Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture - 26
Rayleigh-Taylor 'heavy over light' instability

So, welcome to today's lecture on multiphase flows. What we will do today is, look at another problem in fluid mechanics and the idea is that, this particular problem will be different from the earlier problem of natural convection, in the sense that the earlier problem, we only had one phase okay and although towards the end, we will discuss the hypothetical problem of the liquid with being surrounded by gas. We did not allow the interface to deform, the interface was remaining flat.

Today, what we will do is, we will look at a problem where we will allow the interface to deform okay. And what that means is you would have to use things like the kinematic boundary condition, which we derived a few lectures back and also the other boundary conditions, the normal stress boundary condition. So, in that sense this problem is one level more complicated because we are going to consider a truly multiphase flow problem with 2 liquids and an interface which is actually deforming okay.

And the idea is that we would make assumptions again to try and get a analytical insight into the problem.

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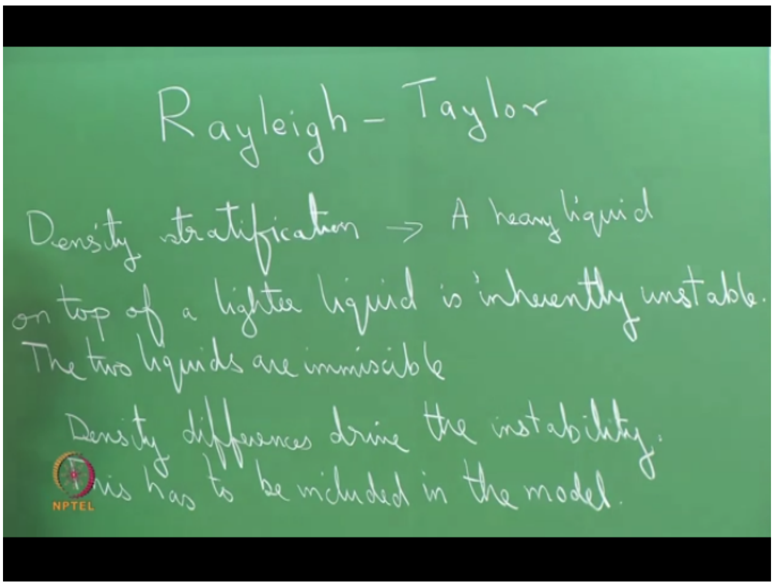

This problem is called the Rayleigh-Taylor problem okay and this is essentially an instability which is going to be driven by a density stratification. So, for example when you have a density stratification and a heavy liquid on top of a lighter liquid is inherently unstable, you all know that. We have 2 liquid layers and these are immiscible liquids okay. So, you can think in terms of water which is usually denser than most of the other liquids like on top of another organic solvent okay.

So, this is inherently unstable and what is going to happen is, the water will have a tendency to come into the phase which is below and then the oil phase is likely to rise up. And if it had been the other way, if the lighter liquid is on top, it is going to stay as it is okay. So, what this means is, the instability is going to be driven primarily by the density differences and that is something which we have to make sure we include in the model okay.

So, basically what I am saying is, density differences drive the instability and this has to be retained and included in the model okay.

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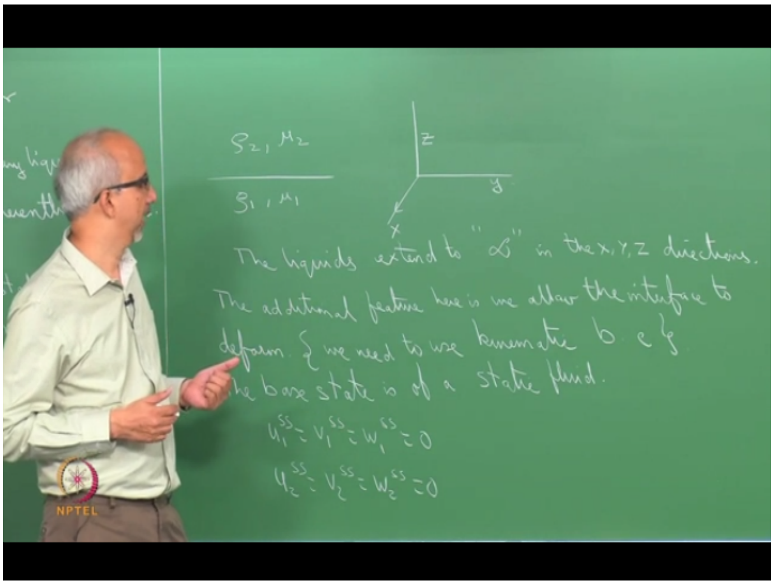

So, this is the geometry that we are going to be looking at. This is liquid one, which has density rho 1 and viscosity mu 1, this is liquid 2, which has density rho 2 viscosity mu 2 and we have a situation where this liquid is just lying on top of the other. I mean you have been very carefully you added this other liquid and then these 2 liquids are having this flat interface.

What we want to do is, we are going to ask the question, if I were to disturb this interface, how exactly is the system going to behave okay. And x, y and z as my coordinate axis, z is the vertical direction. And to keep life simple, what we will do is, we do not want to have to worry about all these boundary conditions. So, we are going to assume that the liquids are going to extend to infinity in all directions in the x direction in the y direction and the z direction okay.

So, the liquids extend to infinity in the x, y and z directions okay. And what I want you to keep in mind as we work out this problem is, the close analysis similarities with what we did for the daily problem the natural convection problem. Because, I think that is basically what you are going to be doing whenever you are solving a problem okay. So, the analysis procedure is the same the only thing is now, we have to worry about the interface deflection.

So, the additional feature here is we allow the interface to deform okay. And which we did not do in the earlier problem. So, what that means is, we need to use the kinematic boundary condition okay for instance. So, that is the level of complexity. What we will do is, we want to consider as usual a base state and then give a perturbation around the base state right. So, what is going to be the base state? The base state is going to be 1, where everything is at rest.

This liquid is at rest and that liquid is at rest. That means all the velocity components are 0 okay. So, the base state is that of a static fluid. I mean both the phases are static. Which means I am going to say u1ss=v1ss=w1ss=0 subscript 1 tells me is this liquid, subscript 2 tells me it corresponds to the velocities of the second liquid okay. And similarly, that means that is the state I have one liquid resting on top of the other completely stationary.

And what we want to do is, we ask the question is this stable or unstable. Of course you already know the answer in the sense that if the guy is denser at the top you expect it to be unstable okay. But, then we want to go through the calculation and see how exactly is this instability going to manifest. Is there a condition which comes because in addition to the density difference, there is also going to be a surface tension of this interface, as this interface, we need to look at.

Does this have a stabilizing influence? So, we are going to look at the influence of the surface tension, the density difference and may be even the viscosity. But, to begin with, we are going to assume the viscosity is not really going to play an important role because viscosity is a friction and what you are focusing on is an instability which is driven by what is happening at the interface. So, what is more important is for you to include the surface tension effect.

So, to begin with, what you will do is, we will assume that the 2 liquids are in visit. Because, what viscosity will do is only going to slow down things. So, viscosity is going to possibly change your growth rate or disturbance and to modify the growth rate. It is not going to really change whether it is going to be positive or negative okay. That we will see.

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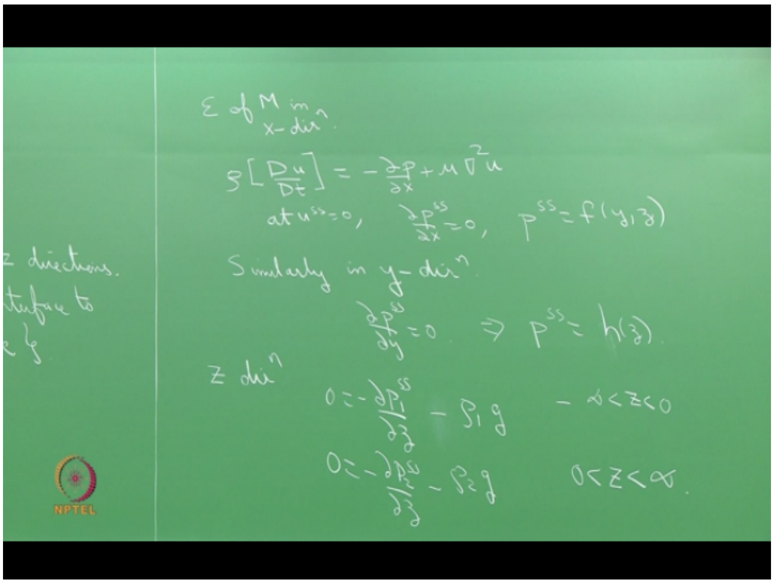

The base state, how do you find the base state? We found the velocity being stationary but we need to get the pressure gradient okay. So, if you want to simplify the equation of motion in the x direction okay, what do we have? Rho du/dt=-dt/dx+mu del square u okay. Since the velocity is 0, this reduces to dp/dx being 0 okay. At uss=0, we get dpss/dx=0, which means pss is a function of y and z only. Because there is no change in the x direction okay. If you look at the y component, similarly, in the y direction we get dpss/dy=0 okay.

Because the gravitational field is in the z direction. And what does this imply? This implies that pss is a function only of z okay. So, in the base state, that is perfectly understandable because you only have the pressure gradient are vertical direction. In the horizontal direction, there is no pressure gradient okay. That is what you conclude from this equations of motion. And as far as z direction is concerned, what do we get? We would get 0=-dp1ss/dz+z I have showing it going upwards okay.

So the gravitational field is in the negative z direction-rho 1g, this is for the first liquid okay. And that means the first liquid is extending from -infinity<z<0, whereas in the second liquid, I have it extends from okay. So, this clearly the hydrostatic pressure gradient which is what everybody understands. So, what I am going to do is I am going to integrate this out and I am going to get that pressure is going to vary linearly with z.

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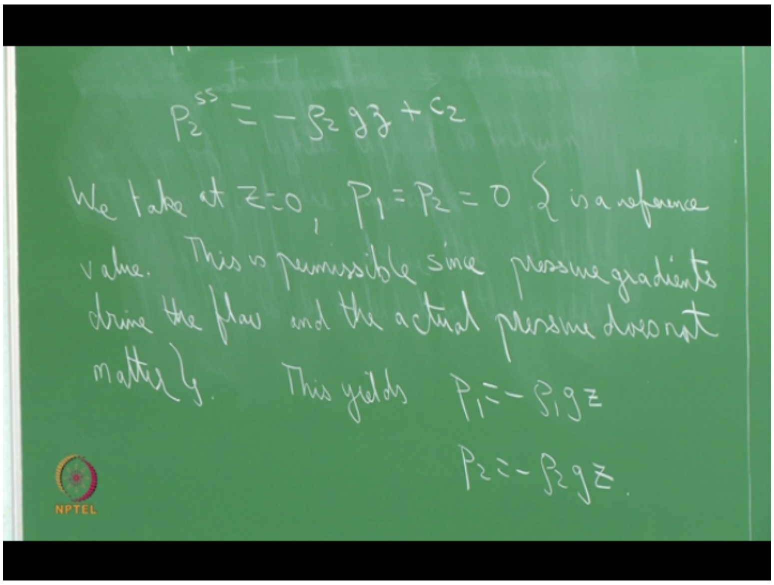

So, from the first equation what do I get? P1ss=-rho 1gz+the constant c1 okay. P2ss=-rho 2gz+ the constant c2 okay. So, that is basically what I get. And you know that the pressure as you go up has to decrease and that is what happening as z increases is becoming more and more negative. Now, we have to determine these constant c1 and c2 okay. And for that what we need to do is have some kind of boundary condition.

So, we are going to take at z=0, p1=p2 of course at the interface the pressures will be equal p1=p2=0. Now, what is the motivation for this? Of course you can take it to be any arbitrary constant okay. The idea is that flows are going to be driven by pressure differences okay. So, it is not the actual value of the pressure which actually is going to drive the flow. Even you have a pipe flow, if you have an inlet pressure of 80 atmospheres outer pressure is 70 atmospheres.

You are going to have a drop of 10 atmospheres okay and if it is a incompressible liquid, you have a particular pressure gradient that is the flow. Suppose you had 50 atmospheres and 40 atmospheres, you again have the same pressure gradient, the flow is going to be the same. Because the pressure gradient is what is important. The absolute value of the pressure does not matter. So, idea is that you can choose one of these pressure points as a reference point and you can calculate what the other value is okay.

So, that is basically what we are doing here. We are just choosing at z=0, the reference value of the pressure to be 0 okay. Keeping this as a reference value, we are going to find the relative value of the pressure at the other points and then see what is going to happen. So, this you can just say, is a reference value okay and this is permissible since pressure gradients drive the flow and the actual pressure does not matter okay. So, with this simplification, at z=0, pressure is 0, what do I get? Basically c1 and c2 will be 0 okay.

This yields p1=-rho 1gz and p2=-rho 2gz okay. Now that we have found the steady state, the steady state is characterized but the velocity and the pressure and we found that. We need to now find that if it is stable. So what do we do to find the stability of this state? We have to give this perturbation right. So, we write the actual variables. So, okay what I am do is, I am going to make one more assumption here.

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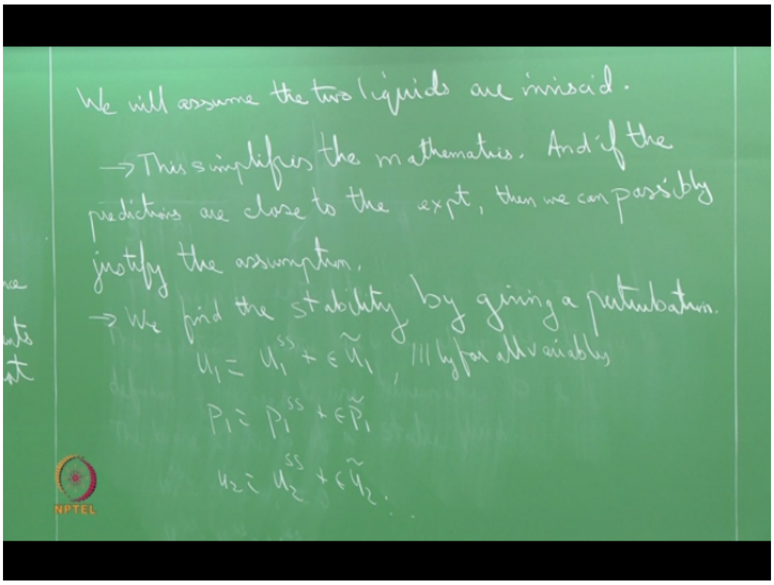

We will assume the 2 liquids are immiscid. So, the way I would justify this is, I like to make this assumption and see what the analysis yields. If I am not happy I will just go back and relax this assumption, include the effect of this cost to be and go through with the analysis okay. So, that is what we should be doing whenever you do any problem, I mean you start with the simple problem and then see if you are getting any insight.

If you are not getting any insight, you have made too much of a simplification, you had a level of complexity and then you keep on building. It would have been stupid for me to assume that the 2 densities are equal and keep the differences in the viscosity. Because what I am expecting to drive the flow is the density difference. So, I am going to retain the density difference but just to make sure that the algebra becomes easier and I am going to neglect the effect of viscosity okay.

So, this basically is going to simplify mathematically okay. And then do my calculation. I get some result. At the end of the day, if the result is anywhere close to the actual problem, then I say fine, I mean may be this assumption was not very bad after all. May be the effect of viscosity is not important okay. But, if it turns out to be different from what I actually see in an experiment, then I come back and say, may be this is the problem, maybe I assume this thing to be immiscid and actually the viscosity effect is important and I have included okay.

That is the only way to go upward to doing it okay. So, right now my motivation is, I do not want a second order equation so I just want to just keep it simple and so that is the motivation. And let us see if I can get away with it okay or get some insight at least into the problem. So, this simplifies the mathematics okay and if the predictions are close to the experiment, then we can possibly justify the assumption okay.

Of course, just because it is close to the experiment, it does not mean it is right, I mean it has just got lucky okay, I mean you have to be careful. If they are matching then you know for sure, something is wrong, if it is matching you are right okay. That is always a problem okay. So, let us not just say that just because this is matching experiment everything is perfect, everything is likely to be perfect, not necessarily perfect okay.

So, now what do we do? We find the stability by giving a perturbation, same step, find a steady state, give a perturbation, find the linearized equations and solve. That is what we did last time. That is what we have been doing for the last 3, four classes okay. So, what is this thing? The actual variable is written in terms of u1ss+epsilon u1 tilde. Similarly, for everything. Now, remember ones denotes the fluid, first fluid or the second fluid. Similarly, for all the variables okay.

And so it is p1 tilde and u2=u2ss+epsilon u2 tilde and so on okay. I am just to tell you that these things are infinite symbol, I have put that epsilon in front of it. So, what do we do now? We substitute all this in my governing equations. Because, the governing equations are going to be valid for u1, v1, u2, v2 okay. It is the actual variable. I am going to go back to my equation of continuity and write wherever u is there as uss + epsilon u tilde okay. And we have to do this for all the equations.

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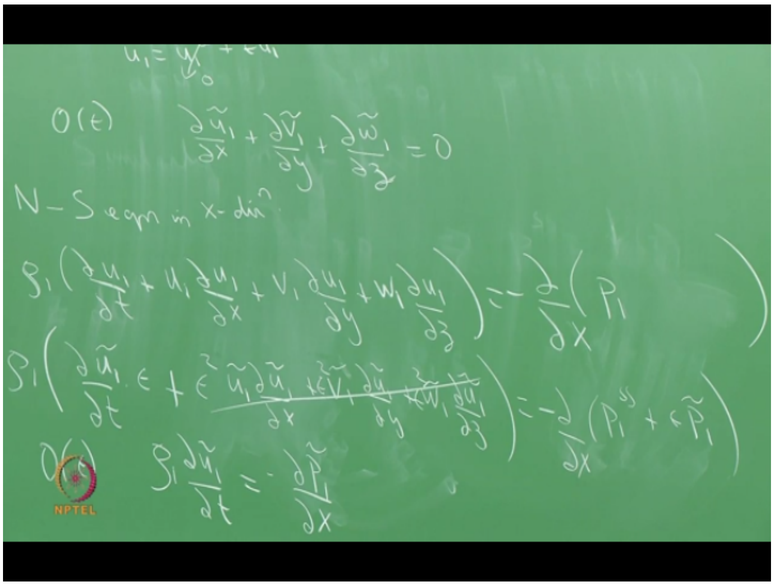

So, the equation of continuity implies d/dx of u+d/dy of v+d/dz of w=0, this is our first fluid. And I am going to substitute u1ss, u1 as u1ss+epsilon u1 tilde and this means an u1 ss is 0, so let us substitute this back inside here at order epsilon, what do I get? Du1 tilde/dx=0 okay. Now, we go the equation of momentum in the x direction, the Navier stokes equation. What is that? That is rho 1 du1/dt+u1du1/dx+x of may be if you just write this as p1, let me write this as p1 and then I will substitute this thing here.

This is in terms of the actual variables, I have not done any perturbation, now, I am going to substitute for all the u1's, u1ss+epsilon u1 okay. So, what do I know, this is going to become rho 1, u1ss of course is 0, so I get u1 tilde/dt multiplied by epsilon. Because, u1 is u1ss+epsilon u1 tilde. So, u1ss is 0 so u1 is epsilon u1 tilde okay. So, I have epsilon u1 tilde here and what about this guy? This is going to give me epsilon u1tilde times a derivative of epsilon u1 tilde, which means this is epsilon squared okay.

So, this is second order term. And this is how you are going to contribute. This is not going to contribute because, it is a second order term. This guy is not going to contribute again to the same reason. All I am saying is, this is epsilon squared u1 tilde du1 tilde/dx + epsilon squared v1 tilde vu1 tilde/dy + epsilon squared w1 tilde du1 tilde/dz okay. And all these guys just go off because they are of higher order okay. And what does this become? -d/dx of t1ss+epsilon t1 tilde.

We know that dp1ss/dx is 0 okay. So, it is not the p1ss is 0, it is that the derivative is 0 and therefore, this reduces to at order epsilon, I get rho 1 du1 tilde/dt=-dt1 tilde/dx okay. Now, you can do the same analysis in the y direction. So, let me just write that thing down neatly a bit.

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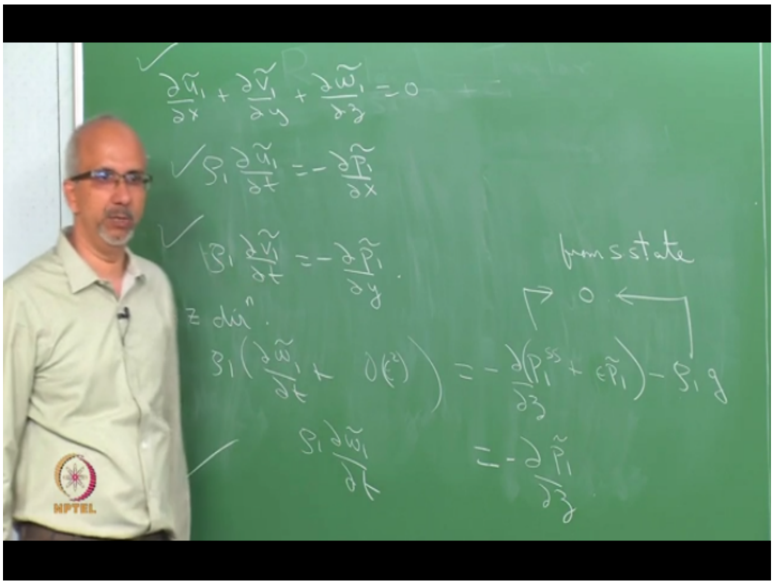

Let me write those 2 equations neatly because I want to preserve this so that I do not make any mistakes. This is just the equation of continuity okay, which I have already derived and then the other one is x. if you did the thing in a y direction, are you guys going to agree with me or if you do not agree with me then you work out the problem and decide you want to agree with me, you get a similar equation. In the y direction, you would get rho 1 dv1 tilde/dt=-dt1 tilde/dy, where everything is going to be the same okay.

The convective terms are going to give you a second order term. So, that is not going to contribute, these guys are going to contribute and order epsilon. And in the z direction, you only have one small complication in the sense that you have the gravity term coming okay but then, the gravity term remember, I will just explain that, and this is similar to the energy what we had for the Rayleigh problem.

In the z direction, you have rho1 dw1 tilde/dt + the order of epsilon square terms is what we do not worry about will be -dp/dz of p1ss+epsilon p1 tilde-rho 1 g. and at steady state, dp1ss/dz= -rho 1g okay. So, this is going to balance of that guy and so that goes of okay. I do not drive the term in the x and y direction, because the guy did not exist in the x and y direction. Here the gravity does exist, but this gravity is going to balance my dp1ss/dz.

Because dp1ss/dz remember was =-rho 1g. So, this again simplifies to, so basically this and this as 0 from my steady state or together okay, is no other rho 1 g0, it is together they are 0. And I get the same equation for w1 tilde also. The point I am trying to make here is, you guys have to sit down and make sure you do each and every term properly and then do the calculation okay and I do not want to just say, you get the same equation, because there is a small settle point here.

This is all for the one phase, you have similar equation for the other phase okay. There are same equations for the other phase. So, basically these are your equations which are going to tell you how the perturbations and pressure are basically relative to each other okay. We have similar equations in phase 2. Now, in the Rayleigh problem, in fact if you remember I tried to write the expansion first, try to convert it to the partial differential equations to ordinary differential equations and then reduce the number of dependent variables okay.

And then we found that you have got stuck because I resumed a sin alpha x dependency and then the 2 velocity components were actually out of phase sin and cosine and then you had a problem. So, I really could not proceed. And then somebody said we should use e power I alpha x as the way out. So, that is what we will do now. So, there what we did was we reduced the number of dependent variables first u, u, w, p and then we convert it to ODE's.

But now, what I am going to do is, I am going to do the expansion in terms of the independent variables first and then do the elimination of this thing. So, what we will do is, we are going to seek the solution. So, both are actually equivalent.

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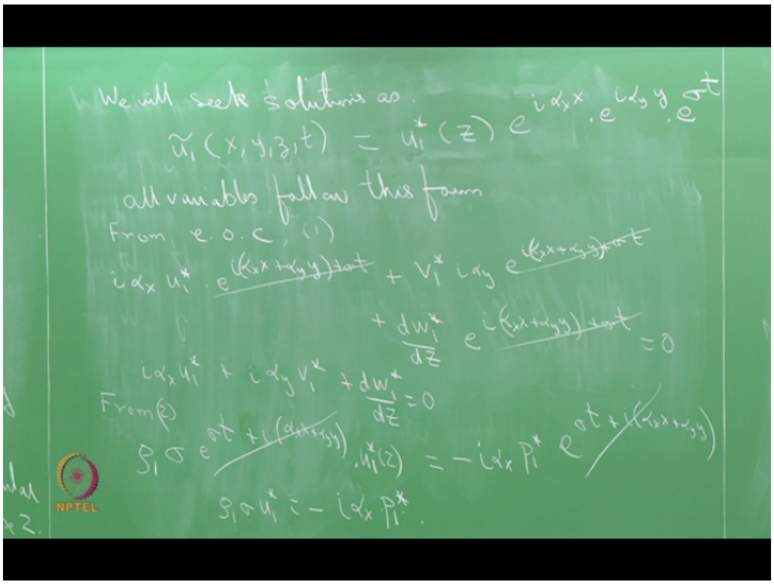

We will seek the solutions as u1 tilde, remember what is u1 tilde a function of x, y, z and t okay. We are going to seek this as u1 star of what variable? I need to impose the boundary conditions in the z direction. What boundary condition? The one at the interface. Some continuity of velocity whatever it is okay. So, I need to have that direction to be determined using the boundary condition okay.

So, that is the guy I am going to keep here and in the other 2 directions, I have assumed them to be spanning to infinity in the x and y directions okay. So, I am going to assume look for periodic solutions there. But, instead of looking for things in the form of sin and cosine, I am going to look for things as e power alpha subscript xx multiplied by e power I alpha subscript yy and the time dependency is of the form e power sigma t okay. So, because these equations are linear, I am assuming that, this is the form of the disturbance.

Our objective again is if I get this relationship between the growth rate and the wave number. We are across the thing we finally got even in the Rayleigh problem, sigma versus alpha or Rayleigh number versus this thing. So, what is alpha x represent? It represents some kind of a spatial frequency in the x direction, alpha y represents the spatial frequency in the y direction, alpha x is in x direction, alpha y is in the y direction okay and that is my growth rate here sigma.

So, this is exactly what we did last time, but the point is instead of using sin and cosine, I am just using a general form of the Fourier transform, not a Fourier sin transform, not a Fourier cosine transform. What I am going to do is, I am going to substitute this, I am going to make the same thing for all the variables okay u1, v1, w1, u2, v2, w2 so all variables follow this form okay all the eight variables, I am just leaving it as it is.

So, now what we need to do is, substitute and get ordinary differential equations for u1, v1,p1 star okay and way we do it is, substitute it in the equation of continuity. When you substitute this in the equation of continuity, I will get the derivative of u1 with respect to x, the derivative of u1 with respect to x is I alpha x multiplied by the entire thing okay. So, this is going to give me from equation of continuity, I get I alpha x times u1 star times e power I alpha xx+ alpha yy+ sigma t okay.

Then, dv/dz, when I have v, I have v1 star okay sorry, dv1/dy. V1 star of z of course and then I differentiate with respect to y, I get I alpha y. when I differentiate with respect to z for w1, I would get dw1 star/dz times e power I sigma t=0. Now, clearly the problem which you had last time of sin and cosine coming in is not that. Because I have the exponential term everywhere, derivative of exponential gives me the exponential, that means this is clearly non 0 and I can mark this thing of and what I have is I alpha x u1 star + I alpha y v1 star+dw1 star/dz=0 okay.

I want to come back and do the same thing for these 3 equations. Basically, my job is to get this ordinary differential equation and use a boundary conditions and find out what the stability condition is going to be okay. So, what we are going to do now is just substitute the form for u1 here, what do I get? Rho 1, when I derivate this with respect to time, I am going to get a sigma multiplied by u1 star etc. etc okay.

So, I am using this condition now some equation number may be this is equation number 2, okay, this is equation number one. So, this is from equation of continuity, which is one and from 2, what do I get? Rho 1 differential with respect to time, I get sigma multiplied by e power sigma t times +I alpha xx + alpha yy times u1 star of z okay=-dp1/dx tilde. That is going to be pressure is also going to be of the same of form okay.

When I differentiate that I get - of I alpha x times t1 star times e power sigma t+ again the fact that form is admissible is coming because of the fact that this particular term is common for both and what this gives me is that rho 1 sigma u1 star=-i alpha xp1 star okay. That is what this is. So, what we can do is, we can extend the same argument to the third equation here and what we will get?

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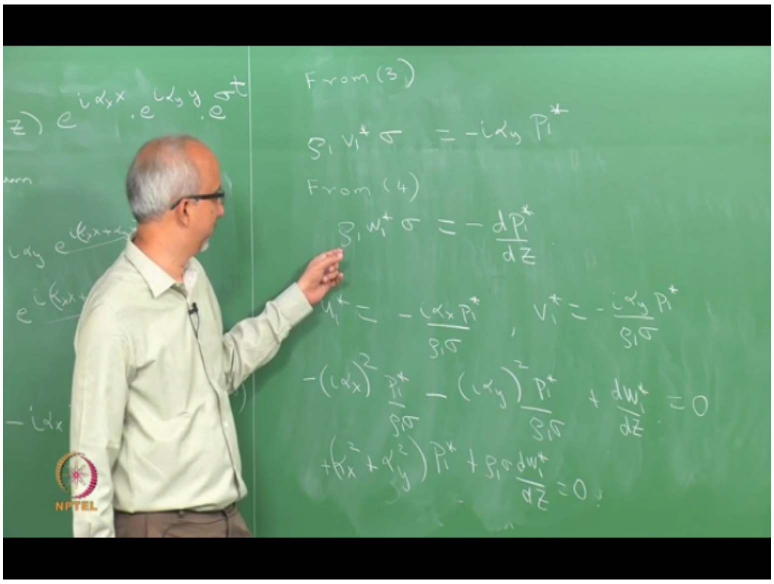

From 3, we are in different state with respect to v1, I will get a rho 1 v1 star sigma and when we differentiate with respect to y, I will get -i alpha y times p1 star okay. That is the equation which I get from this thing. And what about from four, which I have not yet written over there, but that is going to be equation number four. This guy is equation number four okay.

When I differentiate with respect to time I get, rho 1dw1 star sigma= now, I am differentiating with respect to z, so I get -dp1 star/dz. Because exponential terms will not have z. is the p1 star which is unknown which is a function of z. so, I have that here. So, now what I have done is I have converted it into an ordinary differential equation in z. but still have four variables u, v, w and the pressure with a subscript 1 and the star okay. And what I am going to do is, I am going to eliminate okay.

What we can do is, we have to eliminate let us say the u and v component of velocity okay and one way for you to do this elimination of the u and the v component of velocity is, we can write from this what is u1 star and from this what is v1 star in terms of pressure okay. I can find u1 star, v1 star from these equations, substitute it in my equation of continuity, then I will getting rid u and v, get everything in terms of pressure and w.

So, one equation in terms of pressure and w, I have another equation in terms of pressure and w and I can go back and eliminate pressure again may be and get only an equation in w1 okay. So, that is the basic steps that I am going to follow and so what do I get? U1 star=from this equation here -i alpha xp1 star divided by rho 1sigma okay, v1 star is -i alpha y p1 star divided by rho 1 sigma. That is what I found from these 2 equations.

Now, for u1 star and v1 star, I am going to substitute in this equation okay. So, I have i alpha x multiplied by u1 star, which is again an i alpha x with a negative sigh -i alpha x whole squared/rho 1 sigma-i alpha y the whole squared sigma+dw1 star/dz okay =0. I hope that is fine. All I have done is bring a little bit of algebra here and this is going to be -I squared so that is going to be +1, so I get alpha x squared.

I am going to take the rho 1 sigma along this my dw1/dz okay. So, I get alpha x squared +alpha y squared times p1 star + rho 1 sigma dw1 star/dz=0. So, this is the equation which relates pressure and w1 okay. I have also another equation which relates pressure and w1 right here. And what I am going to do is, I like to keep my velocity because I like to have my conditions on my velocity, my kinematic boundary conditions rather to eliminate velocity, I am going to eliminate pressure.

I eliminate pressure by differentiating this with respect to pressure, I will get dp1/dz second derivative. I will substitute for dp1/dz from this equation okay. So, that is what we are going to do. To differentiate the last equation with respect to z.

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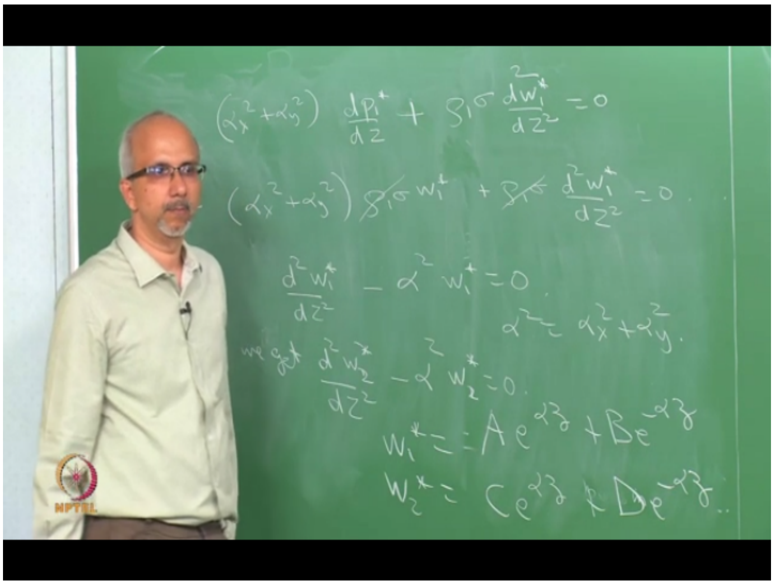

I get alpha x squared +alpha y squared times dp1 star/dz+rho 1 sigma dw1 star/dz=0. Correct? Yes. And you know dp1 star/dz is -rho 1w1 star sigma. So, this gives me. So, clearly rho and sigma cancels of and what I have is d squared w1 star/dz squared-alpha squared w1 star=0 okay. What is alpha squared? Alpha squared is alpha x squared + alpha y squared. So, this again is to tell you, see when we did the Rayleigh problem.

I resumed the things were actually 2 dimensional, we had roles we did not have any pattern in the other direction in one of the directions along which the axis of the role was extending. So, even if I not made an assumption, I would have got it 2 wave numbers like this and finally I would have got a composite wave number alpha okay. So, it really does not matter because there is only a mathematical complexity, it is not a physical complexity. See, what we want to do is, make sure we retain the right physics in the problem.

So, this is my equation for the one phase. If you did the same thing again for the other phase, you would get similarly d squared w2 star/dz squared-alpha squared w2 star=0 okay. This is the other phase. One represents the one phase, 2 represents the other phase. We know the solutions to this equation, alpha is the constant right, this is linear equation with constant coefficient second order. Everybody knows how to solve this problem. What is the solution? W1 star is A e power alpha z+B e power -alpha z, that is the solution to the differential equation.

W2 star is C e power alpha z+ D e power -alpha z. Now, our job is to find the causes A, B, C and D. and for this, we need boundary conditions okay. So, what are the boundary conditions we are going to have? Remember, this first liquid is extending from 0 to -infinity. Clearly, what we expect is when there is some kind of an instability at the interface, we expect that far away at z goes to -infinity in the first fluid, as z goes to +infinity in the other fluid the velocity components are going to go to 0 okay.

So, basically you expect the velocity far away from interface to be finite and bounded. It cannot become infinite. So, that is going to help us determine 2 of these constants. For example, as w2 is from 0 to infinity, where z goes to infinity, I want the velocity to be bounded, which means this guy should be present because this is -sign and this will go to 0, this has to be absent. So, that tells me C has to be 0 here okay.

Similarly, in the other fluid, in the lower fluid, we will have z is from -infinity should be present, this will be present and this will be absent something like that. So, it is not right?

That is right is in it. I thought I said the same thing twice. So, that basically helps you determine 2 of the constants. We need to determine the other 2 constants and that is where the boundary conditions at the interface come in. and that is where we are going to use the kinematic boundary conditions and the one more condition.

So, the other condition which we are going to use, is the normal stress boundary condition. This normal stress boundary condition is going to be preferred over the tangential stress boundary condition. The reason for that is, that we made this thing in visit. See, there are 2 conditions which have to be satisfied at the interface, both the normal stress boundary condition as well as the tangential stress boundary condition.

How is it that we do not need both? We need only one. The reason is, we assume that the fluid is in visit, so, we actually had a second order problem but because I have assumed it be in visit, my problem from second order has become first order. So, I need to let go of one of boundary conditions. The boundary condition which I am going to let go of is the tangential stress boundary condition. Because there is no continuity of tangential stress.

Because, normal stress is going to be present even in the absence of viscosity. So, I retain the normal stress boundary condition, I let go of the tangential stress boundary condition. What we will do is, we will use those conditions, get these constants and get the dispersion curve tomorrow.