**Multiphase Flows: Analytical Solutions and Stability Analysis**
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**Lecture – 24A**
**Rayleigh-Benard convection: Linear stability analysis part 1**

So welcome to today's lecture and what we are going to do today is basically talk about the problem of natural convection and in particular the onset of natural convection. Now this is a very classical problem and it was investigated almost a century ago by Rayleigh and Benard and so this problem also goes by the name of Rayleigh-Benard convection.

What we are going to talk today is how we can determine the onset of natural convection by posing it as a stability problem and this particular problem we are going to discuss where maybe next set of 2 or 3 lectures, which is basically to illustrate the concept behind how to carry out a linearized stability analysis.

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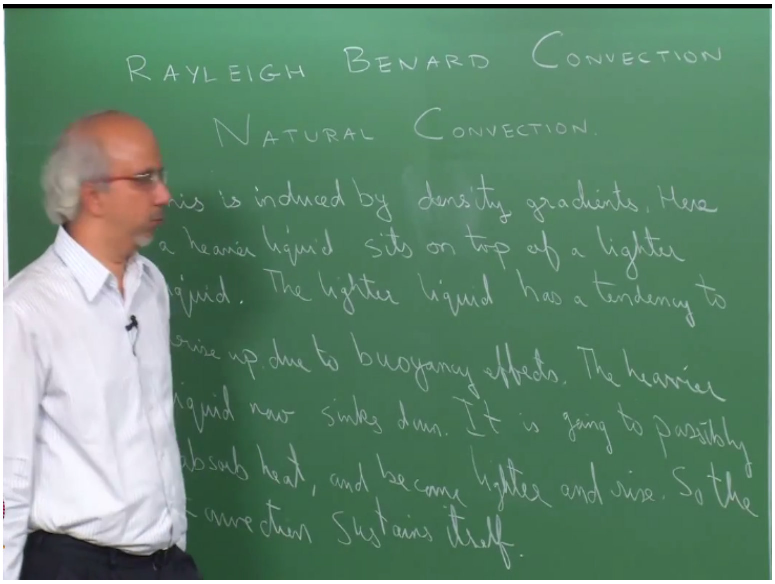


So this is the problem of Rayleigh-Benard convection or the natural convection. The problem of natural convection is the one where you have a density gradient such that the less dense fluid is at the bottom and the more dense fluid is at the top. So because the density of the fluid lying below is lower, this has a tendency to rise up and this is essentially due to the buoyancy force.

When this packet of fluid which is lighter rises up, you have the heavier fluid from the top which is going to sink in and replace this lighter fluid. Now if you have a mechanism by which you can actually sustain let us say a temperature difference so that the temperature of the bottom is higher, the temperature of the top is lower, this heavy fluid which was colder is again going to get heated up and is going to rise.

So what this essentially means is you could have a situation wherein the liquid or the fluid is going to exhibit convection. Now this situation will not arise when you have a density gradient such that you have a heavier liquid at the bottom and a lighter liquid at the top. So that is a stable stratification and the heavier fluid likes to sit at the bottom, the lighter fluid likes to sit at the top and you are not going to see any convection.

So natural convection is one that is induced by density gradients where you have a heavier liquid at the top sitting on a lighter liquid at the bottom okay. So basically what I am trying to say here is this is induced by density gradients. Here a heavier liquid sits on top of a lighter liquid. The lighter liquid has a tendency to rise up okay due to buoyancy effects. So the heavier liquid now sinks down.

It is going to possibly absorb heat and become lighter and rise. So the convection sustains itself.

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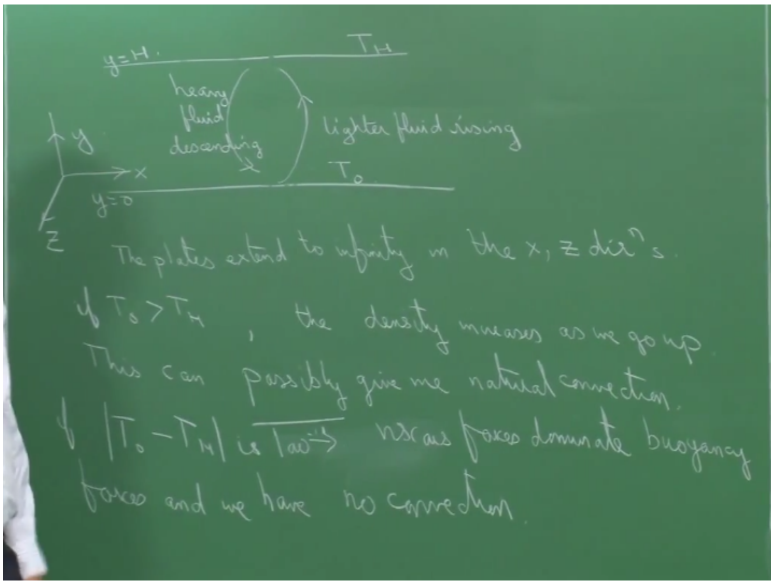


Now to get down to something very specific and in order to do the analysis, we would be looking at fluid which is confined between 2 plates and the coordinate system we are going to use is this is x axis, this is the y axis and this is the z axis. So the z axis is actually coming out of the plane of the board, the x axis is pointing rightwards and the y axis is pointing vertically up. What I have drawn here is basically 2 plates okay.

And these are solid plates, which are impermeable and these are located at y=0 and at y=H. Now to keep life simple, we are going to assume that the plates extend to infinity in the x direction as well as in the z direction okay. So the plates extend to infinity in the x and z directions. You will see later on what the implications of this are but at least in the z direction when I am assuming the extend to be infinity, I am going to end up neglecting changes in the z direction as well as the velocity in the z direction.

Basically, this z direction extending to infinity allows me to solve a 2-dimensional problem in the x-z plane okay. This is something which helps me illustrate the idea to you today in a very clear manner and it helps me simplify things mathematically. I had remembered that is one of the objectives of course, we want to simplify things mathematically but capture the important physics.

And the x direction extending to infinity essentially it allows me to neglect boundary effects in the x direction. Essentially, it means that I would be looking at periodic solutions possibly in the x direction okay. If I had the x direction extending to a finite extent, I would have to worry about the boundary conditions, which I would have to impose and how that is going to affect my system behavior okay.

Now since we are interested in a problem of natural convection and therefore we need to have variation of density and one way I am going to impose this density variation is by imposing a temperature variation. So what this means is I am going to keep the plate which is at the bottom at a temperature T0 and the plate which is at the top at y=H at a temperature TH okay. So there are these 2 plates, which are basically surrounding my liquid or fluid.

And they are maintained at temperatures T0 and TH. So if T0 is>TH, we have a situation where the density increases as we go up. So T0 is higher the fluid here is going to be lighter, the fluid here is heavier, so the density increases as we go up and remember this can possibly give me natural convection. I call this natural convection because I do not have a pump which is actually making my liquid flow and this actually being caused by density gradients.

So what is going to happen and this is one way for you to visualize this. Supposing we do have natural convection then this fluid, which is light will have a tendency to rise up okay and when this fluid packet rises up, the heavier fluid from the top is going to come down. Once this heavier fluid from the top comes down, once it reaches the bottom, it is going to get heated up; it is going to become lighter.

This fluid which was lighter is going to go up okay. It is going to become heavier because this is at a lower temperature and once it becomes heavier at the top this is going to have a tendency to come down and so basically what I am trying to show here is that you have a sustained convection okay. So this is the lighter fluid rising and this is the heavy fluid descending.

And this sustains itself because of my boundary conditions I am maintaining the temperature here higher than the temperature here okay. Now this is the kind of situation which is actually prevalent in the atmosphere where the atmospheric air is going to get heated by the sun's radiation and you would have a situation where you can have a hotter liquid sitting at the bottom on the earth surface and you have you know heavier fluid, heavier air at the top okay.

So basically this convection is going to be induced even in a natural condition like the atmosphere without any pumping effect. So you have you know because of the sun's radiation the surface of the earth being hot, it is going to cool now as you go up and so you have a heavy fluid sitting on top of a lighter fluid okay and you can have natural convection.

So rather than talk about the problem of natural convection in the earth's atmosphere we are going to idealize the situation, try to capture the important effect and try to analyze the problem okay and for this idealization I am going to use this particular geometry of fluid being present between these 2 plates okay and so this is going to allow me to capture the important physics and do the analysis.

So this is idealization of the natural convection problem, which is prevalent in the earth's atmosphere okay. Now I have used the word that this can possibly give me natural convection. Why do I say possibly? Because clearly if the temperature difference T0-TH, if

T0-TH is<a critical value or rather is low, the density variation is going to be less as well and what is going to happen is the liquid is going to be or the fluid is going to be stationary.

Why is that? Because the viscosity of the fluid prevents it from rising up, so you have 2 effects in this problem, you have the buoyancy force, which is trying to push the liquid up okay and you have the viscosity, which is essentially a frictional force, which is going to prevent this liquid from moving. So you have viscous forces and you have buoyancy forces and very low values of T0-TH, the viscous force is dominated the buoyancy forces.

And you have no convection; however, if T0-TH is sufficiently high then the buoyancy forces are going to dominate the viscous forces and you will have convection okay. So the point I am trying to measure is if T0-TH is low, the viscous forces dominate the buoyancy forces and we have no convection. The buoyancy forces are too small and so the liquid cannot rise up.

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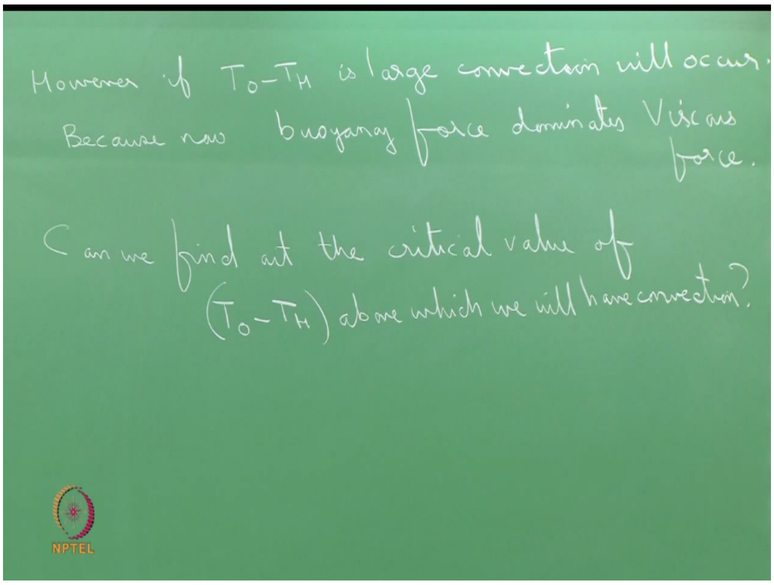


However, if T0-TH is large, the convection will occur and why is that? Because now the buoyancy force dominates the viscous force okay. So clearly what this means is that there is a particular value of this temperature difference and this is going to be decided by the properties of the fluid. This is going to be decided by the spacing between the plates above which you are going to have natural convection.

And below which you will have no convection and the liquid is going to be stationary. So now the question is can we find out this critical value of T0-TH this temperature difference above which we have natural convection okay? So can we find out the critical value of T0-

TH above which we will have convection and the answer is yes, we will be able to determine this value by posing the question as a stability problem okay.

So what we will do is we will look at the state where the liquid is not moving, the liquid is at rest and ask the question whether this particular state is stable or unstable. So if the state were to be stable that means you would be able to experimentally observe it and if the state is unstable you would not be able to experimentally observe it and you would possibly have a different state, which you would observe okay.

So now the question is if I were to pose this as a stability problem, I need to be able to find out the steady state whose stability I am interested in okay. In this particular problem, the steady state which naturally arises is the one where the liquid does not move, where the velocity components are 0 and this happens for low values of the temperature difference okay.

So we will treat this particular steady state as the base state and we will try to find out the stability of this base state and the way we have find out the stability of the base state is by imposing small infinitesimal perturbations and see how these infinitesimal perturbations evolve in space and in time okay. So just by way of trying to explain to you as to the what the complexity level of this problem is.

This problem is complex in the sense that it is going to be governed by partial differential equations, which are essentially the equations of momentum, the Navier-Stokes equations and the equation of continuity and the energy equation, which basically tells you how the temperature varies in the bulk liquid. It is not so complicated because we are focusing our attention only on a single liquid, a single phase okay.

It would be more complicated if I had let us say a layer of 2 liquids between these plates in which case it would be a real multiphase flow system. So we would be looking at those problems later but the point I am trying to make here is that we are right now interested in only a single phase system. So we do not really have to worry about the boundary conditions because I have a problem where I have solid walls okay that is my idealization.

And so the boundary conditions, which naturally arise as the ones where I have no-slip boundary conditions at the walls and the fact that I do not have a perpendicular velocity component okay because the wall is impermeable okay. So that is basically where we are. So what we are going to do now is basically try to solve this idealized problem okay and find out the conditions under which natural convection occurs.

But remember we are going to follow the procedure of having to write down the governing equations, trying to simplify things mathematically so that we capture only the important physical effects, then doing a linearization about the steady state, find out what the steady state stability is and try to answer the question of what is the critical value of this temperature difference above which we are going to see natural convection okay.

So let us look at this idealized problem and let us write the governing equations.

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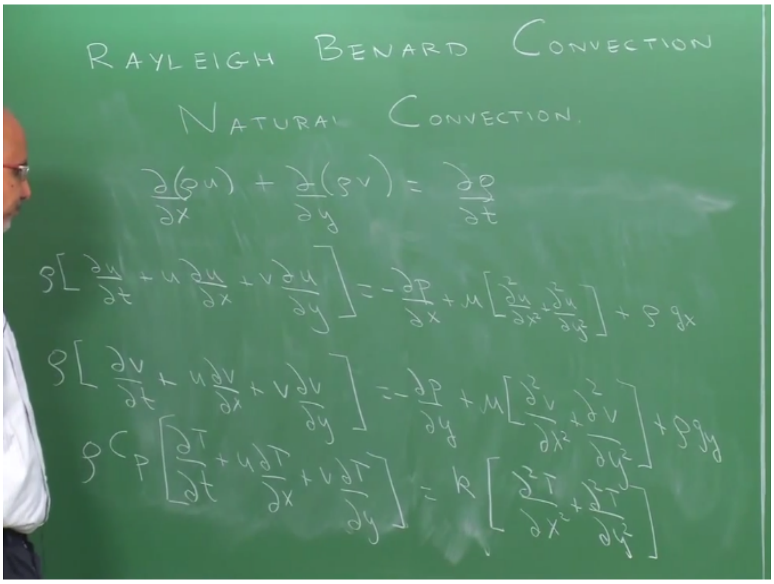


First we have the equation of continuity which says the derivative of I am going to write this in an expanded form equals d rho/dt. Rho is the density, u is the x component of the velocity and v is the y component of the velocity okay and what I am going to do is I am going to keep in mind the fact that the density in this problem is actually going to be varying spatially and so I am writing the equation of continuity in the most general form where the density is inside my derivative operator.

The next equation which I really have to worry about is the momentum equation and that in the x direction tells me rho du/dt+u du/dx+v du/dy=-dp/dx+mu times d squared u/dx squared+d squared u/dy squared+rho the subscript x that is the body force. So I am just writing it in the most general form right now okay and in spite I have written this for the 2-dimensional case where I have not taken into account the velocity in the z direction.

And I have not taken into account the variation of the velocities in the z direction okay. Similarly, the equation in the y direction is going to give me. Now in addition to these equations, I need to write down the energy balance equation because remember the temperature difference is what is actually driving the flow because that is the one which is inducing my density variation.

So when I have to write the equals k times d squared T/dx squared+d squared T/dy squared. So this basically is the accumulation term of temperature, this is the convective flux for heat and this is my conductive flux for heat. What I want to tell you is that we actually have these 4 equations which are coupled to each other, the coupling is going to occur because the temperature is something on which all the properties the density, the specific heat, the viscosity, the thermal conductivity depends upon okay.

So clearly as you can see we have a bunch of partial differential equations and they are nonlinear because of these inertial terms here okay and what we want to do is we want to ask the question rather than solve this by brute-force using some kind of numerical scheme, is it possible for us to actually simplify these equations and get some insight okay. So our philosophy now is to simplify these equations, retain the important physics.

And not to worry about you know unnecessary effects. So for example I can mention the physical properties viscosity k, Cp all of them depend on temperature but what we are interested in is the phenomena of natural convection wherein only the effect of density on temperature has to be retained and is important okay. So what we will do is we will assume all other physical properties are constant as far as temperature is concerned. And only the density is the function of temperature.

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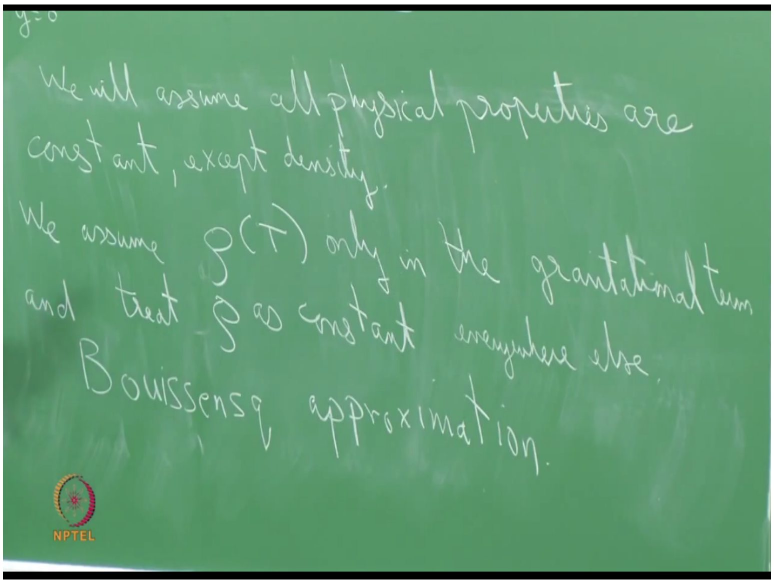


So we will assume all physical properties are constant except the density okay. So density is going to be a function of temperature and I need to retain that to be able to see this natural convection. Now the question is can I simplify it further? So for example density occurs in several places in the governing equation.

It occurs in the continuity equation at these places at 3 places, it occurs in the momentum equations both in the inertial terms as well as in the body force term. So in all these terms is it necessary for you to actually retain the effect of the variation of the dependency of density on temperature? Is it possible for you to say look it is important to retain the effect only in one term and for all practical purposes treat density as a constant in all the other terms?

So can I further simplify my equations that is the question and the answer is yes we can possibly further simplify my equations. I am going to assume that the one term in which the effect of density on temperature has to be retained is the one which is going to give me my buoyancy force and because the buoyancy force is required for my natural convection and so I am going to retain the effect of density on temperature only in my body force terms.

At all other places I am going to assume the density is a constant, some mean average value okay. So this is a further assumption, a further approximation I am making and this is what the cracks of modeling is to try and retain the important physical effect. So what I am saying here is we assume density is a function of temperature only in the gravitational term and treat rho as a constant everywhere else okay.

This particular approximation is called the Boussinesq approximation. So the Boussinesq approximation tells you, you can treat density as a function of temperature only in the gravitational term, assume it to be constant everywhere else okay. Now the question is how valid is this and we can find out the validity of this approach that is the accuracy with which you are going to get results under these assumptions only by validating it with actual experiments.

So it turns out that the experiments tell this as are going to be in close agreement with the prediction under these assumptions then you are justified to make these assumptions. If it turns out that there is a big mismatch between the predictions using these assumptions and the experimental values observed, then clearly this is the bad assumption to make and you need to go back and relax this assumption and work on the more detailed problem.

So the only way I can actually find out if this assumption is actually valid is by comparing the predictions of this model with experiments okay and this is an extremely important aspect but the idea is I am hoping that making this assumption is going to be valid because it helps me simplify my model and it helps me get a deep physical insight into the behavior of the system as to what the role of different parameters are.

Remember if I do not make the simplifications, I have a problem with several parameters and I will not be in a position to understand what the effect of each parameter is in determining the behavior of the system for example what is the effect of the distance between the plates, the viscosity, the thermal conductivity etc., etc. okay.

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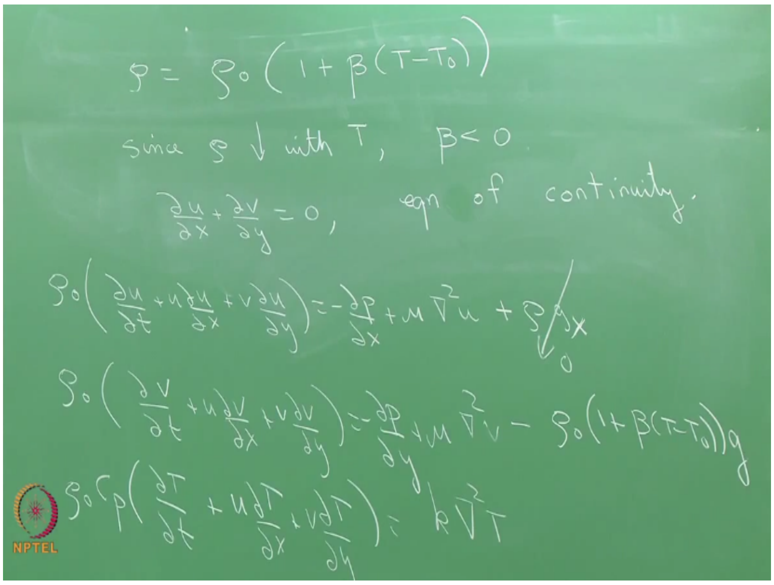


So now what I am going to do is that the density varies linearly with temperature, this is the linear variation with temperature. What I want is the density to actually decrease with temperature and since density decreases with temperature beta has to be negative okay and I am going to assume rho=rho0 in all the other terms. So this basically allows me to simplify my governing equations.

For example, now my equation of continuity is going to be just du/dx+dv/dy=0 because now density is constant in that equation this is my equation of continuity, I need to simplify my x component equation, I am going to write it as rho 0 times du/dt+u du/dx+v du/dy=-dp/dx+mu del squared u where I have just written del squared as a 2-dimensional Laplacian, which has the second derivatives in the x and y direction okay.

Plus, the body force term in the x direction is 0 because the x direction is horizontal and the gravitational force is in the vertical direction and in the y direction I similarly have dp/dt+u dv/dx+v dv/dy=-dp/dy+mu del squared v and now I am going to have to worry about the body force term because it acts in the negative y direction, remember my y direction is pointing in the upward direction.

So y is going to be –g and density now in this gravitational term I am going to retain the dependency on temperature and I am going to write this as –rho 0 times 1+beta times T-T0 times g. So this is the important point here. The negative sign comes because gravity is acting in the negative y direction. I am retaining the linear dependency of density on temperature this term okay.

And that is the assumption we made and then you a have the rho0 times Cp times the time derivative of temperature+u dt/dx+v dt/dy=k del squared T. Again I repeat that the del squared operator is a 2-dimensonal Laplacian with variations considered only in the x and the y directions okay. I like for you to now see that these equations are all coupled, for example the temperature equation has velocity in it.

The velocity equation has temperature in it and the u velocity and the v velocity are all coupled, there is a pressure term. So all these equations are actually interlinked and therefore necessary for you to solve them all together simultaneously okay, but what we will do is we will exploit the fact that we are engineers, we will use our understanding of the physics and try to find steady state to this problem okay.

This steady state that we are interested in is going to be the one which is stationary, the one where the liquid does not move. So that makes our life simple, what it means is that the 2 velocity components are actually going to be 0, u and v are going to be 0 okay and we are going to use this to find out how the temperature is going to vary. We are going to use this steady state to find out how the pressure is going to vary.

So that is going to be the steady state, which we are going to determine. Once we have found what the steady state is, we would then ask the question what is the stability of that steady state.

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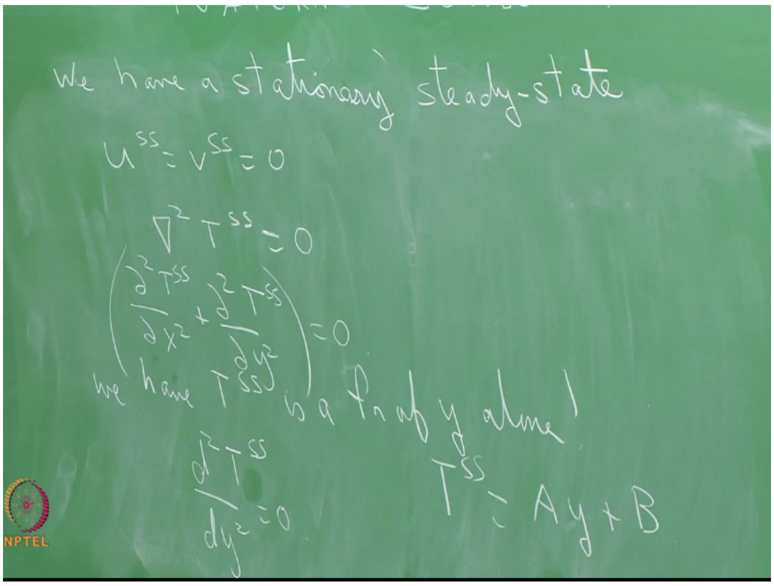

We have a stationary steady state and so uss=vss=0 because the liquid is for moving and so clearly this steady state is going to satisfy your continuity equation. What I would like to do is see when you have such a steady state what is going to be the temperature profile? How does the temperature change? And so for this I am going to substitute this in my energy equation.

And when I substitute it in my energy equation and since I am looking at a steady state the time derivative goes to 0 okay, this is my time derivative term that goes to 0 because we are looking for a steady state. Remember u and v are 0 at the steady state of interest to me and so these terms are going to vanish and what I am left with therefore is that the steady state temperature is going to be governed by the Laplacian of temperature being 0.

So the steady state temperature is going to govern by this and if I were to write it in an expanded form, this is what I am going to get=0 the steady state. Clearly, if you now look at the boundary conditions of this problem, the temperature of the lower wall is the uniform T0, temperature of the upper wall also is uniform. There is no variation in the x direction, so there is nothing in the problem, which is going to induce a variation in the x direction for the steady state.

So what this means is the steady state temperature is not going to be varying with x okay and so we have Tss is a function of y alone and this is therefore going to be governed by d squared Tss/dy squared=0 and clearly you can find Tss as being just a linear profile and we can calculate A and B by looking at the boundary conditions.

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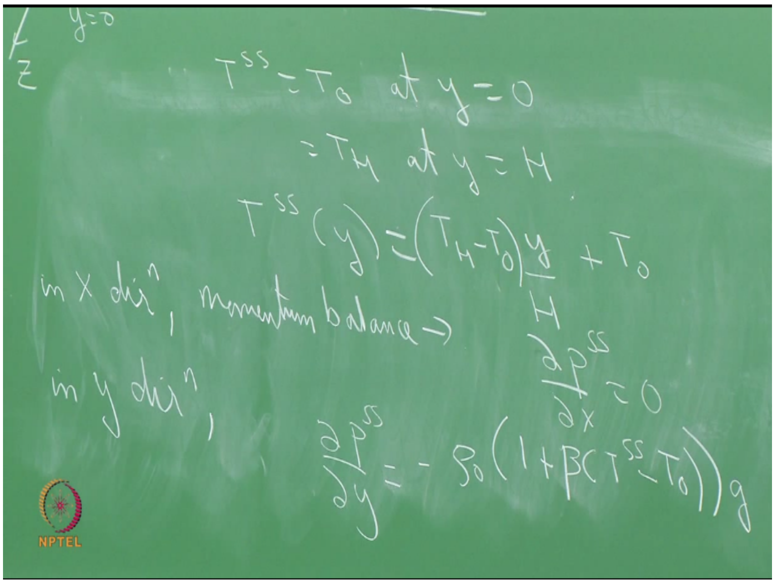


The boundary conditions are that Tss=T0 at y=0=TH at y=H. So these boundary conditions help me evaluate my constants A and B and you can do this exercise and you will find the Tss of y is going to be given by so at y=0 Tss is T0, so I have T0 and then I have y/H. So you can see that when y is 0 I have T0 and when y is H I have TH. So clearly the temperature is varying linearly in the y direction and that is to be expected because the liquid is not moving.

So for all practical purposes it is behaving only as a solid and we only have conduction as the mechanism for heat transport and the temperature is varying linearly okay. Now what information am I going to get form my momentum balance equation in the x and y direction? So if I were to now use the momentum balance equation in the x direction, in the x direction my momentum balance what does it yield?

It tells you that the velocities are 0 and so I get something like a trivial relationship dpss/dx=0 this means that there is no variation of pressure in the x direction and this is something we expect because we actually have hydrostatic phenomena. You have hydrostatics, liquid is not moving and you expect that there is no variation of pressure in the x direction okay. What about the equation in the y direction?

In the y direction, we are going to get dpss/dy=-rho0 times beta times Tss-T0 times g. So here the pressure variation in the y direction is going to be present because as the vertical direction and since the density is not a constant, I want to incorporate that. I see that the temperature that I am actually going to be using is that of the steady state temperature, which I have already determined as my linear profile.

So I know how the temperature varies in the y direction, I can substitute that inside here and I can calculate how the pressure varies in the y direction okay. So that is the basic steady state that we are going to find the stability of. So let me just write down what the steady state is whose stability we are interested in.

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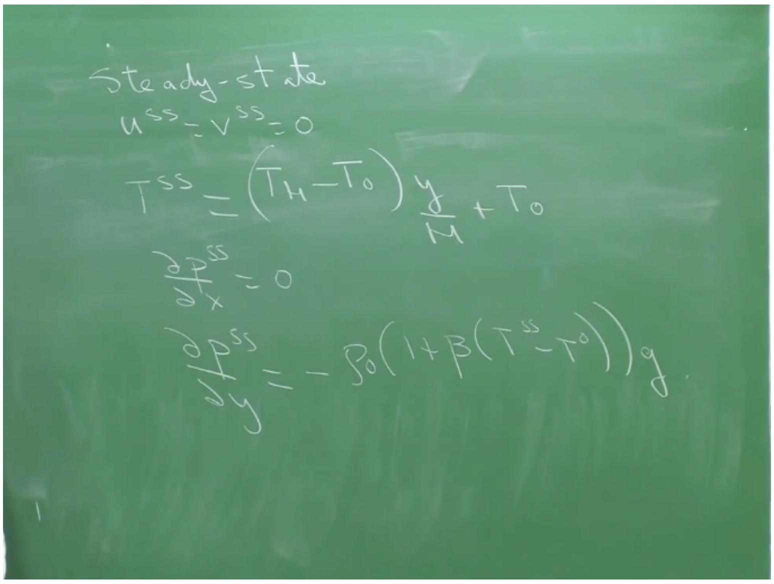


Steady state is uss=vss=0, Tss is TH-T0 times y/H+T0 and dpss/dx is 0, dpss/dy=-rho0 times 1+beta times Tss-T0 times g and that is the steady state whose stability we will be finding out in the next class. Thank you.