this axis. So in this case I am going to take a beautiful leap and write instead of viscosity 1 and 2 separately I write their ratio.

Instead of h and capital H separate I write again their ratio. And now you can ask me how do I know that the viscosity will always play together as a ratio and the height would play together as a ratio. Well the only answer I have right now is again based on our intuition and what we figured out from the equations themselves that it seems that it is just really about which viscosity is greater and which thickness is greater.

To be sure what you would have to do is go back to the solution that you have and try to derive this result with the expressions and see whether you can get all your results in terms of viscosity ratio and thickness issue rather than a viscosity separately somewhere. So more than that I cannot say now but once again when we look at dimension analysis and scaling which we will do to tomorrow we can make a positive statement about these 2 parameters.

In other words, I can tell you that the viscosity of fluid 1 will not come somewhere in some term by itself. It has to come with the viscosity of fluid, so right now we will proceed in this manner. So what I know from here is that if the viscosity of fluid 1 is high which is this case then I will have a profile that looks like that. On the other hand, if the viscosity of fluid 1 is low I will have a profile that look something like that. And somewhere with h I can also switch between the 2.

So what I really want to do is divide these 2 guys and the best way to divide them is to somehow track the case where the velocity is maximum at the interface. So the question is how I will arrive at that condition?

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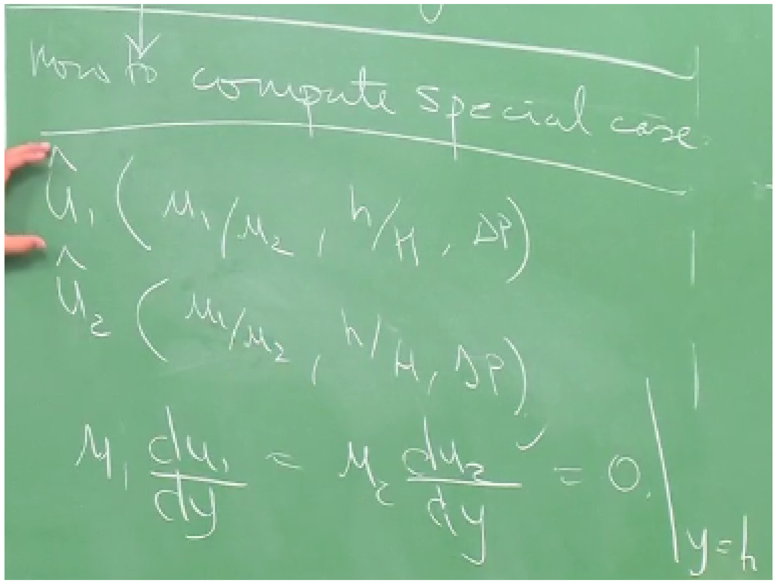

So how to compute the special case? So what we have already with us – so now we need the solution. So we have reached at certain points where we have got this whole picture in our mind and now we want to specific the answer. So by looking at the equations we have arrived at a specific question rather than you know simply generating results with the expression we have solved. So the specific question is for what combinations of viscosity ratio and thickness will have the velocity maximum with interface.

So to answer that question I use the velocity profile that I have, so I have u1 I put the hat to note that it is a function of the viscosities. And I am ascertaining that expression can be written in this way. And of course I have Delta P. But as I pointed the Delta P but as I pointed the delta P has to somewhere kept knocked out because it is not going to change a qualitative. I have also u2. So the simple way to determine where this is going to happen.

If you try and identify something special about this case where the maxima are at the interface something in addition to what we already said. So what we have already said is given as 2 expressions, I need to get something out of this by imposing another condition. So the condition, can anyone tell me what the condition will be at this interface? So let us look at the slopes for example.

So from the boundary condition that we had where the stresses were equated we had a viscosity ratio. So therefore the slopes are always going to be different when mu1 != mu2. See you see here the slope magnitude jumps and the same case happens here. The only case where the slope will continuous smoothly through for unequal viscosities is if we actually have the derivative going to 0, because both size will balance out and that is precisely what will happen when we have the velocity maximum at the interface.

So that is nothing but saying that my derivative with y has to be 0 at the interface. And that would already satisfy this condition. So what I am trying to say is that I will not changing anything because I have the balance of shear stresses. So this already holds in this problem. In addition, what I am saying is that both of these things must be = 0, of course at y=h. So now what you would find is if you take your solution for any of the expression because this part is already satisfied.

You take u1 take its derivative and put it = 0 and rare in that condition you will get precisely some curve in this play which will be some function of viscosity ratio as definition a critical curve.

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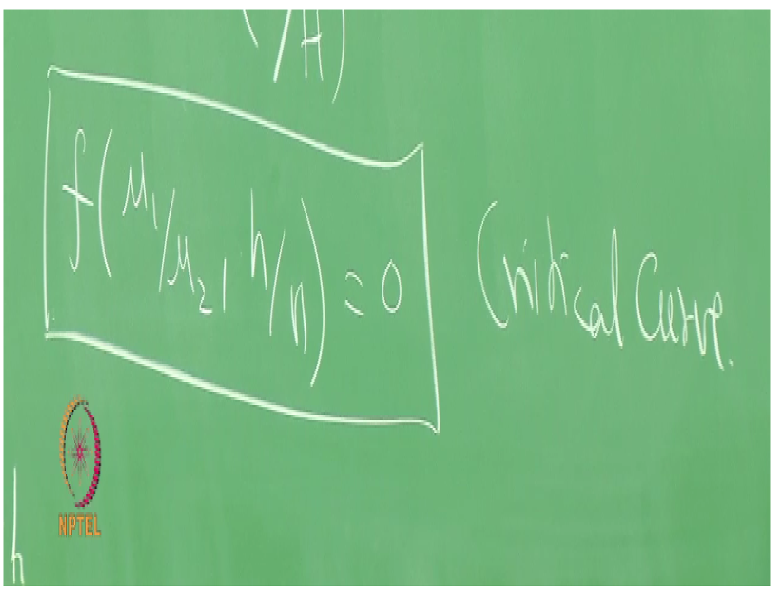

And that will follow when you equate the derivative to 0. Now of course this might be an explicit relation or implicit one which means that I could maybe directly write mu1/mu2 at some function of h over capital H or I might have to solve this numerically but that should be relatively easy to do using MATLAB or mathematical. And what you will find is ultimately we will get some curve that splits this to (()) (35:38) down.

"Professor - student conversation starts" I leave this as a bit of mystery for you to think about whether it is going to reach 0 or not. Again do this without actually computing. **"Professor - student conversation ends"** So and exactly along this curve I will have whatever be the height being exactly 0 there. I mean the derivative being 0 the maximum located at the interface. So this simple curve has now split entire parameter regime into the different profile that we know exist in the problem.

And this is the kind of curve which you can now fold into your pocket and walk into an experimental laboratory, that actually give some information and some new understanding to the people who are doing those experiments.

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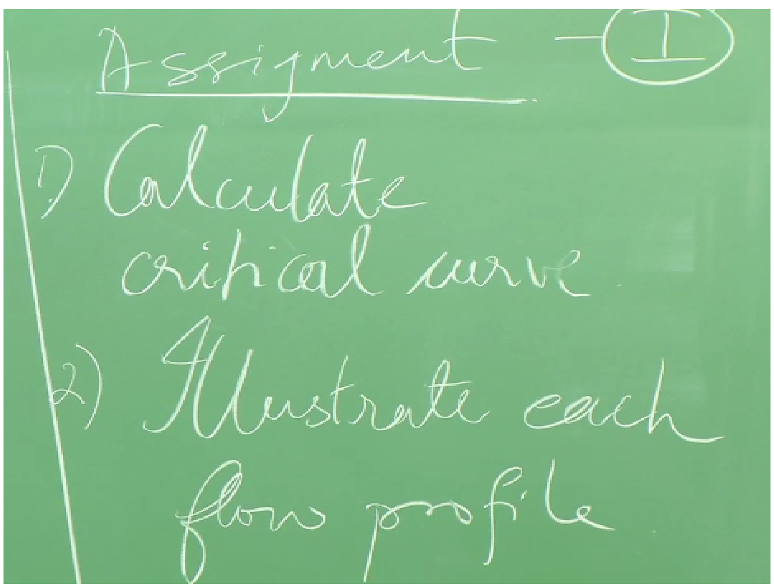

"Professor - student conversation starts" So this brings us to your first assignment the first of many hopefully interesting one to come. So since I am the TA I will ask the Assistant TA to just copy down the assignment and maybe send it to the (()) (36:57). **"Professor - student conversation ends"** So the assignment is to calculate the critical curve by which I mean this curve by the procedure.

I will just illustrate. And you might have to get the curve by using MATLAB or Mathematically. Once you have this curve you can take some points here, here maybe one guy there and illustrate each flow profile. So by that I mean that once you have the critical curve you choose some parameter values plug them in u1 and u2 and plot MATLAB plots of the velocity profile. So that is part 1 and part 2 of Assignment 1.

When you do these assignments be sure to pay some attention to specific details by which I mean choose some cases that are statically pleasing after all I need to correct or my assistant TA needs to correct some 20-30 odd assignments. So what I mean is when you draw this plots make sure that the axis labels are clearly visible and make sure that you plot those lines properly, you know why not put a dash line along the interface and so on.

So these things will help you first of all understand the expression themselves better and it will be—it is a much nicer way of getting a feel of the problem. Okay so in the last few minutes of this lecture I want to introduce or talk about for a little something one the more practical aspect of this whole problem which is--.

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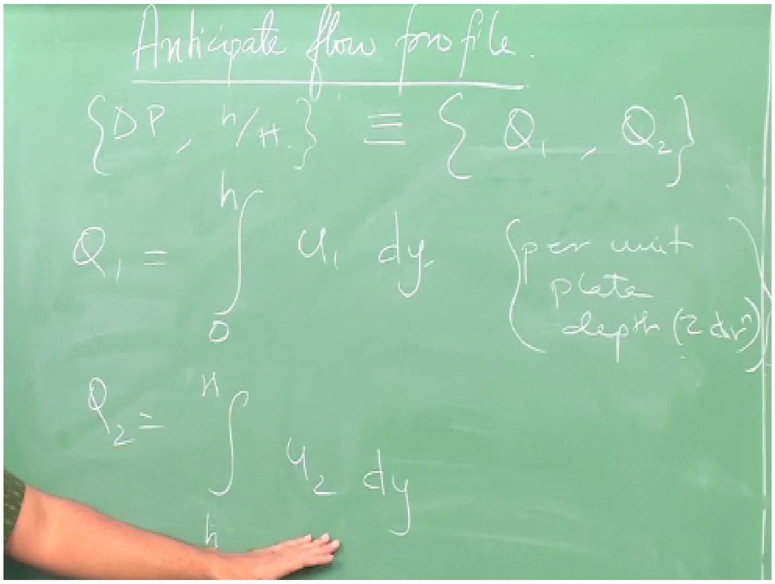

How do you really relate the pressure drop and height the problem? So the pressure drop was something we applied, and throughout this problem I have implicitly assume that I have control over where the interface is located and that is why I was able to – look at different cases where the interface is on top 1 and bottom and so on. But actually in the experiment how may I going to control the interfaces?

So in fact these 2 global operating parameters for a given specific fluid which has specific mu1 and mu2, these are the 2 things I am saying I can control. But in fact what I can actually control is not this it is related quantities which are the flow rates. So as an experimentalist I have 2 syringe pumps and I will pump in this fluids at certain flow rates that I can control. And based on the viscosities I will get some interface location and some Delta P.

So the reason we calculated it this way because it is way for calculation but ultimately if you want to achieve this we need to come from the flow rates. So the only way we can do this is to of course compute the flow rates using the expressions we have for u1 and u2. So that is not too difficult. So Q1 will integrate from 0 to the interface u1 dy. Q2 will integrate from the interface of the top wall u2 dy.

Of course these are per unit plate; let us say depth by which I mean there is a z direction which I have not considered. So that is pretty straightforward to do. And once we do this what we will have essentially is a relationship involving 2 equations between Q1 and Q2 and Delta P and the thickness ratio.

So now if I know my Q1 and Q2 I can go back and calculate this or alternatively if I want a certain Delta P and H, I can use these 2 equations and figure out which flow rates I need to pump in. And of course that will change as the viscosity is of the fluid change.

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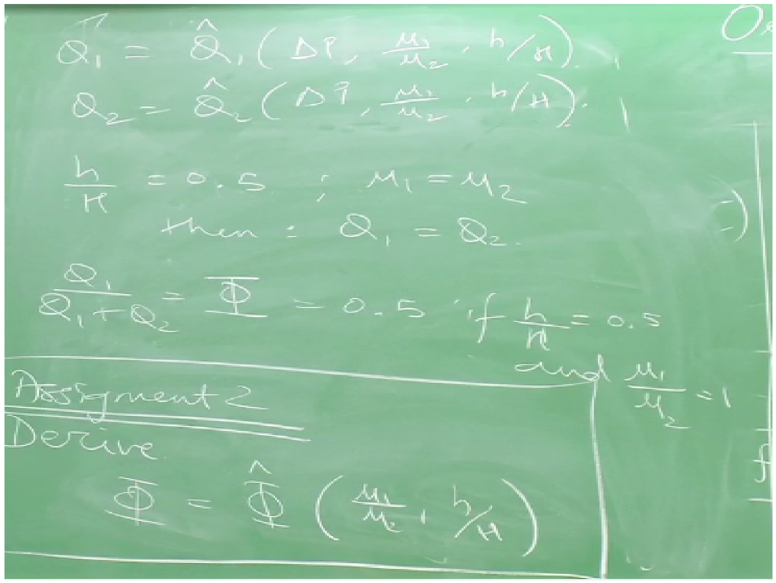

So from this what I will ultimately get is some relationship Q1 as a function of Delta P mu1/mu2, h/H. So now once again let us see if we can make at least one a periodic prediction about what these equations will tell us. And how we will do that we will try to fallback so once you get this result you try and fallback to case you know which you would say the single phase case. So if we put h/H to 0.5 and we say mu1=mu2.

Then it was trying to (()) (42:38) that that a flow rate should be equal because then you will have a symmetric and the problem is symmetric the reason I have done this is more than to fallback to single phase is to look at a symmetric case when the top and bottom are same viscosities same heights so there is no option for the problem about to give me Q1/Q2, no matter what is going on in that problem. And other way to state is this is that the flow rate fraction which I will define as capital C = 0. 5; where if h/H=0.5 and mu1/mu2=1.

Right, so this should be clear. It is a symmetric case. Now once again now that we know the symmetric case we need to ask ourselves what will happen if I deviate from this special case. In other words, suppose I make fluid 1 more viscous, what is going to happen? So probably if you make fluid 1 more viscous the flow rate of fluid 1 would drop based on our understanding of the problem, and the reverse would happen if you make fluid 2 more viscous.

But suppose my interface position wants to move, now then which flow rate would change? Now from the experimentalist point of few if he pumps in equal flow rates of fluid that have the same viscosities he is confident the interface would be at the center. But now he changes one of the fluids from water to glycerol and then pumps them through the channel, his interface is not going to be at the center anymore because they have different viscosity. You know then maybe he has to adjust for that in whatever reactions he carrying on.

So the next task which is the task of assignment 2 is to basically derive this expression. So in assignment 2 you use the u1 and u2 you have plug them in here get expressions for Q1 and Q2 in this form. And you should be able to get a viscosity ratios and the thickness ratios. Then compute the flow rate fraction and ultimately give me the flow rate fraction as some function of the viscosity ratio and the thickness ratio.

Somewhere in this process the pressure drop will again drop off because it just a magnitude parameter. So assignment 2 is to derive the relationship between flow rate fraction, viscosity ratio and a term that is know in Chemical Engineering (()) (45:31) is the holdup is something that you come across during the course. So the holdup which tells us what is the relative volume fraction of the 2 phases or which phase occupies larger part of a channel and so on.

And as we will show you in some in the future classes that this relationship plays a very important role, because in some 2 phase flows you will want to estimate an average property like someone wants to know how to apply the Moody's chart say for a 2 phase flow, and he does not have a chart with him he has only the chart for a single phase problem, that an empirical relationship we have for single phase flow, I think he needs to make a quick estimate for 2 phase.

So one way is you can say let me get the average viscosity. And then if I take it average viscosity of the problem and plugged into Moody's chart I may get some ballpark figure. But now you have a problem because the guy who is pumping the fluids only knows is Q1/Q2. He does not know the actual volume fraction which is h/H. So he may naively think that my flow rate fraction will be equal to my volume fraction or my Holdup fraction and actually weight the viscosities with the flow rates. But that would be wrong.

If our viscosities are not equal and if the height is not 0.5, so that you would actually have to weighted with holdup, so this is—it is an important thing. And what we will do is wind up the class for today and come back to this in the future class with some experimental results and try and see whether our calculation thus far and the assumptions, we have made holdup with the real world.