Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 24C
Rayleigh-Benard convection: Linear stability analysis part 3

Welcome to today lecture where we will be discussing the Rayleigh-Benard convection problem. We will just continue from where we left off, the last class. And just to make sure that we have everything in place what I have done is I have written down the equation which describes the perturbation v star talks about the functional dependency of y of the perturbation. And v star is going to be governed by this 6th order ordinary differential equation.

Now, this particular ordinary differential equation was obtained by eliminating the u component of velocity the pressure and the temperature. What we did was we linearized the governing equations about the steady state. And then we converted the partial differential equation to ordinary differential equal to ordinary differential equations and then we got this 6 order ordinary differential equation.

So, all though the (()) (01:38) equations were only second order to begin with because we eliminated the other variables the u components of the velocity, the x component of the velocity and the temperature. The second order equation became a 6 order equation. This equation is subject to these boundary conditions. The boundary conditions are the same at both the surfaces, the lower wall and the upper wall because both of them are solid walls.

We have the velocity being 0 this comes from the fact that the walls are impermeable and therefore there is no vertical component of velocity the fact that you have a no slip boundary condition which tells you that the x component of velocity is 0 we use that to transform the boundary condition to the vertical component of velocity v. And for this we use the continuity equation and we got that the first derivative of velocity in the y direction is 0.

Remember D is the operator d/dy and this boundary conditions come from the temperature boundary condition. So, you see that the boundary conditions which were there for the temperature. The boundary conditions which were there for the x component of velocity all of that has been converted to boundary conditions on the vertical component of velocity v. Now, what we like to do is take a close look at this equation.

And I want you to see that this is a linear homogeneous equation with homogeneous boundary conditions. What does this mean? This means that v star =0 is always a solution to this problem.

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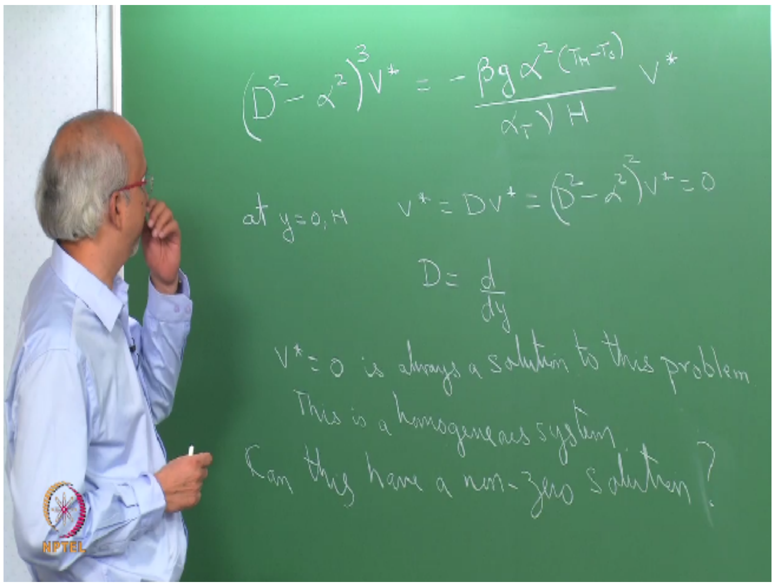

It satisfies the boundary conditions and it satisfies this differential equation. So, in some sense this is like an eigenvalue problem. You have ax=lambda x where a is my differential operator now lambda is something like this quantity and x is my velocity vector. What we are interested in doing is we are interested in finding out conditions for which this equation has a non-zero solution. We know that v star = 0 is always a solution to this problem. Why?

Because this is a homogeneous system but can this have a non-zero solution that is the question? Or under what conditions can this have a non-zero solution? Clearly, the only parameters which are at your disposal are the temperature of the wall the gap between the 2 plates, the properties of the fluid the thermal diffusivity, the kinematic viscosity, the density dependency on temperature.

And so for some combinations of these parameters it is likely that this system will have a non-zero solution. Now if this system were to have a non-zero solution then for that combination of parameters remember we are on the point of marginal stability or neutral stability. Why is that? Because we have derived this equation by imposing the condition that sigma the growth rate in time for the perturbation is 0.

Now, if you recall what we did in the last class this equation was derived by imposing the condition that the real part of the growth rate and since growth rate turn out to be real and not complex the growth rate was 0. When the growth rate is 0 you are on the point of marginal stability or just that transition point, the critical point where it is going to change from stable to unstable.

So, our neutral stability and the fact that this has got a particular combination this is going to have a non-zero solution is going to correspond to the point where you have the transition of steady state from a stable steady state to an unstable steady state. What I want to do is I want to point out here that this particular group of parameters leads me to you know seek a dimensionless number.

Now, I have unfortunately not made the equation dimensionless in the problem so far. And what I like to do is I like to now make this equation dimensionless and then I will possibly reiterate what I have just told you to make my point very clear. So, remember v tilde the perturbation was given as v star*y times e power sigma t times sin alpha x.

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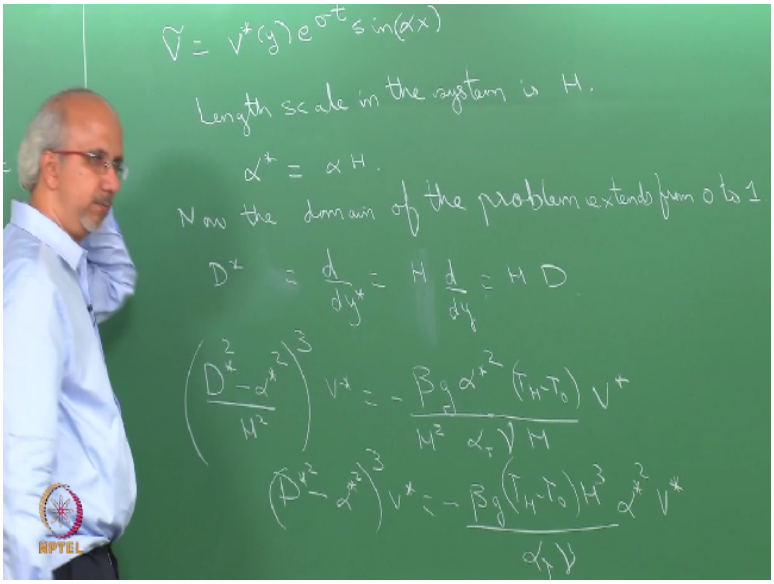

Now, since we have the same variable on both sides we are not particularly worried about making the velocity dimensionless because we would choose a characteristic scale for the velocity and that would occur both on the left hand side and the right hand and that is going to cancel off. So, imagine you had a characteristic velocity and you made this dimensionless. It would not really effect. But what we have here is this term here which possibly has some units.

So, remember this term D the capital D is given by d/dy. So, this has units of reciprocal length. The alpha here also has units of reciprocal length for example the alpha occurs as a wave number. The wave number is such that it is a reciprocal of wave length. So, alpha is something like 2 pi/ lambda. So, the point I am trying to make here is that alpha is actually has units of reciprocal of length and there is only one length scale in this system.

And that is the height or the gap between the 2 plates. So, the length scale in the system is H. I am going to use this length scale and make my equation dimensionless. What does that mean? I am going to define alpha star as alpha*h. So, this has (()) (10:15) of reciprocal length and this has units of length so this is dimensionless. And I am going to write and if were to make my equations dimensionless by scaling it with H.

Remember the domain of the problem now is going to go from 0 to 1. So, now the domain of the problem extends from 0 to 1 instead of 0 to H and clearly what I am going to do is I am going to now write the alpha here in terms of my dimensionless wave number alpha star I am going to write this as alpha star divided by H. The D square is going to be the second derivative with respect to y and when I am making y dimensionless.

I am going to have an h coming out in the denominator. So, what I am going to do is I am going to define D star as d/dy star which is =1/H d/dy sorry =H times d/dy. This is a dimensionless derivative and this is what I get. So, this is H times D. So remember this is the dimensionless derivative with respect to the dimensionless coordinate and I am converting this to dimensional coordinate. I am going to substitute for D and alpha star in this equation here.

So D I am going to write as D star divided by H, alpha I am going to write as alpha star/H. Now I get D star –alpha star and this is all square divided by H square whole cube v star = -beta g and alpha square I like to write in terms of alpha I like to write in terms of star. So, I am going to get alpha star square divided by H square times TH-T0/ alpha T times nu times H times v star. So, all I have done is converted that equation to a dimensionless form.

And remember now alpha star is dimensionless because I have scaled it with H and d star is also dimensionless. I would have gotten this if I had scaled my equations made a dimensionless at the beginning. And all I am going to do is recognize that this is H to the power 6, take H to the power 6 side and remember I have an H square and an H here. So, I have a H cube. So, I have H to the power 6 when I take it to this side I get H cube.

So, I am going to write this as D star square –alpha star square whole cube v star = - beta g Th – T0 times H cube/ alpha T times nu times alpha star square times v star. I have D square star square – alpha star square whole cube operating on v star equal to this. I would like to group these terms together and I want you to realize that beta in general is negative and Th – T0 is in general negative.

Because the upper plate temperature is lower than the lower plate temperature. So, this is negative and this is negative so the quantity inside my brackets is a positive quantity. And those of you who have done a course in heat transfer will remember that this is extremely similar to this number called (()) (16:07) which you have come across. Only thing is in the context of this problem this number is called the Rayleigh number.

So, I am going to define this Rayleigh number as beta time g times TH-T0 times alpha T times nu times, times H cube.

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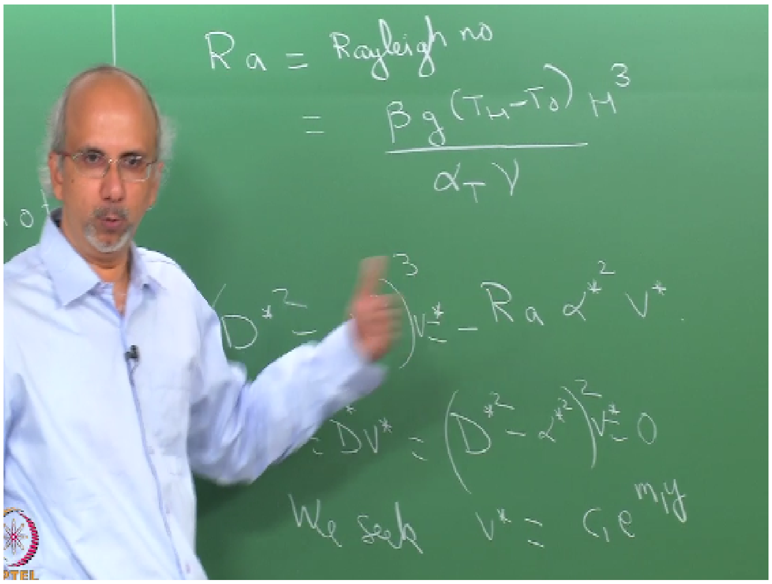

And this is a dimensionless number just like you have the Reynolds number, the Prandtl number and the (()) (16:50) number. And the whole idea of making this thing dimensionless was to get a dimensionless number so we could talk things in terms of this dimensionless number, Rayleigh number. So, now I would like to recast this equation as D star square –alpha star square whole cube = -*the Rayleigh number times alpha star square v star.

And the boundary conditions to which the subject to are v star =dv star or d star v star =d star square –alpha star square whole square =0 all I have done is converted the boundary conditions also to the dimensionless form. And this is straight forward because you just get a factor of capital H in the denominator and since that non-zero this has to be zero. So, now to go back to what I was saying we are going to ask the question.

What is the Rayleigh number for which this has a non-zero solution? If I can find the Rayleigh number for which this has a non-zero solution then I will know that for that groups of parameters equal to that number, equal to that value I have the marginal stability of the transition from stable steady state to a unstable steady state. How do I go about solving this problem? Remember alpha star is a constant so what you really have is a 6th order equation with constant coefficient.

Now you know how to solve this problem. So what you will normally do is you would seek a solution in the form of an exponential function. So, the way you go about solving this problem is we seek the solution v star as, sorry there is a v star here. There is a v star there and a v star here. We seek v star as some function times some constant multiplied by an exponential function you would substitute it in this equation and you would get a characteristic equation which is a 6th order polynomial.

You will get 6 routes which would give you 6 values for m and those would be your possible solutions to the homogeneous equation. Clearly, the values of m would depend upon the Rayleigh number and the alpha star. Then what you would do is you would impose these boundary conditions and try to find a non-zero combination of C1 for which you will be able to satisfy the boundary conditions.

So, we do not want a situation where the constants C1 and all identical is 0 because then you would get v star as 0 remember we want v star to be non-zero. So, the strategy is to try and solve this equation by seeking solutions of this kind, getting a 6th order polynomial and then imposing the boundary conditions. Since you have 6 boundary conditions and you have 6 constants C1, C2, C3 maybe I should just write this.

C1e to the power m1y, C2e power m2y you want to impose a boundary conditions and then you would find that you have 6 equations and 6 unknowns. The 6 unknowns are the arbitrary constants Ci, we want a non-zero Ci and what you would do is, you would formulate the problem as if it is a matrix problem. As if it is a matrix multiplying a vector. The vector contains my Ci's and look for a solution for which you get a non-zero solution.

Look for a condition for which you get a non-zero solution. And from linear algebra you know that this is given by the data minute of the matrix being zero. So, let me just write this down.

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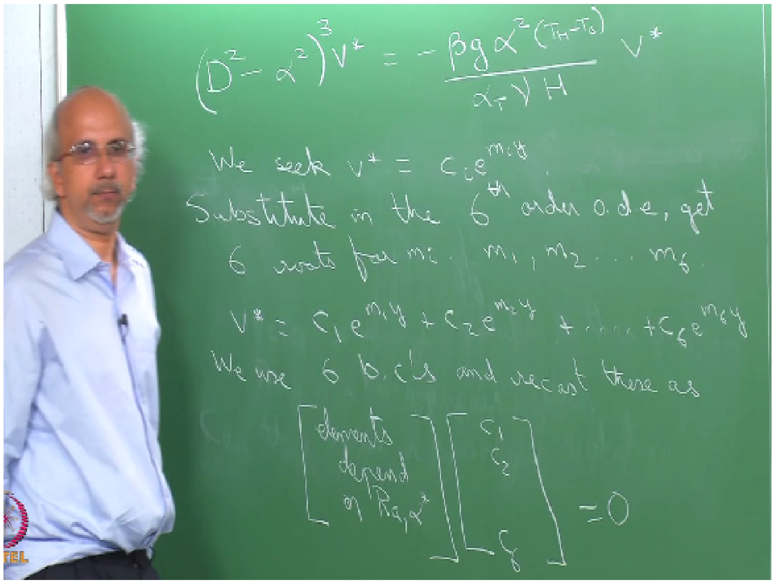

We seek v star as Ci e power miy substitute in the 6th order o.d.e and get 6th roots for the mi that is I will have m1, m2 up till m6. And you know that v star is going to be given by C1e to the power m1y + C2 e to the power m2y + etcetera up still C6 e to the power m6y. What we do is we use the 6 boundary conditions and recast these as the matrix which multiplies C1, C2 up till C6 =0.

The right hand side is going to be 0 because remember my boundary conditions are homogenous. These equations are going to be linear in the C1, C2, C3, C4 because my boundary conditions were linear. So, I can actually write this as a matrix multiplied by a vector and remember the elements here are going to be functions of depend on Rayleigh number and alpha star. So, you would get a non-zero solution if the determinant of this matrix is 0.

So, as you can see there are 2 unknown one is the wave number, dimensionless wave number and the Rayleigh number. So, what you would do is you will actually fix alpha star add some value and find the Rayleigh number for which the determinant is 0 that is the idea. And you would do it for different alpha starts. You will do it for different wave numbers and you would get different values of Rayleigh number.

And you will make a plot of this curve. Therefor every wave number you would know what the

Rayleigh number is for which you have a non-zero solution. Now, you would have to possibly write a small computer code to actually do this problem in (()) (24:52). But what I like to do now is refer you to this book by Subramanian Chandrasekhar on Hydrodynamics and Hydromagnetic instabilities.

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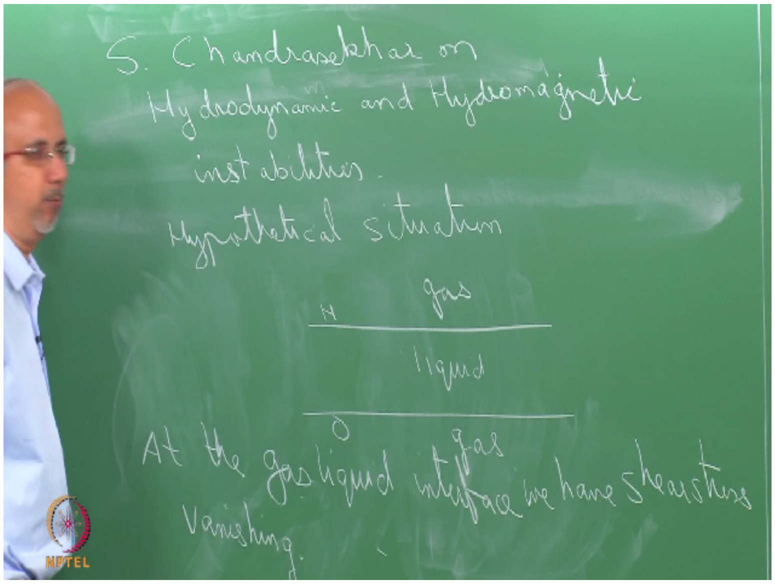

And the primary reason for me to refer you to this book is for you to see that how about 100 years back when scientist did not have access to the kind of software tools that you have access to they were still able to solve these problems. So, reading this particular section on Rayleigh-Benard problem and Chandrasekhar you will see how he uses arguments like symmetry and tries to answer this question and that I have just talked about.

How to go about finding a non-zero solution? What I want to do is I want to tell you that we can still get some insight about this problem and get a feel for how this curve, how the Rayleigh number depends on the wave number by solving a hypothetical problem. And so at the being when I started I would tell you that we possibly get an analytical solution. So, I am not going to disappoint you I am going to still go about getting an analytical solution.

But then it will not be for the problem that we were talking about where we had the fluid between 2 solid walls, 2 rigid walls. Imagine now hypothetical situation where you have a layer of fluid situation. Let us say you have a layer of liquid which is suspended in the atmosphere.

This is exactly what we had last time only thing is in the sense that I have a liquid film which is extending from 0 to H or a dimensionless from 0 to 1.

Instead of having a solid wall I have a gas liquid interface here. I have a gas liquid interphase here. Remember a gas liquid interphase the boundary condition is that of the shear stress being 0. So, at the gas liquid interface we have the sheer stress vanishing. I am going to keep my life simple and I am going to restrict myself to the case where the interphase is going to remain flat. The interface does not deform.

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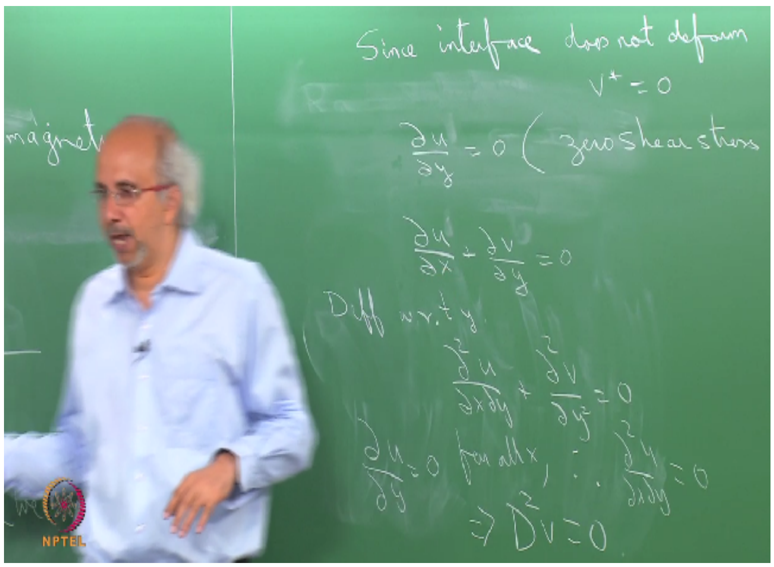

So, now since the interface does not deform I am going to still have d star=0. So, since the interface does not deform v*=0 but now the sheer stress is 0 you do not have a (()) (29:19) boundary condition where I put u as 0 but you would now have du star/dy =0. So, this is the 0 sheer stress condition and remember what I want to do is I want to convert this condition on u to the condition on velocity v star.

So, what we do is we use equation of continuity d mu star/dx + dv star/dy =0 that is my equation of continuity I can use this condition only if I want to differentiate this with respect to y. So, if I differentiate it with respect to y, I would get d square u star/ dx dy + d square v star. You know I am really bit more careful I could leave this as u and (()) (30:41) y square=0. So, this is my 0 sheer stress condition on the actual velocity.

And also on the perturbation and this is my equation of continuity and all I am trying to tell you is this condition of the 0 sheer stress this is going to be true for all x. du/dy is 0 for all x. So, du/dy=0 for all x therefor t square u/dx dy =0 and which this condition implies not the second derivative of v with respect to y is 0. So, this implies that t square v is 0. Remember for the rigid wall I got the first derivative = 0.

Now, I am getting the second derivative as being = 0. So, this hypothetical problem results in v star being 0. The second derivative being 0 and also the other boundary condition which is D star square –alpha star square whole square D star being 0. So, this condition remember comes from a temperature equation. My temperature equation is such that my temperature is fixed. So, my temperature perturbations is 0 at the lower wall.

Temperature perturbations is 0 at the upper wall and therefor that still remains the same. Therefor this boundary condition remains the same only thing is the change from the solid rigid wall to a gas liquid interphase has resulted in the change from the first derivative to the second derivative. Now, what I like to do is come back to the dimensionless form of the equation which is D star square –alpha star square whole cube = -Rayleigh number times alpha star square v star.

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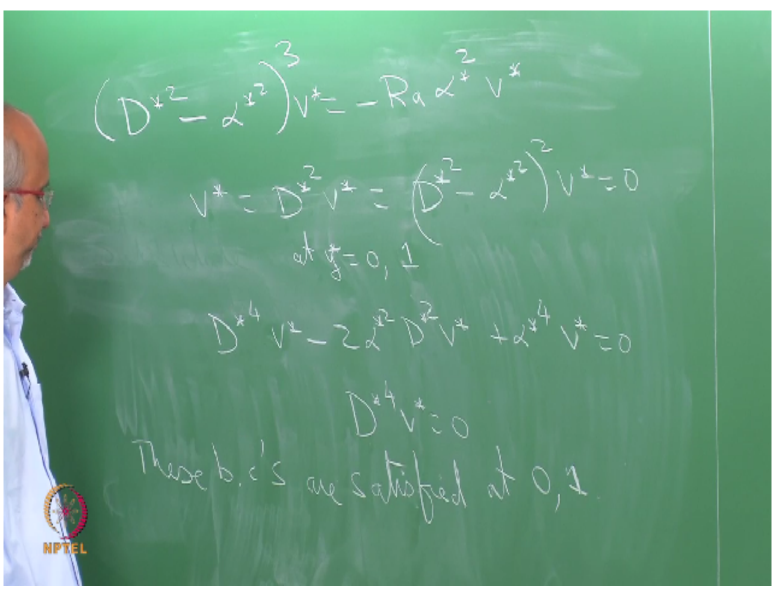

Subject to v star = and this is true at y=0 and H. I want to expand this and I am going to get D star power 4*v star -2 alpha star square D star square v star + alpha star power 4 v star =0. Since v star and the second derivative are both 0 this equation simplifies to this equation and it further simplifies to the fourth derivative being 0. Because this is 0 and this is 0. And so what I have is the fourth derivative =0.

Now, therefor what we have now is a problem where in my v star is a function which satisfies all my even derivatives being 0. The function the even derivative the second derivative and the fourth derivative are 0. And remember these are going to be 0 at 0 and 1. These boundary conditions are satisfied at 0, 1. Okay, since made dimensionless I am going to write this at 0, 1 and not at 0, H. So, these are the dimensionless version.

So, now can you think of a function which actually has this property? So, let me give you a clue this is a trigonometric function and the function which satisfies this is the sinusoidal function.

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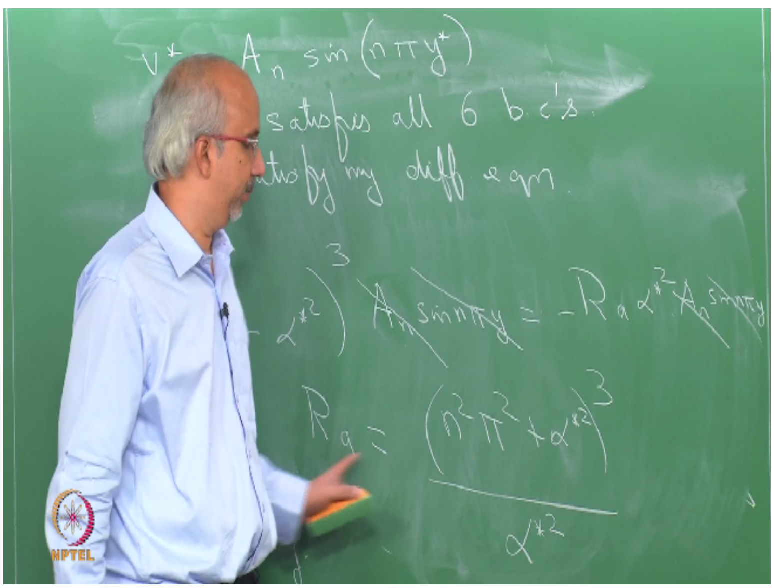

Clearly An sin n pi y satisfies the boundary conditions v* is 0. The second derivative gives me the same thing. Something multiplied by sin. The fourth derivative gives me this. So automatically this function satisfies the boundary conditions. But we want a function which satisfies not only the boundary conditions but also which satisfies the differential equation. So, I want to make sure that this satisfies the differential equation. So, satisfies all 6 boundary conditions.

Do not worry about the boundary conditions but in order to satisfies my differential equation I am going to substitute this functional form into my differential equation here. And remember there is actually n pi y star because I made it dimensionless. When I do this I am going to differentiate it 2 times I am going to get n square pi square with a negative value sin. So, I get –n square pi square – alpha star square whole cube.

So v star I would get I have already substituted this so I am going to get An sin n pi y = - Rayleigh number times alpha star square times An sin n pi y. All I have done is substituted this functional form in my 6 order differential equation. Differentiating this 2 times give me n square pi square and this is whole cube this is what I get. Clearly, we want a non-zero An sin pi y and I can take out this negative outside.

I will get negative of this thing whole cube and so this basically gives me the condition Rayleigh number is of the form n square pi square + alpha square whole cube/ alpha star square. So when I was talking about the problem with the 6 constants and the determinant and told you that you get an expression for Rayleigh number as a function of alpha star. We were not able to do it because it involves you know possibly I think a computer program.

But these hypothetical problem gives me explicitly what the dependency is of Rayleigh number on alpha (()) (38:45) square. And clearly it also depends upon n what we will do now is make a plot of this particular function. I am going to plot Rayleigh number (()) (39:02) alpha square.

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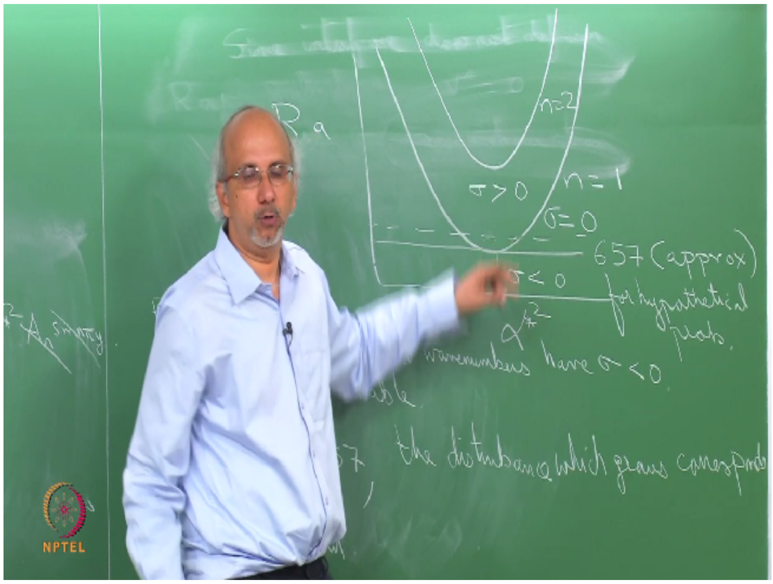

You realize that as alpha star tends to 0. This function is going to go to infinity is going to be unbounded because of the denominator. And so for lower values of alpha star this is going to be negligible compared to this. This is low and that is going to make my Rayleigh number go to infinity. For very large values of alpha star this is going to dominate over this I can neglect this I will have alpha star power 6 divided alpha star square so it goes as alpha star power 4 again is going to go to infinity.

You also realized that this is all going to be positive so I have a curve which is going to be confined only to the first coordinate and which has a shape of this kind. Point I am trying to make here is that this curve is going to have a shape of this kind and I am going to draw this for different values of n. I will have a curve like this for n =1. I would have a similar curve for n=2 clearly the curve for n=2 is going to be having a magnitude or larger for Rayleigh number.

For every value of alpha square. Because of the way n is appearing here. So, the point I am trying to make here is that n=1 curve is the one which is lowest. And what we want to do is we want you to realize that along this curve remember the growth rate is 0 because that was the condition which we had imposed on getting the solution to the equation when I was simplifying. So the growth rate of the curve sigma.

The growth rate of the disturbances is 0 along this curve. So, let me try to tell you what we have done. I started solving a real problem of a fluid between 2 rigid wall and I said look I have to do it numerically and you have to do it numerically in the sense you have to solve this problem numerically and is going to be one of the assignments. But I do not want to give up I want to explain to you something about the physics of this problem.

And to allow me to do that I have gone to this hypothetical problem where my boundary condition now has changed from a (()) (41:52) boundary condition to a 0 sheer stress boundary condition this helps me get all even derivative 0 hence we guess a solution to my 6 order equation. The solution I have guessed is the sinusoidal curve and then I have substituted this sinusoidal curve and got in into a situation where my boundary conditions are satisfied.

And I just want to make sure that the differential equation is satisfied and that is how Rayleigh number depends on alpha star square. What we want to do is find out for n=1 find the point at which this minimum occurs and I am going to leave this as an exercise for you to calculate the value of alpha star at which you have the minimum and the corresponding value of the Rayleigh number. I am going to tell you what the corresponding value of Rayleigh number is.

And this value of the Rayleigh number is 657 approximately. And this is remembered for the hypothetical problem. So, far we have not bothered about the alpha the wave number of the disturbance. We have not told you what is wave numbers is we have just done the problem for a general arbitrary wave number that is how this analysis has gone. Now, how do I know or how does the system decides what the wave number is of disturbance that is the question.

And while looking at this picture we can get some insight. When you give a perturbation to a system this perturbation is going to be composed of different wave numbers. It is going to be an arbitrary perturbation. You cannot in a real situation impose a perturbation of a fixed wave number in general. So, you had given arbitrary perturbation and this arbitrary perturbation can be decomposed using something like a (()) (44:32) series analysis into different modes.

So, those different modes correspond to the sin alpha x. So, when I have an arbitrary perturbation is the function of x I would then say this has some component sin alpha 1x some components in alpha 2x. This is like decomposing a vector into a set of basis vector components. Now, what we are asking is I want to find out how does each wave number behave? How does the disturbance in each wave number behave?

Clearly, I have told you along this curve sigma is 0. Which means this curve is dividing this space into 2 regions on one side of the curve sigma is going to be negative and on the other side sigma is going to be positive. Now, we can use common sense to figure out where sigma is going to be negative. Clearly, for low values of Rayleigh number this corresponds to low values of the temperature gradient or the temperature difference. I expect the system to be stable.

So, if a system is going to be stable this corresponds to sigma being negative and the region above this curve is going to correspond to sigma being positive. Now, this curve therefor divides this into 2 portions where one sigma is negative, one where sigma is > 0. Now, if the Rayleigh number is sufficiently small it turns out that the growth rate sigma is negative for all the wave numbers.

Which means if I give an arbitrary disturbance if I have decomposed it into different wave numbers into components of different wave numbers each of them is going to have sigma negative so in this region the system is stable. So, for Rayleigh number < 657 and all wave numbers have sigma negative so system is stable. For Rayleigh number which is just > 657 so at just slightly above this line.

You see that if I give an arbitrary disturbance I am going to resolve it into again different wave numbers. Wave numbers on this side are stable, wave numbers of this side are stable. But the wave numbers here are unstable. So, you would therefor see in your real system the disturbance which grows and that is going to correspond to a disturbance which has a wave number closer to the value where this minimum is occurring.

So, the disturbance which grows corresponds to the alpha star of the minimum. The minimum value of alpha star and this one remember is the one which is going to have the fastest growth rate because close to this sigma 0. So, if you have a Rayleigh number given by this value corresponding to this alpha star this is furthest from this boundary. So, the sigma value is going to be largest.

As you come closer to this boundary sigma is going to be 0 and so this disturbance with this wave number is going to be the one which is going to be the furthest. So, what I am saying is the one disturbance which is going to manifest itself in a real situation is going to correspond to the one with this alpha star. And this is the one which the system is going to decide by itself. So, the non-linear interaction of the system.

Tell you that the wave number which you are going to observe when it becomes unstable is going to be given by this alpha star. We will see more of this in the next class where I will discuss this in more detail. Thank you.