Multiphase Flows: Analytical Solutions and Stability Analysis

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Lecture – 24B
Rayleigh-Benard convection: Linear stability analysis part 2

Welcome to today lecture where we are going to do is continue from where we left off in the last lecture when we were analyzing the linear stability of the Rayleigh-Benard problem. So, just for a brief recollect we had taken the stationary solution and we are imposed small perturbations on this stationary solution. And these perturbations are actually being denoted by the tilde variable. So, T tilde represent the perturbations on temperature.

v tilde represents the perturbations on velocity and we started with the original non-linear equations and we linearized the equation about the steady state. So, what it basically means is we assume the perturbation to be of order epsilon and retain terms only of order epsilon to the power 1. We do not consider terms of order epsilon square because they are higher order terms and they are negligibly small. So, when you do that you get a bunch of linearized equations.

So, what they had is if you remember you had the perturbations on u, the (()) (01:44) perturbation velocity, the perturbation of pressure v tilde. And we have eliminated those 2 variables and you had 4 perturbations on the 2 velocity components of the pressure and on temperature. What we did is we eliminate 2 and we have reduced it to a system of 2 equations. One on the temperature perturbation and one on the vertical velocity component perturbation.

And this is what we have derived in the last class. What we want to do now is take the analysis further and in our quest for getting an analytical solution. You observed that these 2 equations are linear and they are also couple the temperature equation has velocity in it. The velocity equation has temperature in it. So, you have to solve them simultaneously. And you also observed that they are homogeneous.

That is every term present in each of the equation contains the perturbation variable or the derivative of it to the first power. So, they are linear and there is no non homogeneity and these are important characteristic of a linearized problem. So, what we want to do now is remember that the temperature and the velocity perturbation are actually functions of are dependent on x, y and t the time.

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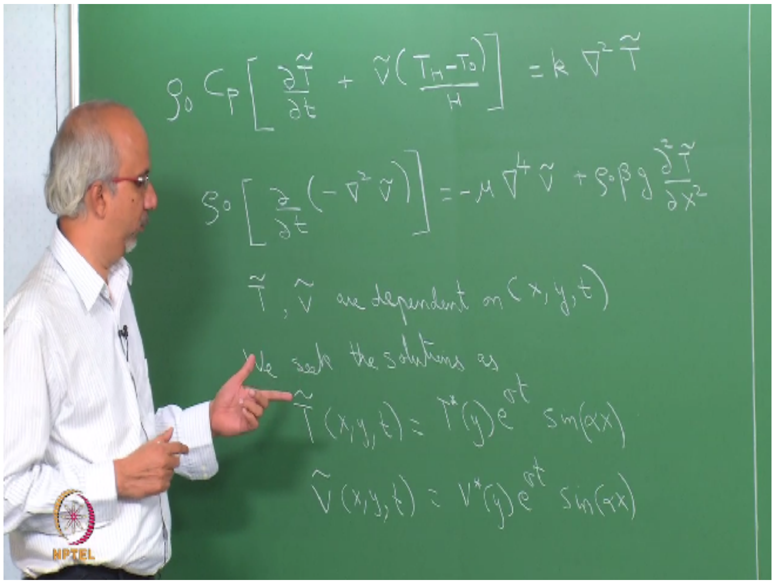

They are independent on z while we assume the span to be infinity and which basically means that these are partial differential equations. So, we will do is we would like to convert these 2 a set of ordinary differential equations. So, we could seek the solution as T tilde of x, y, t =T star *y times e power sigma t times sin alpha x. And v tilde *x, y, t as V star of y times e power sigma t times sin alpha x. What I have done here?

I am going to give 2 interpretations to this functional form that I am seeking. One is the physical interpretation. The physical interpretation is I am going to seek solutions which are period in the x direction. Now, the fact that the x direction extend to infinity allows me to actually seek period solutions to x direction. I do not have to worry about boundary conditions in x direction and the fact that this is linear equation.

Which is first order in time allows me to seek the time dependency to be exponential and sigma is the growth rate of the disturbance in time. The y dependency is captured in t star and v star. So, clearly if you have a system which is stable then the real part of sigma is going to be negative and that implies stability. And the real part of the sigma is going to be positive this implies the semi-stable and stable. So, this is the physical interpretation.

The other interpretation is a mathematical interpretation. The mathematical interpretation comes from the techniques you have learned in your mathematical courses where you have been talking about taking Laplace transform and Fourier transforms. So, when you are taking a Fourier transform or Fourier sine transforms or Fourier cosine transform you are essentially seeking a periodic solution in the x direction.

When you take a Laplace transform you are essentially seeking time dependency in the Laplace domain and which is also going to be exponential. Our objective is to you know seek solution of this kind and get the T star of y and V star of y that is what we are going to do is we are going to convert this bunch of partial differential equations here to a bunch of ordinary differential equations which are going to describe T star and V star.

So, remember that is the object, convert partial differential equations to ordinary differential equations. We can do that by using a physical argument or we can use a mathematical argument. So, let us write this down.

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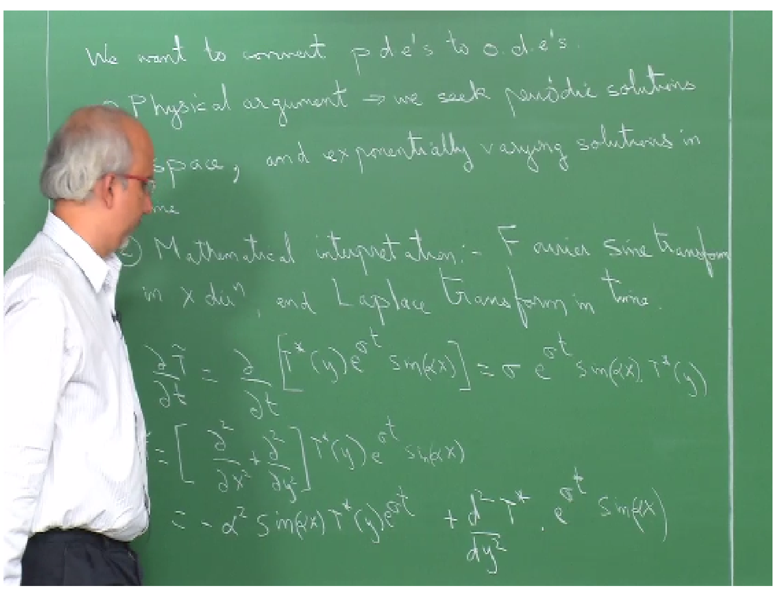

We want to convert PDE’S to ODE’S, ordinary differential equations. The first is physical argument where we seek periodic solutions in space; and exponentially varying solutions in time.

The second is more of a mathematical argument, mathematical interpretation. We take a Fourier F sign transform in the x direction and a Laplace transform in time. The fact that we have actually assumed x, to x into infinity is what is allowing us to do a Fourier sin transform.

And this is where the assumption of x being large comes in handy. The form assumed for the temperature and the velocity here implicitly assumes that the temperature and the velocity are in phase as far as the spatial dependence is concerned that is both of them are varying as sinusoidal. It is not that one is varying as a sin and the other is varying as cosine. And whether they are actually going to be in phase or not we can find out only by substituting these equations.

These forms for the perturbations in these governing equation. So, if turns out that by substituting these forms in these equations the a power sigma t and sin alpha x occurs in every term. Then you can essentially cancel off the a power sigma t and sin alpha x from every term and which means that such a solution is possible and this has to be true for both the equations. So, that is what actually we are going to do now.

We are going to substitute these forms for the perturbation in these equations and find out how this equation can be reduced to an ordinary differential equation. So, let us do that. But before we proceed I am going to make a small calculation so that it will allow me to proceed me faster. So, let us see what is the time derivative of the temperature? So, suppose we were interested in calculating d/dt of T tilde. I need to get T/dt of T star of y, e power sigma t sin alpha x.

This is a function of y that is the function of x. This depends on time so when I differentiate I am just going to differentiate only this term and this is going to give me sigma* e power sigma t times sin alpha x times t star of y. That is how the time derivative is found. Now, let us look at how one can calculate delta square of T tilde. Remember delta square is 2 dimensional it only has variations in x and y.

And so delta square is essentially D square/D x squared + D square/ Dy square* T tilde which is T star of y e power sigma t sin alpha x. Now, when I am going to differentiate this term with respect to x these are for all practical purpose is constant and the derivative of sig alpha x the first derivative is going to be alpha times cosine, alpha x. And when I differentiate one more time I am going to get minus alpha square when I get back sin.

So, please understand that this is going to be the same as the first term. When I look at the first term I am going to get –alpha square times sin alpha x times t star/y times e power sigma t. And now when I am going to differentiate this with respect to y these are going to be constant and what I have is essentially the second derivative of T star versus T star is only a function of y is not going to be a partial derivate any more but is going to be a total derivative.

And this is going to be written as d square/ dy square *T star times e power sigma t times sin alpha x. I can take out e power sigma t and sin alpha x, common from these 2 terms and I can write this in a slightly more compact way as d square/ dy square – alpha square *T star multiplied by e to the power sigma t sin alpha x.

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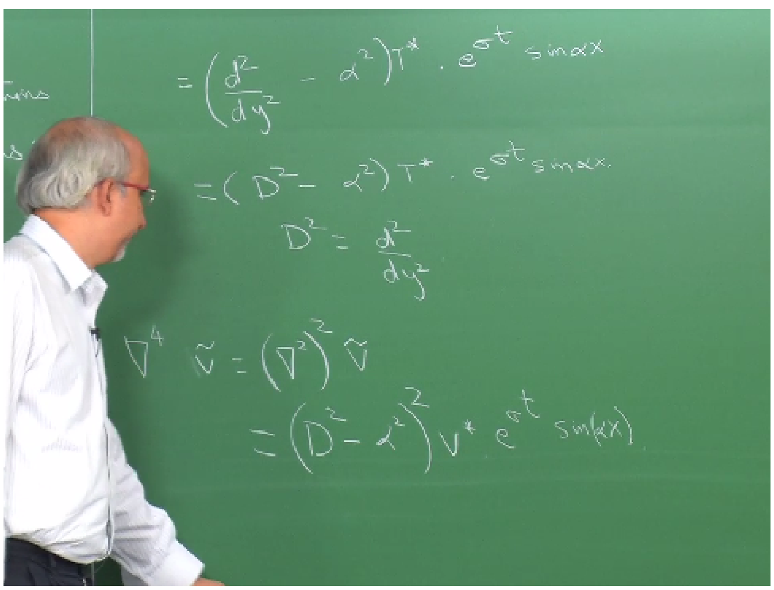

The derivative operator is only in the y direction. This alpha square is of course the wave number of the periodic disturbance. It is the reciprocal of a wave length. This is the wave number which is reciprocal of the spatial wavelength and these are the time dependency. So, what is normally done is to make things a bit compact I am going to write this as T star times e power sigma T time sin alpha x where D square is nothing but d square/ dy square.

So, remember this is the Laplace stand of t tilde. You see that there is also delta to the power 4 operator occurring in the velocity equation. So, the delta to the power 4 operator of the velocity equation is going to be of the velocity variable is going to be nothing but delta square of velocity. And you already know that delta square of velocity is nothing but D square –alpha square * V star e power sigma t sin alpha x because now it is velocity is of T star we have V star.

And so this is going to basically reduce to T square –alpha square the whole square*V star e power sigma T sin alpha x. Again this operator is operating only on V star. Remember V start is the function only of y and this derivative D here is a derivate only with respect to y. So, that is basically what we have done we have just made sure that the partial differential operator delta to the power 4 can be converted to an ordinary differential operator D square.

Now, it may appear to you that I have actually jumped a step but if you understood how you have done this I think what you would do is just go through the algebra and you can verify for yourself that this is indeed correct that is one, when you apply delta square you get the D square – alpha square. When you apply it again you get another D square – alpha square and that basically gives you this.

So, if you are not comfortable with this I suggest you work this out in your home and make sure that this is indeed right. Our job now is to basically substitute these expressions in my partial differential equation and convert it to an ordinary differential equation. So, let us do that.

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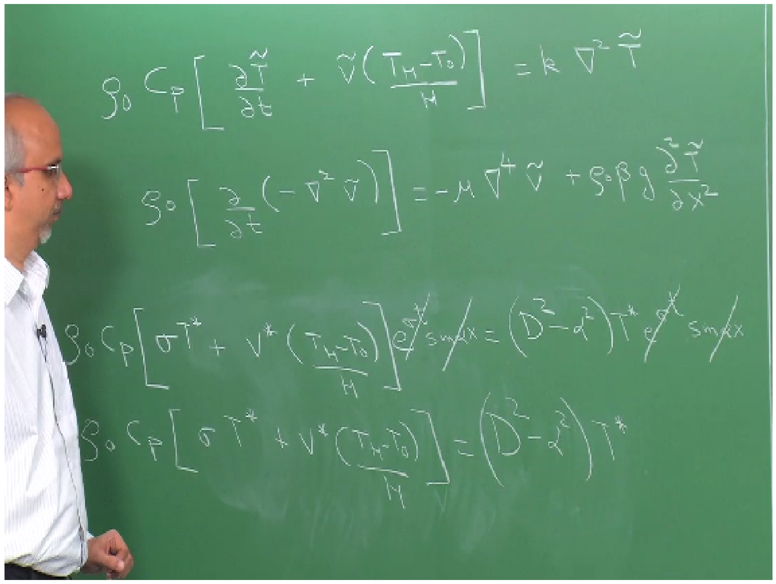

So, the first equation here on temperature now when I were to look at the time derivative with respect to temperature I would only get sigma. And this is got V tilde I am going to write this as V star of Th –T0/H and what I have done is we placed all my tilde variables in terms of my star variables. So, this is going to be sigma times T star because remember the temperature derivative with time is nothing but sigma times T star.

And you also have the e to the power sigma t times sin alpha x. And the right hand side is nothing but the D square – alpha square *T star times e to the power sigma t times sin alpha x and that is what we just did. Make sure that delta square can be reduced to D square – alpha square *T (()) (20:15) sin alpha x. And V tilde is V star times that. When you differentiate this I get a sigma time T star and this.

Now, remember we have made this assumption of things being in phase. Now, the fact that we are on the right track that the velocity and the temperature are indeed in phase is going to be conform by the fact that every term here I have this exponential term and the sin term. So, this basically an indication that indeed that those variables are in phase so this now reduces to rho 0 Cp times sigma T star + V star times TH –T0 /H =D square – alpha square * T star.

So, this is a linear equation but now it is linear ordinary differential equation. What I like to do next is take the second equation. Here this equation and convert it to an ordinary differential equation. So, when I do that I get rho 0 times.

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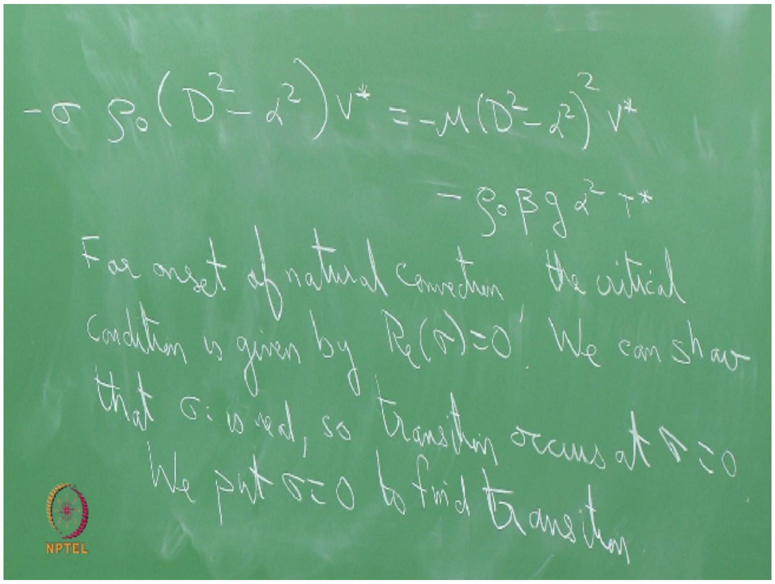

The time derivative gives me a sigma and the delta square operator gives me a D square –alpha square. And there is already a minus sign so that gives me this minus sign. So, what I am doing is I am looking at this term here. I am looking at this term and I am getting a sigma because of this time derivative. This delta square gives me the D square – alpha square and V tilde remember is now going to get converted to V star.

And this gives me V star *e power sigma t times sin alpha x that is my left hand side. On the right hand side, I have 2 terms the first term is the delta to the power 4 operator which is nothing but D square – alpha square whole square * V star times e to the power sigma t times sin alpha x. So, this is a viscous term which goes with the fourth order delta or delta to the power 4. And that gives me my D square –alpha square, whole square times V star times e to the power sigma t sin alpha x.

And then you have the body force term which remember is this term here. This term here is the body force term and this is associated with the second derivative with respect to x. This is something which I had not done earlier but remember the x dependency is sinusoidal. So, when I am differentiating it twice I am going to get a –alpha square multiplying this. And so this term now is going to be reduced to –rho 0 beta g alpha square times T star times e power sigma t times sin alpha x.

All I have done is said that invoked the fact that the second derivative of T tilde is nothing but –alpha square of T star. Again what we see is that the exponential term and the sinusoidal term cancel off because they are present in all the terms. They cannot be zero because if they were 0 then my perturbation itself is 0. Because I have assumed the perturbation to be of the form e (()) (24:33) sin alpha x.

So, they are non 0 and that basically justifies and allows me to cancel them and the fact that they are occurring in all the terms tells me that the assumed form of the special dependance for temperature and the velocity in the x direction the periodic sin alpha x in phase is indeed right. If they are not cancelled off, then it means that those velocity components are actually out of phase with the temperature component.

And there are situations where variables can be out of phase. So, this equation now simplifies to –sigma times rho 0 times D square –alpha square times V star = - mu times D square –alpha square whole square V star –rho 0 beta g alpha square T star. So, what I have done is converted partial differential equation to ordinary differential equations and the idea is I know how to solve ordinary differential equal especially because these equations are linear.

What I am going to do now is tell you that the objective we have is to find this point of onset of natural convection. When exactly is natural convection going to start? That is this critical value of the temperature gradient for a fixed fluid and the geometry. So, when the temperature gradient is less then this critical value if you were to impose any disturbance this disturbance would actually decay and you would have the system going back to the stationary solution where the fluid does not move.

If the temperature difference is more than this critical value when you are going to be giving a disturbance, the disturbance will get amplified such that you would actually see convection. So, the transition between the stable to the unstable is going to take place by looking at the real part of the growth constant in time sigma. If the real part is negative I have a stable system. If the real part is positive I have an unstable system.

So, the critical point where you have the change from stable to unstable is going to be given by the condition that the real part of sigma is 0. So, for the onset of natural convection the critical condition is given by the real part of sigma =0. And one of the things which we can establish I am not going to do that in this course is show that for this particular problem the sigma is real that is the sigma is not complex. There is no imaginary component.

And so rather than talk about the real part of sigma being 0 I am going to talk about the sigma being 0. We can show that sigma equal real so transition occurs at sigma =0. Now, I just give you an inkling of how we can go about proving that sigma is indeed real. And this arises because you know matrices which are real symmetric they have the property that their eigenvalue are real and what we can do is generalize this idea of a real symmetric metrics.

To that of a Hermitian metrics to that of what is known as a self-adjoint operator. So, rather than talk about matrices we can look upon this as an operator a matrix takes a vector who convert it to another vector. This is an operator which takes a function, converts it to another function and we can talk in terms of eigenvalues of this operator and for this particular system we can look at the fact that whether it is self-adjoint joint or not and establish a sigma 0.

So, this is just some piece of information I am giving you for those who are interested in pursuing this. Otherwise you just accept what I am saying sigma is indeed real and so the transition occurs at sigma being zero. So, since I am interested only in the onset of the natural convection what I am going to do is I am going to further simplify my equations by putting sigma =0 in these ordinary differential equations.

That I have just derived which describes my V star and T star. So, we put sigma =0 to find the transition and that basically means this particular equation reduces to mu*D square –alpha square whole squared times V star = -rho 0 beta g alpha square T star.

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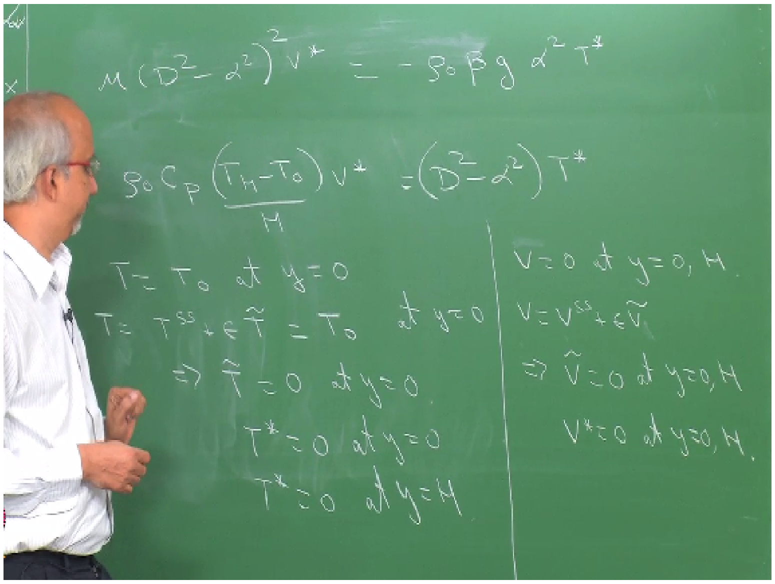

I just want to make sure that I am not messing up the negative sign anywhere because I might into trouble later. And the other equation becomes rho 0 Cp* Th-T0/H*V star =D square-alpha square* T star. See, I have 2 equations now well these are ordinary differential equations and they have a velocity and temperature and you can see that the velocity and the temperature again coupled to each other.

Remember all these are constants which I know for a given experimental system. What I like to do now is write down the boundary conditions for this system of equations that we have. The condition on temperature is going to be obtained from the boundary conditions on temperature that is going to be telling me what the boundary conditions are for the perturbation. So, remember T=T0 at y=0 and you know that T was written as Tss +epsilon* T tilde.

And what we want to do is we know the Tss=T0 and y=0 therefor you know Tss =T0 at y=0 which basically implies that T tilde =0 at y=0. So, basically since the steady state satisfies the boundary condition of the original problem the perturbation is going to vanish at y=o. Similarly, and what we have done is we have decomposed T tilde to T star of y and so this is actually the condition on T star *y. T star =0 at y=0.

Because T tilde is nothing but T star multiplied by e power sigma T times alpha x and those are independent of y. So, the only way T tilde can vanish is if T start can vanish. You can similarly have stablished that T star =0 at y=H. These are the boundary conditions on the temperature perturbation. Look at the equation for velocity. This is a fourth order equation for velocity and therefor any four conditions 2 on the upper plate and 2 on the lower plate.

Remember v star is vertical component of velocity. So, my plate is impermeable the liquid cannot penetrate my upper plate. So the velocity component is going to be 0. V is going to be 0 at both y=0 and h and so I am just going to write this here that V tilde =0 at v=0 at y=0 and H. This follows from the fact that the liquid cannot penetrate the wall. And v remembered this nothing but we vss + epsilon times v1 tilde and this is 0 at y =0 and h and therefor this implies v tilde is 0 at y=0, H.

And in another words v star is 0 at y=0, H. So, I have v star also being 0 so I have Dirichlet conditions on temperature and on velocity v star. But do I have enough conditions to solve the problem? The answer is no since I actually have a fourth order equation remember in v star. So, I need four boundary conditions and what I have is only 2 boundary conditions. So, I need 2 more boundary conditions where am I going to get this from?

I am going to get this from the conditions on the x components of velocity u, remember what we have done is we have converted this problem simplified it by eliminating the x components of velocity. In this process we have not used the boundary conditions on the x component of velocity. So, we have to figure out a way for converting the boundary conditions on the x component of velocity to conditions on the y component of velocity v star.

So, let us see how we can do that and let me give you a clue we are going to use the equation of continuity to accomplish this. So, what I want to do is I want to get 2 more boundary conditions for v star by using the boundary conditions on the x component of velocity u star.

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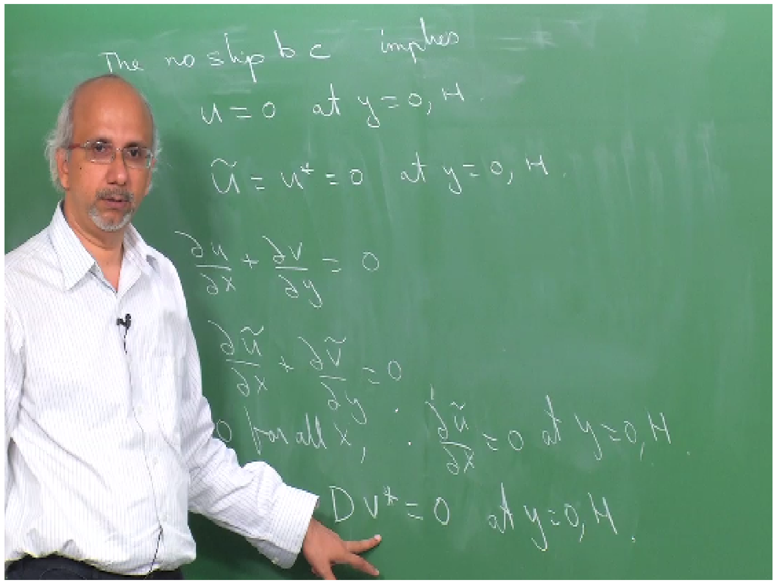

So, the no slip boundary condition implies that u=0 at y=0 at H and you can make the same argument but the steady state velocity is 0 at 0 and H and so u tilde which =u star =0 at y=0, at H and this is my no slip. So, this is the boundary condition on my original velocity. This the boundary condition on my x component of the perturbation velocity. So, this is for the original and this is for perturbation. I want to convert this to a boundary condition on v star.

Remember du/dx + dv/dy =0 that is my equation of continuity. And I am going to write this again in terms of my perturbation variables. So, I get du tilde/ dx+ dv tilde/dy =0 you can convert this to star variables. But remember u tilde is nothing but 0 and y=0 and H. So, that means all along for all x this is true y=0 and H sorry this is not true y=0 and H. This is equation of continuity which is always valid but u tilde is 0 at y=0 and H.

Which means that u tilde does not change with x for no matter what the position is in the horizontal direction, no matter what the x position is u tilde is 0. So, not only u tilde 0 but du tilde/ dx is also 0. u tilde =0 for all x therefore du tilde/dx =0 at y=0 at H. So, that means if du tilde/ dx is 0 that means the first derivative of v tilde is 0 at y =0 and H. And this I can use to say that Dv star =0 at y=0 and H.

So, this the boundary condition which I have on velocity. So, what I have done is basically these are the extra 2 boundary conditions which I was talking about earlier which I need to solve my problem and this basically tells me that the first derivative of the velocity perturbation v star is 0 and the 2 walls. So, now I have 6 boundary conditions and I am all set to solve the problem. However, what we would now do is do a further simplification.

And this simplification is going to come by converting the system of 2 equations that we have to only one equation, one variable. So, what we have here is a system of 2 equations which are coupled to each other in 2 variables v star and T star. I like to write this as a system of equations or only one equation in one and on v star. That is I want to eliminate my temperature perturbation between these 2 equations.

I like to keep my equation as if it is an equation which describes only the velocity perturbation without brining into account the temperature perturbation. So, let us do that by operating on both sides by D square –alpha square so when I do that these are all constants. I have D square – alpha square of T star. I can use this equation and substitute for that expression from here and that way I can eliminate T star. So operating on this equation by D square –alpha square.

What I get is mu times D square –alpha square whole cube*v star = - rho 0 beta g alpha square of D square –alpha square T star.

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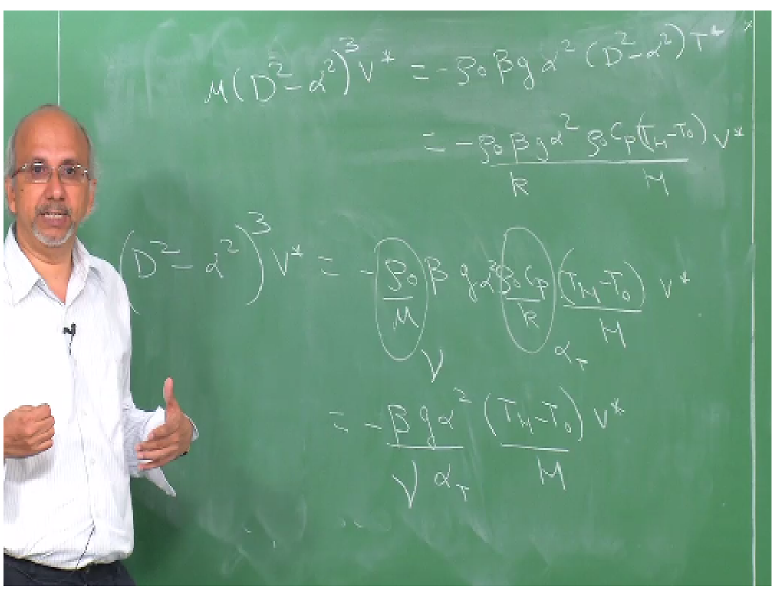

So, that is what I have done I have just operated on that using D square – alpha square. And now

I am going to use the fact that D square – alpha square of T star is given by my velocity perturbation from the second equation to write this as –rho 0 beta g alpha square. And S square – alpha square T square is nothing but rho 0 Cp times TH- T0/H. You know, I think I have missed a thermal conductivity somewhere. I have missed a thermal conductivity here.

I have missed a thermal conductivity in that equation. So, this thermal conductivity is important because remember the delta square comes with the thermal conductivity and so I would have a k at the bottom here. And I can write this equation now as D square – alpha square whole cube of V star = - rho 0/ mu times beta times g times rho 0 Cp/k times TH –T0/H and I have missed a V star here that is going to be a v star here.

So, I have done is rewritten this as it is here and that is an alpha square which is important. I brought the mu down here at the denominator. I am going to remember that this is nothing but my kinematic viscosity and this is nothing but my thermal diffusivity alpha T. And I am going to write this as –beta g alpha square/mu the kinematic viscosity times the thermal diffusivity. Times TH-T0/H times v star.

So, this is my 6 order equation and like I said I need 6 boundary conditions. I have found 4 boundary conditions on velocity that is the velocity perturbation and the first derivative must be 0. But remember my other 2 boundary conditions are on temperature. So, again what I want to do is? I want to convert my temperature boundary condition to a velocity boundary condition. And how do I do that? I am going to use the fact that T star is 0 and y =0 and h.

See if T star =0 and y=0 and H that means this term has to be 0 and y =0H.

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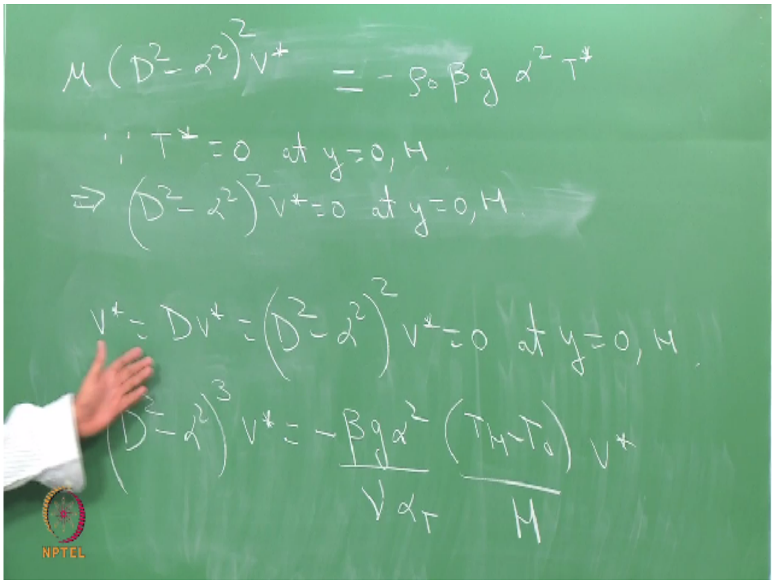

Since T star =0 at y=0 and H. This implies D square –alpha square whole square of v star =0 at y=0 and H. So, what I have done is I have converted my boundary condition on temperature to my boundary condition on velocity. So, this 6 order equation that I have just written is going to basically need six boundary conditions. And the 6 boundary conditions are V star =Dv star = D square – alpha square whole square of v star =0 at y =0 and h.

And the differential equation is D square –alpha square whole cube*v star = -beta g alpha square times mu/alpha t times TH-T0/H v star. So, this is the differential equation. This as a boundary conditions and what we have to do is see how we can solve this. We will do this in the next class. Thank you.