Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 20
Introduction to stability of dynamical systems: ODEs

Good morning everyone and welcome to today's lecture. So today we will be looking at short introduction to the stability of dynamical systems.

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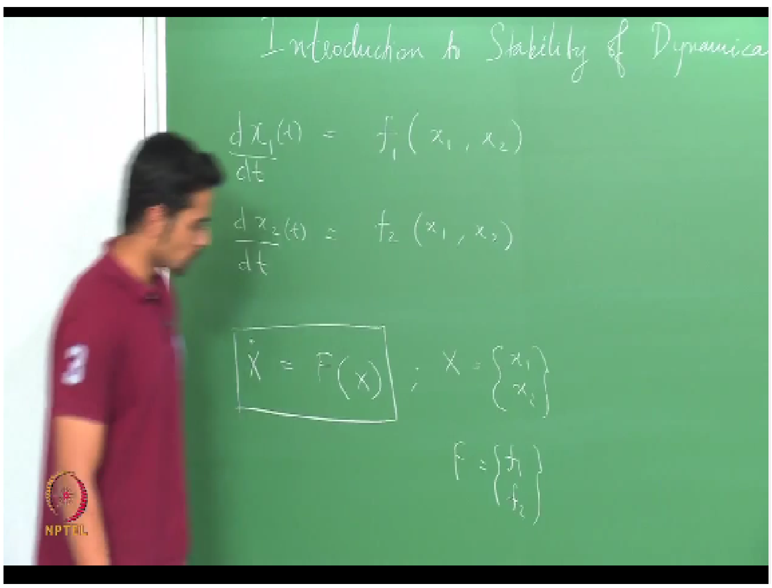

So in the previous lectures so far in this course, we have seen how to formulate problems in fluid mechanics including 2-face flows and how to obtain a solution for the case of a steady-state in most of the problems. Ultimately we are building up to a position where we want to be able to study some complex 2-face flows which have time dynamics. So how it fits in with the previous part of the course is that so far we have seen how to find one particular solution under certain assumptions to the Navier-Stokes Equations.

But the problem is that that solution is not always stable which is what we discussed at the end of the previous lecture that often there will be multiple solutions to the Navier-Stokes equations because they are highly non-linear and what will happen physically is that for some set of parameter and Reynold's numbers and other parameters, you will have one solution stable and that is what you will see in experiment.

But on the other set of conditions when you increase the Reynold's number or in terms of an experimentary parameter if you turn up the flow rate, at some stage you will have a transition to a new type of behaviour and specially you will see a new kind of pattern. So what has happened is a solution has become unstable and a new solution has become stable. So to understand these transitions, we can draw on the theory of diameter systems and the stability of these different steady states.

So in this class, I will take a very simple second-order 2-dimensional dynamical system and we will track, go through how we can understand the stability of a steady-state and how the transition happens. So before I start, how many of you all are familiar with the idea of linearization and then stability of systems in ODE's. You can have a show of hands. So I can gauze where to pitch select, alright.

So for the audience watching this lecture also, this is the basics of what we were doing course like process control as well and I will just focus on the simple ideas here. For those of you all who already know some of this what I will do is try to give it a spin that allows me to generalise what I am going to do today to the partial differential equation setting which is what we are really interested in this course.

So let us begin. Let us consider very simple system. So I have 2 variables x1 and x2 which are dependent on time and system is governed by 2 equations that tell me what the time dynamics are going to be and in general this first derivative will be some function, F1 of the 2 dependent variables and since the right-hand side does not have time in it, it tells us that the system is, basically the derivatives are time independent or how the system evolves depends on the x1 and x2, its current state.

I can rewrite these 2 ODE's in a more condensed form where the dot represents the derivative. F is the vector function and this capital X is a vector of the 2 quantities x1 and x2 and capital F is… So this is a standard representation of a system in a condensed notation. So what I will try to do today is layout the steps that one would need to follow to analyse the system for steady-states and stability.

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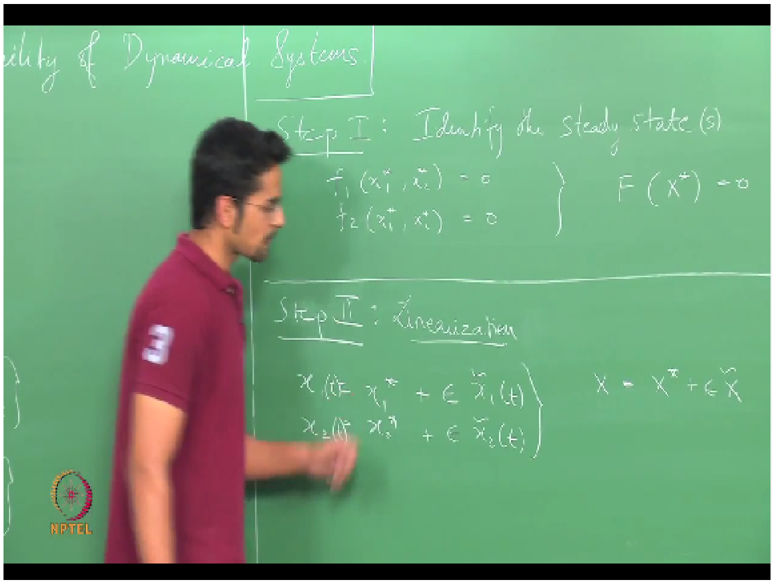

So the very first step is to identify the steady-states and generally it will be plural for non-linear systems. So in order to identify the steady-state, what we need to do is we are looking for x1 and x2 values of the system such that the left-hand side derivatives go to 0 or there is no evolution in time. So these states are simply given by setting x1.. So x1 star, x2 star are the steady-states of the system which must satisfy these 2 equations because only then my evolution in time will be 0 or rather time independent steady-state.

Again in vectorial form, I can write this as…, right. So this first step is what we have already covered in the course so far. So if this were then looking at the Navier-Stokes equations, we have already seen how to calculate steady-states for some condition and generally the idea will be that this x star, this steady-state is something that is quite simple. I mean it is an obvious steady-state solution of the problem for some parameter values.

So for example it could be a situation where there is no flow or for example Poiseuille flow in a pipe, just a simple Poiseuille flow solutions at steady-state and then we want to understand how that solution undergoes a transition. So in the ultimately case transition to turbulence in some simpler cases that we are looking at, this course maybe some convection cells will set in and different features may arise.

So this will be a simple base state that we are going to start with and now we want to see whether that x star is stable or not. So that brings us to step 2, here. So step 2 is called linearization and the idea here is that because you want to see whether the system is stable around x1 star and x2 star, we give small perturbations to the current position and see whether the system returns back to the original steady-state or whether it deviates off and this ties in very well with the perturbation theory.

We have just completed where we have seen how to derive equations when you have small changes epsilon order changes. So that is what we will do here. So the variable x1 which is my dependent variable, I am going to give it a small perturbation about x1 star. So that x1=x1 star+epsilon which has the amplitude of my perturbation*x1~. So here ~ represents the perturbation that I am giving but this x1~ is bounded by 1 so that its magnitude is an epsilon. So this entire quantity is very small compared to x1 star.

This quantity is the small quantity. Similar I can do that for x2. So x1~, x2~ tells me in this 2-D case, the direction in which I am pushing the system or what kind of deviation I am going to give to the system. An epsilon just reminds me that that has to be a small deviation because I am looking at small perturbations. So again I can write this in vectorial form as x, which was the vector of my 2 dependent variables, as the steady-state, +epsilon*x~, where x~ is the deviation variables.

So this is the expansion and now like any perturbations, calculation that we have done, we need to substitute this into the originally equations and then obtain simplify the equations for the evolution of x1~ and x2~ because that is what we want to identify with stability. We are perturbations of this form and we want to see whether these, how these grow in time. So to be explicit, I have x1 of t and x2 of t, they are both time dependent.

And for small deviations about the steady-state, this time dependence is simply going to be of this form. So we will have the steady-state part and to the steady-state, there is some time-dependent deviation and the idea is to find out whether this time-dependent deviation is growing in time or whether it is going to decay back down to 0 and give me my steady-state. So to do that, we come back here and substitute it in these equations.

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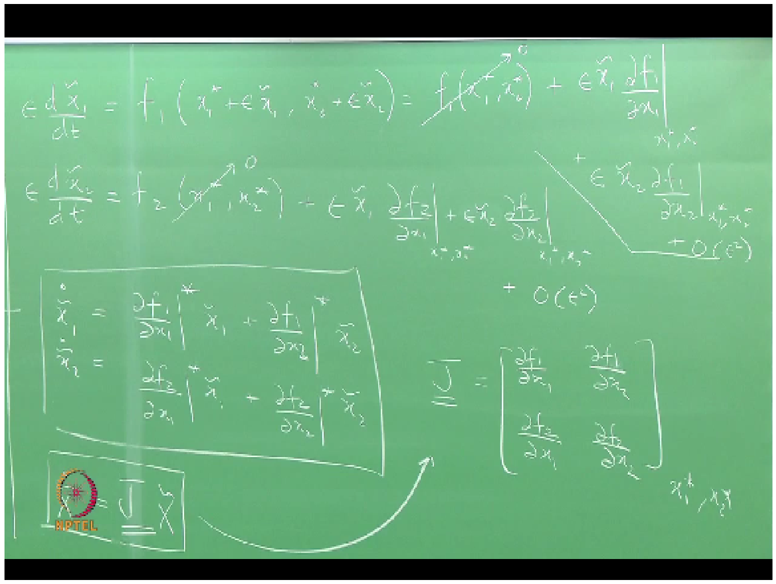

So if you put it into the left-hand side, we will get simply the derivatives of the deviation variables to order epsilon naturally because these are of course time independent. Now coming to the right-hand side, here F1 and F2 is any general function. So while I cannot make the substitution mechanically, what I can do is use Taylor series and that is what we have been doing in the course when we do not have simple functions.

So we can expand F1 as a Taylor series, so let me write it down here explicitly. So x1 is simply x1 star which is some constant+epsilon x1~… So that is F1 and now if I expand it in the Taylor series, about x1~ and x2~ because it is a 2-dimensional function. Then I will have the function F1 at x1 star, the value of the function at the steady-state, +the deviation into the derivative of the function with respect to x1, of course evaluated at x1 star x2 star.

Same thing for the variable x2 and I can write the same over here. So that is pretty clear. I just used the Taylor series up to first order epsilon. So to be exact, I should add the fact that there will be order of epsilon square terms which I have neglected to write down at the moment. So now the first thing you can see is that the function value F1x1 star x2 star is naturally going to be 0 because that was how we got x1 star x2 star at the first place.

In other words, these 2 are steady-states. So naturally they will evaluate to 0 on the functions F1 and F2. So these guys are just knocked out and I mean that is by construction. It is not a miracle, that is going to happen every time naturally. So what we are left with if you see is just a linearized system, that is why it is called linearization. So the derivative of x1~, which is now the growth of the deviation, is going to be related linearly to x1 because this is simply a constant evaluated at the steady-state.

So this is a function, the derivative but we have to evaluate it at the steady-state. So it is constant factor multiplying x1~. Now of course you will ask me that there is order of epsilon square terms as well. So if I go to epsilon square, I will get high order corrections which will involve x1~ square and so on. But then we know from the perturbation theory that we have already looked at, that since the left-hand side is order epsilon, so naturally we will equate it to the right-hand side terms of order epsilon itself because these order epsilon square terms will be much smaller.

So if we look at it as a perturbation problem, we will just equate the coefficients of epsilon on both sides and what will be left with, where again that dots represent derivatives, is simply, I will just put the star here to indicate I am evaluating at steady-state. So that is what I get after I equate terms toward epsilon. And now I can write this in my vectorized notation, then I will get the growth of x~ is simply = a matrix which I will call J and use the double under bar to signify matrix*x~ itself which is a classic first order linear algebraic system.

And this J is called a Jacobian matrix and it is simply given by the partial derivatives. So the matrix J has got for 4 partial derivatives, all evaluated at the steady-state and the note that it is F1 in the row and F2 in the second row. Of course, have you ever forgot that you just have to come back and rederive this which is quite straight forward. So ultimately what we have seen here, we should not get too caught up in the formal notation. I want you to remember from the beginning that we have some dynamical system.

This could for all you know be partial differential equations also on the right-hand side but the point is, we found the steady-state and then we linearized about the steady-state by the basic idea is giving small perturbations. Now in the case of this simple 2-D system though small perturbation procedure ultimately led to this very simple x.=Jx form.

This same form may not arise in an immediately obvious fashion when we are PDE's but the idea will be the same, we will have to give small perturbations and we know how to deal with those perturbations using perturbation theory. So ultimately we will land up in some linearized description of the system. So that is the point about step 2.

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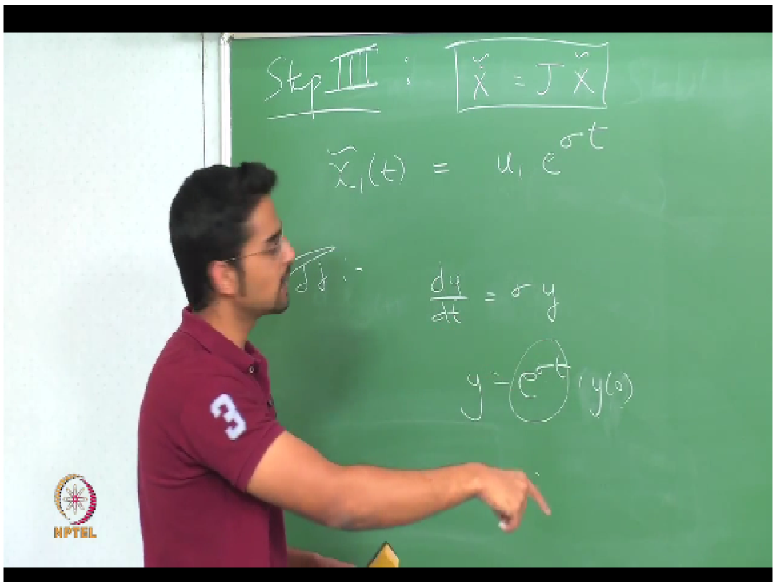

So moving on to step 3 and step 3 is where we now want to ask the question given my linearized equation, how is x~ going to grow in time. So imagining that we have absolutely no clue how to solve the system by any formal rules, what we would do is assume a solution for the time dependence. So if you look back there, you can see the full matrix form and since it is linear, I can assume that x~, which is a function of time, is going to be, I am sorry x1~, is going to be some constant u1, it is a number like 2 3 -5, some number which I have yet to determine.

And the time dependence can grow as e to the power sigma t or exponential growth. The reason I would do e to the power sigma t is quick note, that suppose I have a system dy/dt= say lambda, sigma y, yes. This is your classic first order single equation ODE, right and what is the solution with this problem? **"Professor - student conversation starts"** e to the power sigma. **"Professor - student conversation ends"**

It is e to the power sigma t*the value at t=0. So it is just an exponential growth. So you can see that the time dependence as e to the power sigma t. So in analogy to this, we are looking for a similar solution when we have more than 1 equation and I am postulating beforehand that this growth rate for all the variables that is your x1~ and x2~ have the same sigma. So right now let us look at it like a hypothesis and let us proceed and then if things work out in the end, we can look back and think about (()) (20:36).

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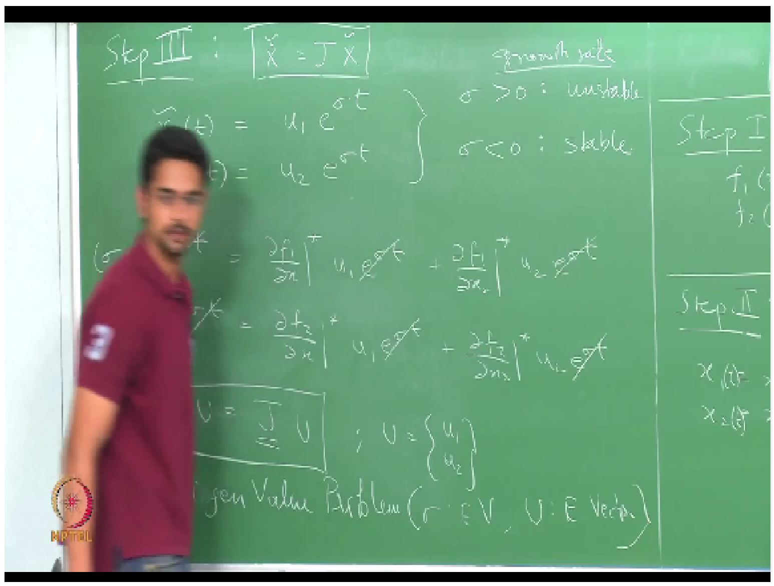

So right now I am saying that this has e to the power sigma t and so does x2, alright. So what I have done now is made a statement about how the time dependence is going to be and what this tells me now is that suppose my sigma is positive, then I will have this exponential term growing unboundedly which means that x1~ and x2~ will both grow. So if sigma is positive, I will naturally have an unstable system, right because my initial perturbations here x1~ x2~ are growing.

So I give some small initial perturbation. That initial perturbation you will see here is just u1 and u2, correct, in the form that I have shown if time t=0, the initial condition I just have u1 and u2. So u1 and u2 represent that small perturbation I am giving. Now the question is, whether this factor will grow? So if sigma is positive, that initial u1 and u2 is going to grow without bound if sigma is positive and this x1 and x2 of table very rapidly move away from the steady-state because of these terms.

On the other hand, if sigma is negative, then e to the power sigma t dies off because sigma is negative. So then the system will be stable because this portion will rapidly decay to 0 with time and a return back to simply x` star and x2 star. So in this form of my perturbation, sigma is crucial and that is why sigma is often called the growth rate. So typically again in these problem that step 3, we look at the linearized equations.

And these equations as you will notice will be homogeneous which means that if I say x~=0, it will satisfy the equation and that it has to happen because saying x~ is 0, basically is saying that I am on the steady-state and I know that steady-state satisfies the original equations. So I will have a linearized homogeneous system for the evolution of the disturbance variables and the next step is always to assume a form e to the power sigma t for the growth where sigma is the growth rate.

So now for this particular system, we will see where this leads us. So if I substitute this again into the equation that I have here, so let us look at the expanded notation and let us see what we are going to get. So what will the right-hand side be? x1~ derivative is simply d/dt of this and e to the power sigma t gives me back e to the power sigma t. So I will have u1, but with the sigma, right and my right-hand side, this is of course a constant. I have x1~ which I will substitute from here.

So I will get u1e to the power sigma t back, right. We will do the same thing for the second equation and now you see that because the equations were linear, I have this e to the power sigma t term common everywhere, right. So naturally I can knock that off. So what am I left with. I am left with an equation for u1 and u2, which of course are unknown at this point, that is one way of looking at it.

"Professor - student conversation starts" Is sigma (()) (25:20) that is what we are trying to find. **"Professor - student conversation ends"** So this is what I have. So now if we again look at this in terms of the matrix notation remembering that those 4 partial derivatives are J, I will just have sigma*U, capital U, where capital U is, wait I will write that at the end. So sigma*U, where capital U is the vector of u1 and u2, is simply the Jacobian matrix, *u itself, where here U is u1 and u2, alright.

So this process of assuming this form e to the power sigma t has led us to this equation in this particular system. And now the question is how do we use this to evaluate sigma or what does this tell us about sigma or what does it tells us about U. So suppose sigma is some random value, may be 5, right. Then I need to find U such that 5U is the same as Jacobian times u. One sure solution is if I take U=0, right. So U=0 will always solve the system, that is because it is the homogeneous set of equations and saying U=0 means I have not given any disturbance.

If U is non-zero, this equation may not be solved. There is no guarantee that for any non-zero values of U, I will have this equation satisfied but there could be some values of sigma such that a non-zero value of U will satisfy this equation. So my objective here is to find those values of sigma and those non-zero disturbances, such that this equation is satisfied and this is the classic eigenvalue problem that we have solved in general dynamical systems theory.

This is what an eigenvalue problem looks like where you have a homogeneous system of equations which always has the 0 solution but there will be some parameter like sigma and in this case, the growth rate it is. So there will be some values of the growth rate such that you will have non-zero disturbances propagate into the system and that is why we want to solve. Now in this particular 2-D system, it takes the simple matrix eigenvalue form which we can solve it the standard theory we already know.

But in the PDE system, we will also reach the stage. We will have an eigenvalue problem to solve and then we need to proceed with that solution either analytically or numerically. So that brings us to the end of step 3 is the eigenvalue problem. So in this case, sigma is the eigenvalue and the disturbance is the eigenvector.

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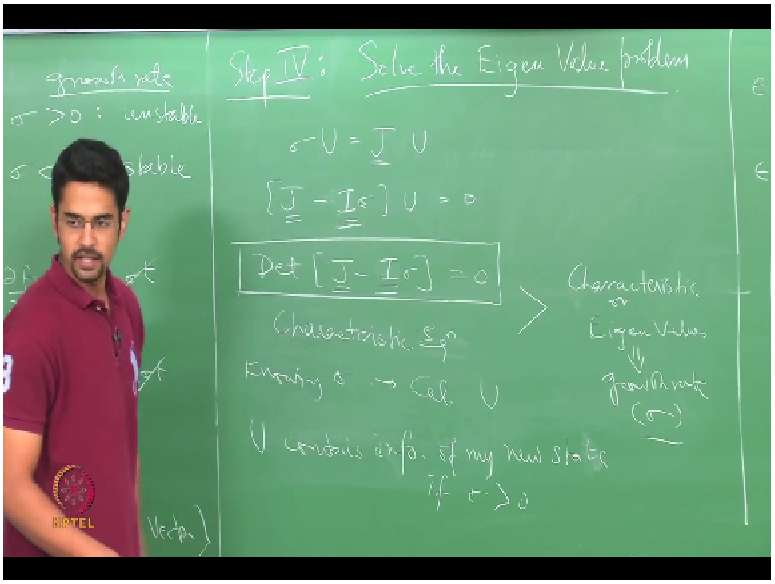

So that brings us to step 4. Step 4 is to solve the eigenvalue problem and determine the growth rate and determine the disturbance, right and how do we solve classic matrix eigenvalue problem? We realise that this is equal to saying…, right. So this itself is a matrix and for this to have non-zero solutions U, this matrix has to be singular which means that the determinant should be equal to 0.

So for a second-order system like the one we are looking at, this will lead me to a quadratic equation and I will get 2 values of sigma. In general, this equation can be some polynomial nth order. If my system is nth order, we have a partial differential equation, then this can be some extremely complicated function, more complex than we can imagines sometimes but it is always solvable.

If you cannot do it analytically, we can do it numerically but this is the equation, it is called the characteristic equation. Equation that gives us, sorry, I do not know where lambda pop into my head but it should be the identity matrix naturally. So this is the characteristic equation that gives me the characteristic values or eigenvalues which in this case is simply the growth rate and you will have many values of sigma in general.

For a 2-D system, you will get 2 growth rates. For a partial differential equation, you potentially have an infinite number of sigmas or infinite number of growth rates. So what that means is that there are various key directions in which you can disturb the system and if sigma is positive along even one of those directions or if even one value of sigma from here is > 0, then the system is unstable and the eigenvector corresponding to that eigenvalue sigma, the positive guy, will be the direction along which the system will grow and that eigenvector will contain information of the new pattern or the new state.

So this equation can usually be solved and we can obtain the values of sigma and once we know sigma, we can come back here and calculate the eigenvector. Knowing sigma, we then calculate U which is the eigenvector and that U will contain information. U contains the information of my new state if sigma>0. So if sigma>0, the previous state has become unstable and this is the mass to go somewhere.

So where it is going to go, what will be the new pattern that I see will that in some of the information at least is there in U and we can get a lot of information if we do a slightly nonlinear calculation but from linear calculations, we can tell if it is stable or not to very small perturbations because that was the initial hypothesis, it has to be infinitely small. So only to those step are perturbations we can tell whether the system is stable or unstable.

So that was step 4 where we solved the eigenvalue problem and that will immediately tell us, answer all our questions about stability. Are there any questions at this point? **"Professor - student conversation starts**" Yes. There will be multiple sigmas. So how are we going to look upon this (()) (33:34). Yes, then you did the same thing that you do in the case of solving these linear algebraic equations because they are linear, you will recognize that each of those sigma's and each of those U's correspond to linearly independent solutions.

So then the general solution will be given by the sum of them. So because it is a second-order system, you will have 2 solutions and you will have 2 different directions in which the system can grow and in the general case, your actual growth will be some linear combination of the 2. So I will write that down formally. **"Professor - student conversation ends"**

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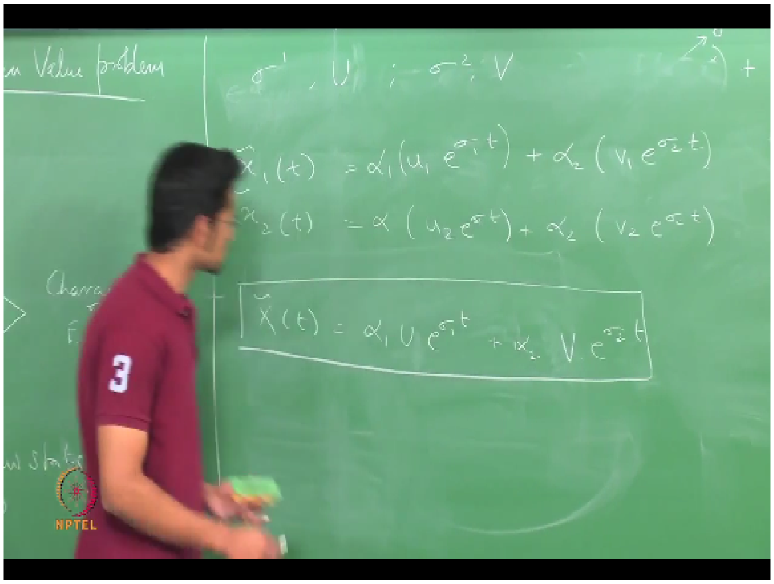

The question is that after solving this problem, I have a value of sigma 1 and a value of U1, right and I have also calculated another sigma 2 and another U2. That is not a good idea. Let me call it sigma 1, 2… So U is eigenvector corresponding to sigma 1, the first eigenvalue. Second eigenvalue has another eigenvector V and you will have only 2 for a 2-dimensional system and if I have high-dimensional system, I will get more of these.

So then I can represent x1 of t for example as the value u1 that comes from here, e to the power sigma 1t, right and multiplying with some constant alpha, some linear combination. Sorry…v1 or in terms of my, this is a ~. So all this is some linear combination of the 2 fundamental solutions I got from the eigenvalue problem and you see that make sense because it is a second-order system.

So I need 2, I mean, I can have 2 unknown constants because I have 2 initial conditions, one for the initial value of x1 and for the initial value of x2. So there are 2 constants here which I can determine from our initial condition. Now so then this is definitely a solution because if I put it back in the equations, they will each satisfy the equation on each half. So the first part will be 0, second part will be 0 independently and I will get a solution.

So a standard superposition of solutions. So this is why if sigma 1, if either of sigma 1 or sigma 2 are positive, it is unstable. So even if sigma 1 is negative, this part will go to 0 but this guy will blow up if sigma 2 is positive. So for stability, I need all the sigma's to be negative and even if one of them is positive, it will grow. So if everything else is reasonably clear at this point, what we can do is look at a simple example of a 2-dimensional problem where we will get a feel for the actual mechanics and see how it plays out in a real system.

So there are a lot of certainties that I have left out here, for examples what would happen if you get sigma=0. So that would mean that my Jacobian matrix is, I mean it has a 0 eigenvalue, so then I have some problems I cannot calculate the eigenvector and the eigenvector can be anything basically. So in such systems you actually end up at a contradiction in the mechanics because the fundamental theorem that lets me do this linearization fails if you have a steady-state where you have a 0 eigenvalue.

And those points are very special actually and that is usually what happens, the problematic points are the ones that should be paid special attention to. So in such situations honestly this theory completely fails. So if the sigma=0, I cannot tell what will the dynamics be, whether it will be stable or unstable, I have absolutely no idea. So to find that out, we need to go to the next order of epsilon.

So here we stopped at first order epsilon, we need to go to the epsilon square order to get more information and the formal theory is called center manifold theory and we discussed more of these higher order dynamics in the course that we do in the alternate semester which is steady-state in dynamical systems. So next semester probably we will be having that course again. But for our purpose in this course, we will be mainly wanting to identify the transition point.

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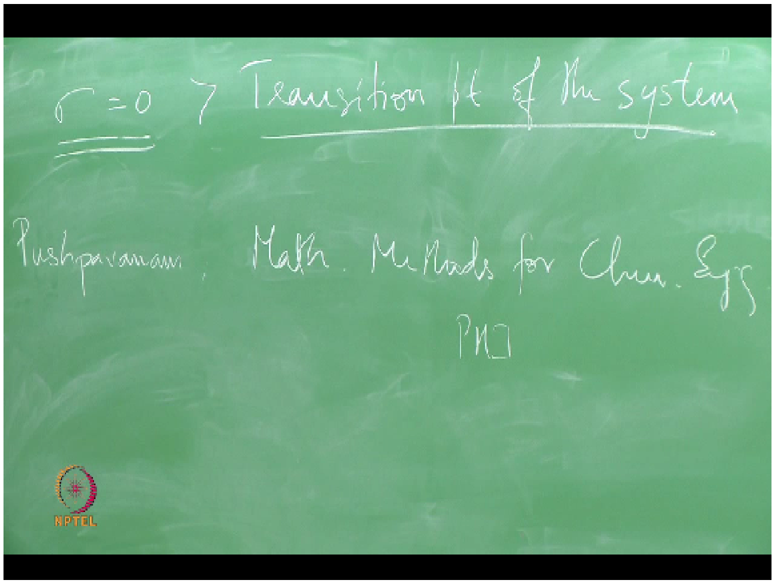

So sigma=0 is the transition point of the system, right. So we will be varying some parameter like the Reynold's number and for small Reynold's numbers say the sigma will be negative, system is stable. For higher Reynold's number, suddenly one of those sigma's in the calculation will become positive. So that point where the 0 marks the transition and that is very important for us because that will tell us a critical parameter values for which I am going to have a change in my flow.

And how to investigate the dynamics over there, we need to go to high order in a perturbation. So to read more about the ODE systems theory, dynamical systems theory and go through some more links I have said, you can refer to Professors book and it is mathematical methods for chemical engineers. I think it is Prentice-Hall India, so this book is available in the library.

So I think in his last chapter, he goes through this stability theory in some detail with a lot of practice problems and even answers some of the questions about high order dynamics. So that is basically I would refer for you down today. So we will now look at specific example to which we can apply the past steps that we have just discussed and analyze the stability of a dynamical system.

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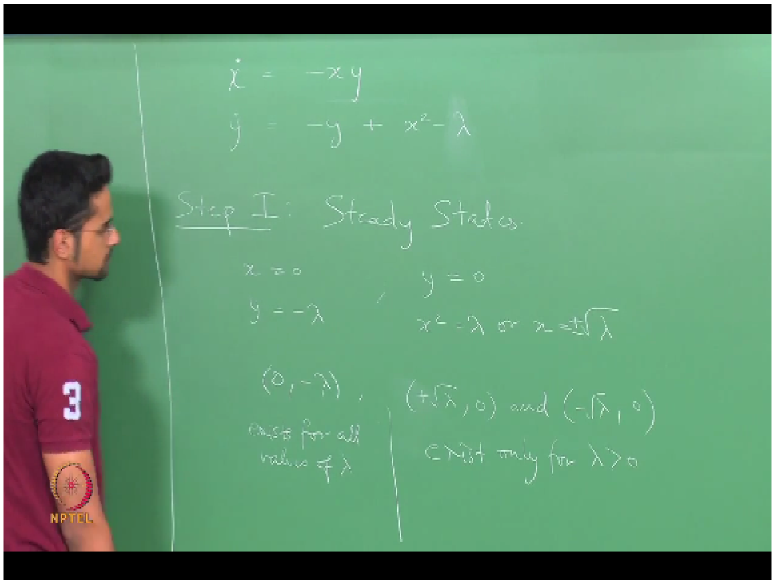

So now I will work in x y z variables rather than subscripts. The change in x where the dot denotes the derivative is -xy and the change in y=-y+x square-lambda and here lambda is a parameter, write that clearly. So look a bit carefully at these equations, you will see that for positive values of both x and y, the change in x pulls it back towards 0 as does this term in y but the x square term takes y further away from 0.

So there is a chance for some instability in some dynamics in the system and we will get right to the very first step of our analysis which is to find the steady-states. So to find the steady-states, we put the right-hand sides of both these equations to 0. So from the first equation, we can get that say x=0, then from the second equation, I would immediately find that y=-lambda, right. So this is my first steady-state and the other option form here is that y=0 in which case I am going to get x square=lambda or x=+-square root of lambda.

So you see that I actually have 3 different possibilities, one is 0, -lambda and the other 2 are --and you see another interesting thing here that these 2 steady-states are possible only if lambda is positive because if lambda is negative, naturally both of these would be imaginary and our vector field to begin with is real. So these exist only for lambda>0 whereas this steady-state exists for all lambda, right.

So this is the picture that we have and clearly at lambda=0, I have a transition from one steady-state to 3 steady-states and this is something that we should keep in mind as we go on and we will get back to this at the end after we analyze stability. So now that we have looked at steady-state, so we will move on to step 2.

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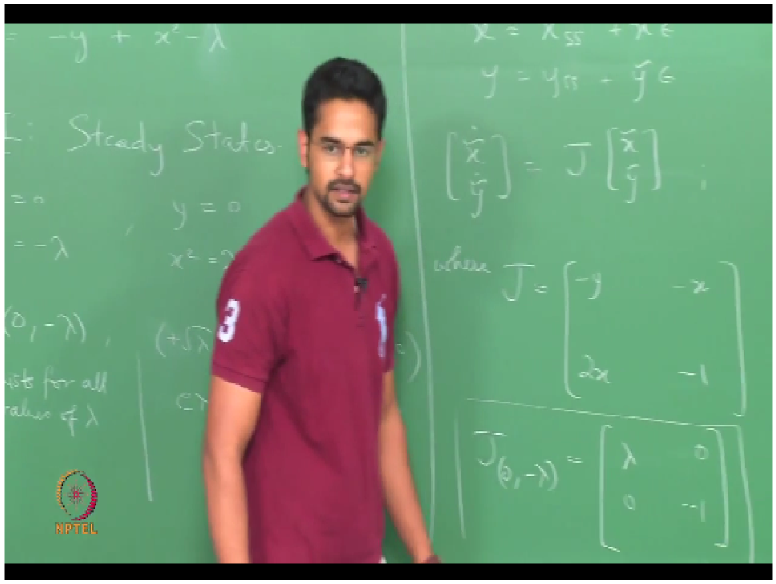

And that was of course linearization. So in this step, we write the variables as the steady-state value+a perturbation of order epsilon and substituting this into our original system of dynamic equations and linearizing, we already know that we will get equations of the following form, right, where the Jacobian matrix contains the partial derivatives of the 2 right-hand side vector fields.

So why do not we compute the Jacobian matrix now. So here I will have derivative of the first function with respect to x, then moving on to the second term, right and now that we have the Jacobian, we want to evaluate the Jacobian at the base solution which in this case the base solution that we want to study the stability is the one that exists for all values of lambda and that is when x=0 and y is -lambda, right and we will store this neat result, okay.

So you might ask the question why are we picking this particular steady-state to look at? Of course, we should study the stability of these as well but for this lecture, I am going to focus on this for 2 reasons, firstly this is the steady-state that exists for all values of this parameter lambda and so in some sense it is the trivial steady-state that is always a solution and in this sense which comes to the second point, this steady-state is very similar to the kind of simple steady-states that we find in the fluid systems that we will be studying in this course.

So for example a Rayleigh-Benard convection experiment, when the fluid is at rest without any convection, that is the simplest possible steady-state of the system which exists for all values of the parameter like the Rayleigh number or the amount of temperature difference of the plates and then after a certain amount of heat input, you get new steady-states arising which is the convection cells which correspond in this analogy to these new states.

So that is why right now we are going to focus on this trivial simple steady-state which exists for all lambda and we can see that the Jacobian at that point at 0, -lambda is simply this diagonal matrix. So having completed the first 2 steps, we now move onto the crucial step 3 which involves the eigenvalues problem.

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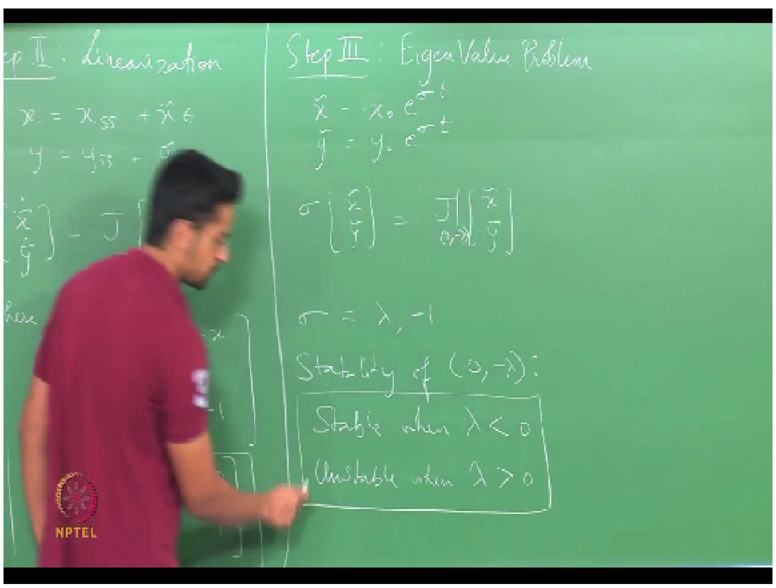

So if you remember at this stage, we make a substitution for our variables, some constant which is governed by its initial condition or at this point itself I can say x0 e to the power sigma t. So we assume an exponential growth having the same growth parameter sigma for both variables and we can make this substitution into a linearized system where the Jacobian about a base state is lambda and -1, the diagonal matrix.

So after doing this as we know, we are going to get sigma*, after simply substituting this in this expression, we will get sigma*the vector=the Jacobian * the vector and the classic eigenvalue form matrix*vector gives me the eigenvalue which in this case is sigma * the vector. So directly as we had already said, sigma is going to take the values of the eigenvalues of the matrix J, of course evaluated at 0, -lambda at the steady-state of interest.

So in this case, sigma can take 2 values, +lambda and -1. So in a 2-dimensional system, the 2 eigenvalues or growth rates are lambda and -1. So we can see here directly that this -1 term, this eigenvalue is only going to lead to a decay, an exponential decay of the solution. Lambda on the other hand, can make the system unstable and lead to exponential deviation away from the base steady-state, 0, -lambda if the values of lambda are positive.

So at this stage, now we can conclude about the stability of the steady-state of interest, 0, -lambda and say that it is stable when lambda<0 and it is unstable when lambda>0 and this is now very interesting result because you see it fits in very well with the multiplicity of steady-states that we had observed right at the beginning. So if you come here, you will see that for negative values of lambda, only 1 solution exists because these 2 solutions would become complex.

So if lambda < 0, I have only my base steady-state and in that case, the base steady-state is stable; however, when lambda > 0, my base steady-state becomes unstable and that is exactly when the 2 new steady-states arise with + - square root of lambda, 0 and now these come in so to speak to take the place of the original steady-state. So I can represent all of this in terms of a diagram of the steady-states plotted against the parameter lambda which in the literature has been called the bifurcation diagram.

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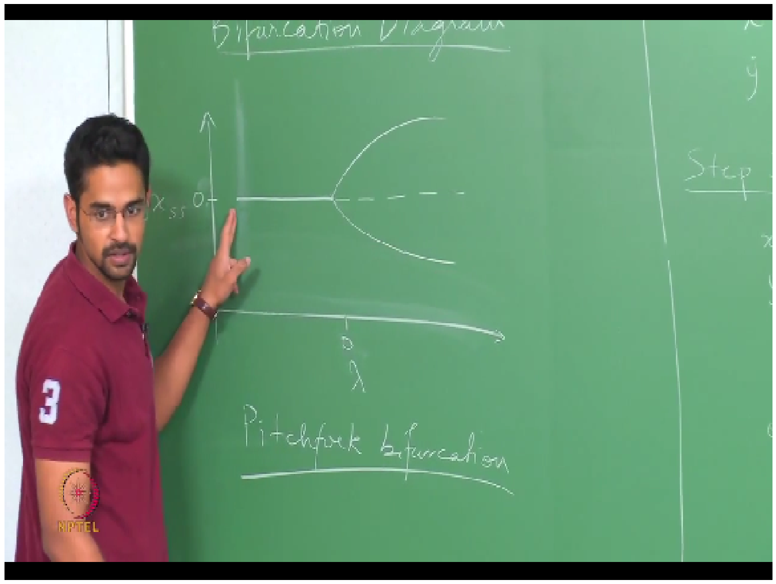

So in the bifurcation diagram now, so on this axis, I am going to plot values of lambda and on this axis, I am going to plot x steady-state and here lambda is 0. So on this side, lambda is negative and here lambda is positive and if I plot the steady-states now, I will get a situation where I have only 1 solution all the way up to lambda=0 and then exactly at this point, this original solution which has simply x=0, becomes unstable which I have denoted by the dash line.

And instead of this, I will get the 2 new solutions which grow as the square root of lambda and you can see that they will be symmetric because they are +- square root of lambda. This is 0 and positive and negative. Now I will leave it up as a homework exercise to do the linear stability analysis about these 2 steady-states which are +- square root of lambda, 0. So you can linearize the system about these 2 steady-states, calculate the Jacobian, its eigenvalues and find whether these are stable or unstable.

So that is the homework problem. But I can tell you now that in fact these will turn out to be stable and this is a classic exchange of stability concept where one system, one state becomes unstable and 2 new steady-state solutions emerge which are actually stable. So for lambda<0, the system will be at this steady-state globally and as soon as lambda crosses 0, it is going to leave this steady-state now which has become unstable and move to either the one above or the one below and this bifurcation diagram is called the Pitchfork bifurcation.

And is in fact a classic, one of the classic bifurcations in 1-dimensional systems and is in fact seen in high dimensions as well as in pattern forming systems involving partial differential equations and there is special reason for that and that is the inherent symmetry of this bifurcation diagram and the reason for this symmetry you can trace back to the vector field which if you see is symmetric for values of plus or minus x.

So what that means is if I substitute x=-x, I will get a - sign out from here as well us from here which will leave the equation unchanged and you can see the same thing happens here because of x square. So this equation also remains unchanged if I make the substitution x goes to -x and that is the reason why along the x axis, I have this symmetry about 0. So you will find that in physical systems, this is quite common where the inversion symmetry about positive and negative values holds.

And that is why the Pitchfork bifurcation is quite common in many of our systems. So we will be seeing this later on in the course again in a more complex context. But it is good to remember this right here. So with this, we have analyzed today the key stages in a stability analysis and I have shown it to you in ODE's but the same ideas propagate forward when we look at partial differential equations.

And therefore very often even though we may not be able to find out these steady-states because in this case, we have a simple vector field, so we could directly obtain the solutions analytic but in complex partial differential equations which have many solutions, it is very difficult to find out to solve those non-linear PDE's and get all the steady-states in the system. However, we will usually be able to find a simple steady-state which exists for all values which in this case is this, base steady-state, x=0.

So we would know this and by doing a stability analysis, we can find when this becomes unstable and possibly leads to new solutions and so during the next, the latter part of this course, we will be focusing on analysing these systems and studying their pattern formation.