# Multiphase Flows: Analytical Solutions and Stability Analysis

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## Lecture – 19
## Flow Between Wavy Walls: Velocity Profile

So, welcome to today's lecture on where what we are going to do is wrap up the problem in some sense, not completely, slightly wrap up because the rest of the things you guys with wrap up as a homework. What I want to do is just take you through the process of finding the base solution as well as the first order solution, okay. Then, we will make some observations and then we will keep moving.

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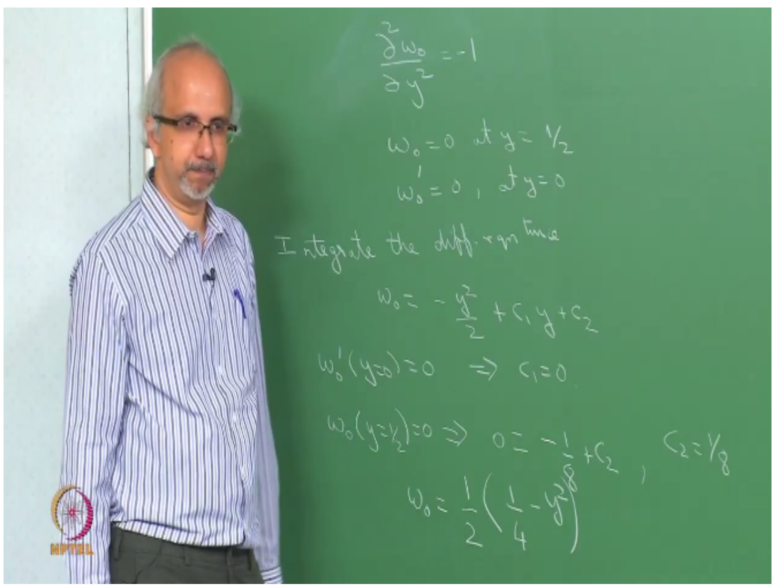


So, let us proceed with finding the solution here. The base solution remember corresponds to the problem when Epsilon=0. So, if our method is right, you expect that the base solution falls back, collapses to the flow between two infinite parallel plates, okay, the flat and that is exactly what is happening. This upper half is what we are looking at and at y=0 you have the (()) (01:15) boundary condition and y=0 you have the symmetry condition and that is differential equation which has to be satisfied.

This is actually a total derivative. It is not a partial derivative and you can therefore integrate it out directly, right. So, before somebody points it out and makes this remark. So, now dw0/dy we integrate the differential equation twice and what do we get w0=-y square/2+C1y+C2 and when I impose these boundary conditions, I have w0 prime at y=0=0 and this implies that c1=0.

When I differentiate this, substitute y=0, I get C1 and if that has to be 0, C1 has to be 0 and from the other boundary condition, at y=1/2, I have w0=0, so I have w0 of y=1/2=0 implies 0=1/2-1/8+C2 or C2 is 1/8 and what this means is that w0 is actually 1/2 of 1/4-y square. So, that is my parabolic profile, that is my base solution and I think that is perfect. So, we want to do now is we want to go ahead and construct the solution for w1, okay and remember I need this information to find out w1 and that is how it is.

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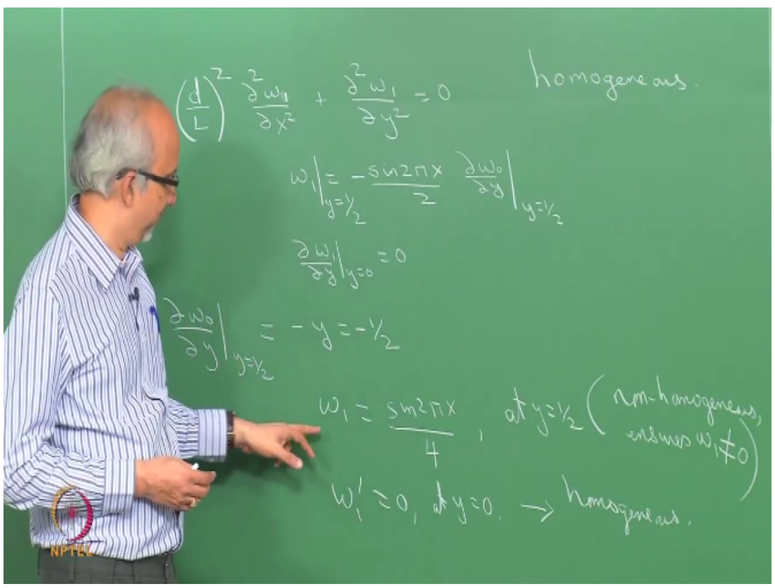


Whenever you are doing a perturbation series, what you are doing is you are converting a problem into a hierarchy of problems, use this information to construct the next fellow, right and where does this information come in. This information comes in here. So, our need to calculate dw0/dy and y=1/2 and this is evaluated y=1/2. So, basically what I am going to do is calculate dw0/dy and y=0 and what is dw0/dy, is -2y/2 is nothing buy –y, is nothing but -1/2, okay.

So, I substitute -1/2 here because this entire thing is evaluated at y=1/2. Therefore, evaluated at y=1/2 and dw0/dy I have just found out is -1/2, okay. I know w0 from my earlier solution and so, this boundary condition basically reduces to w1=sin2 pi x/4 at y=1/2. The other fellow remains as it is. This guy w1 dash=0 at y=0. What I want you to observe is that now we have a homogenous differential equation, okay.

This differential equation is homogenous when right hand side is zero, but the non-homogeneity is in the boundary condition here. This boundary condition is homogenous. If this also had been zero, what would have happened, the solution to this system of differential equation and this boundary condition would be the trigger solution w1=0. The fact that this guy is non-zero gives me a non-zero value for w1, okay. So, I just want to point out that this is non-homogenous and ensures w1 is != 0.

This is homogenous and this is also homogenous. The other nice thing of course is that this problem is linear. There is no non-linearity in it. So, now the question arises I have this guy evaluate y=1/2 which is nice because I can now substitute this. What I have to do now is remember w1 depends upon both x and y, okay and if you remember the physical problem, the periodicity was in x direction, the flow is in the z direction, okay.

So, one of the things that we can visualized is that this particular velocity at the boundary is periodic is possibly going to persist throughout the domain, throughout y, that is whatever is the periodicity of the velocity in the x direction, this periodicity is going to be present everywhere but w1 is a function of y as well.

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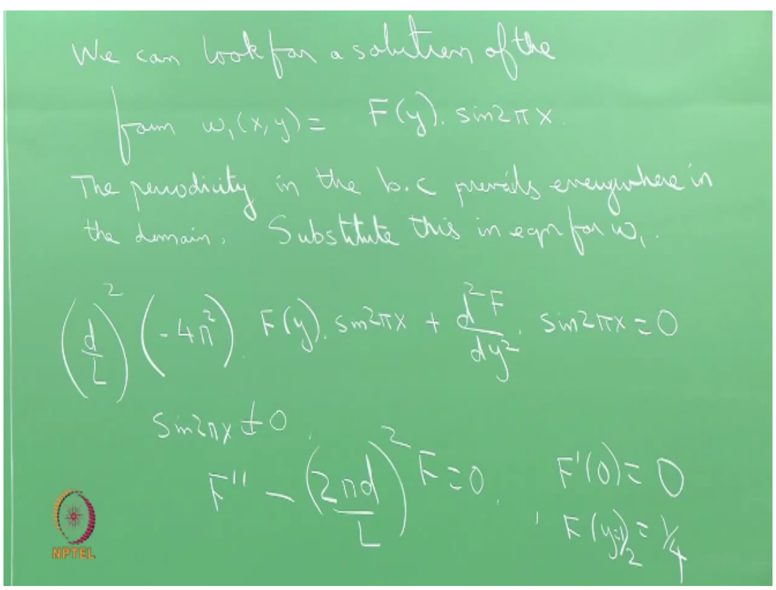

So, we can look for a solution of the form where w1 of x, y is of the form sum F of y*sin2 pi x, okay. That means the periodicity in the boundary condition prevails everywhere in the domain. See I have fully developed (()) (08:43) in the z direction, the dependence in x and y and when x direction if it is getting periodicity at the boundary, I am just going to say that the same thing happens everywhere.

Now, first of all what does this mean, what I can do is whether this is correct or not or whether such a solution is possible or not, I can find by substituting this form there and see if I can actually get an F of y. Supposing I am not able to proceed further that means this assumption is wrong, okay. Then, I have to come back and do something else. So, that is one way to actually find out if this is indeed possible. So, you know you make an assumption of this kind, you proceed further, you get stuck, you come back and then you make a change.

So, clearly what we are going to do is we are going to substitute this here and what does this means in terms of differential equation. Let us substitute this form of the solution in this equation. I get d/l whole square, d square w1/dx square, when I do a second derivative, I get 2 pi cosine, I will get -2 pi again cosine. So, I get -4pi square*F of y*sin 2 pi x+d square F/dy square*sin 2 pi x=0. All I have done is substituted this form in my equation for w, okay.

So, substitute w1. So, that is perfectly fine. Now, remember so what happening is sin 2 pi x is present and sin 2 pi x is not zero, actually knock it off. Supposing these have not been second derivative, your equation actually was with the first derivative, then such a solution would not have exited. Because on differentiating we got cosine and cosine and sin you could not have factored out, okay. So, the fact that I have a diffusive process, viscosity actually have a second derivative process, I actually can get this.

So, that is something which I want to point out to you, okay. If it is a first order if it is just conviction, then this would not have happened, okay. So, this sin2 pi x is != 0 and therefore what I get is F double prime is that term -2 pi d/l whole square F=0, okay, that is my equation for F and what is boundary condition on F. I need to use these boundary conditions, w1 dash=0 at y=0 with derivative with respect to y.

So, F dash of 0=0, okay and w1 is sin2 pi x and y=1/2, so if w1 is sin2 pi x/4. So, this must be = sin2 pi x/4. I have F of y=1/4 at y=1/2, okay. So, at y=1/2 w1 sin2 pi x/4, so F of y has to be that, so F has to be 1/4. So, basically what I have done and this is something which I want you to see because later on when there would be other problems, we will be following the same strategy. You have started up with a partial differential equation and we are going to reduce it to an ordinary differential equation, okay.

That is one way to help find out analytical solution, okay. We will see this in more detail later but basically that is only one periodic mode which is present in the system. Since my equation is linear, okay, I am looking for a solution which has only that mode, okay. If my equation was non-linear, then the different modes would have interacted and I could have got in different waveforms, not just sin2 pi x, I may have got sin4 pi x, I may have got sin8 pi x, okay.

But because my equation is linear and I am giving this kind of a disturbance which is having a shape of sin2 pi x, I am expecting my response to also have sin2 pi x, but if my equation has been non-linear, I would have got in other terms, okay. So, one of the things which allows me to do this is actually the fact that the equation is linear, okay. Now, this equation is something which everybody can solve with our eyes closed, except me, alright.

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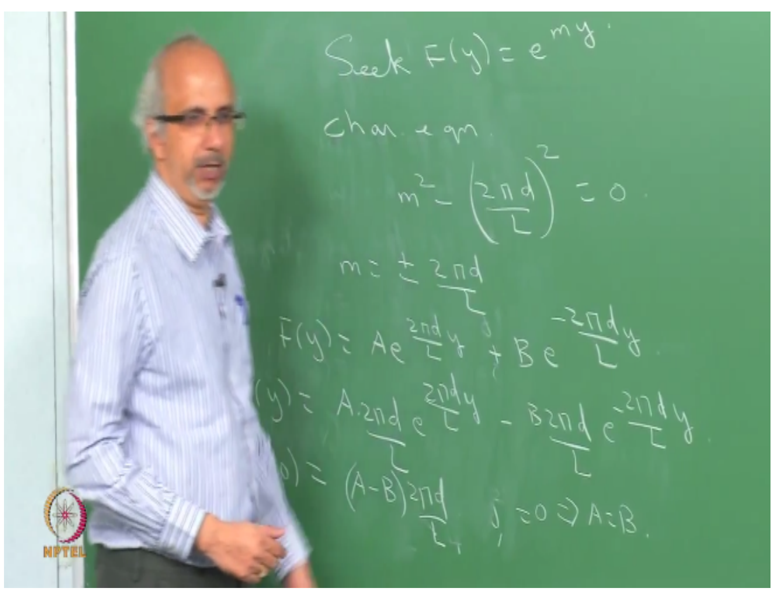

So, how do you solve that equation. I mean you just seek a solution of the form E par mx, 2pi d/l number is constant. So, seek F of y as e power m y and what would you get. You get the characteristic equation just as m square-2pi d/l whole square=0. So, ms plus or minus 2pi d/l, okay and F or y is of the form Ae power 2pi d/ly+Be power 2pi d/ly, okay. That is your thing from calculus, whatever you have learned.

Now, I need to find A and B and if I know A and B, then I can go back, I know F, I know w1 and that is what I wanted to get. So, how to get A and B, you have to use those boundary conditions, let us use the boundary condition which is homogenous which is F dash of 0=0, okay. So, F dash of y is A*2pi d/l e power 2pi d/ly-B 2 pi d/l e power -2pi d/l*y, that is F dash of y, okay. I want to evaluate this at 0. So, F dash of 0 is nothing but A-B*2pi d/l.

This has to be 0 which means A has to be = B. This equals 0 implies A=B. So, that is good, I got one constant and now I need to put the other boundary condition and get the constant, okay. So, let us do that.

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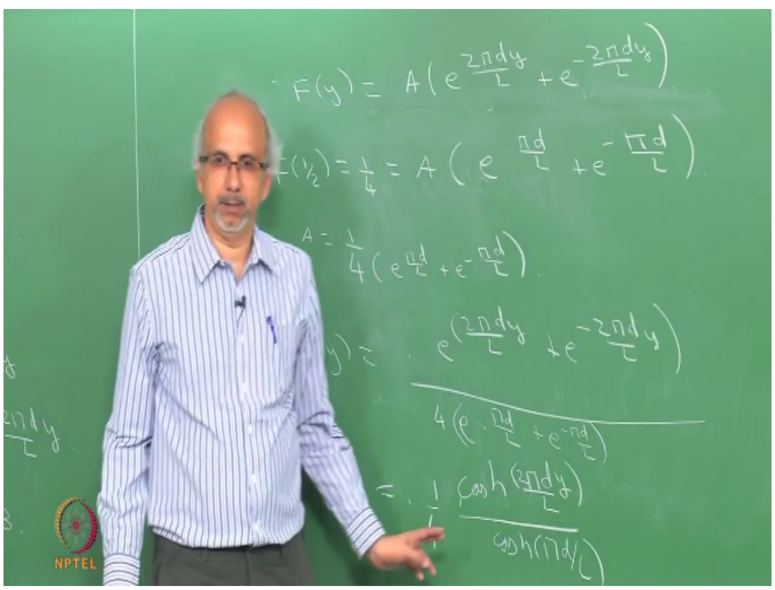

So, from the other boundary condition, so I have F of y is now therefore A=B right. So, A*e power 2pi dy/l+A power -2pi dy/left, okay. So, what is the funda now. F of 1/2 is 1/4. I have to evaluate this at 1/2. So, F of 1/2=1/4=A*e power pi d/l+e power –pi d/l. I am sticking to exponentials as we people are comfortable, you could have worked with hyperbolic functions as well. So, that tells me what is A, okay. I can use this to get A.

I can go back and substitute it by here and get this thing. So, this implies that A is 1/4 of e power pi d/l+e power –pi d/l, okay and F of y is therefore A, okay. So, you can just observe that this is cos hyperbolic, okay and you will get cos hyperbolic and this also (()) (18:32) cos hyperbolic so that the factor of 2 which comes it gets cancelled off with the definition of cos hyperbolic. So, this is 1/4 of cos hyperbolic 2pi dy/l/cos hyperbolic pi d/l, okay, that is what we have.

So, this is w1 for you and now you have found the correction to the velocity to the first order because now w is w0 which was your parabolic profile plus epsilon w1. You should similarly go ahead and find w2, okay. Again, you have homogenous equation, one of the boundary condition is homogenous. One boundary condition will be non-homogenous but for this boundary condition, you need the information from w0 and w1 and this information you would use to go find a solution, okay.

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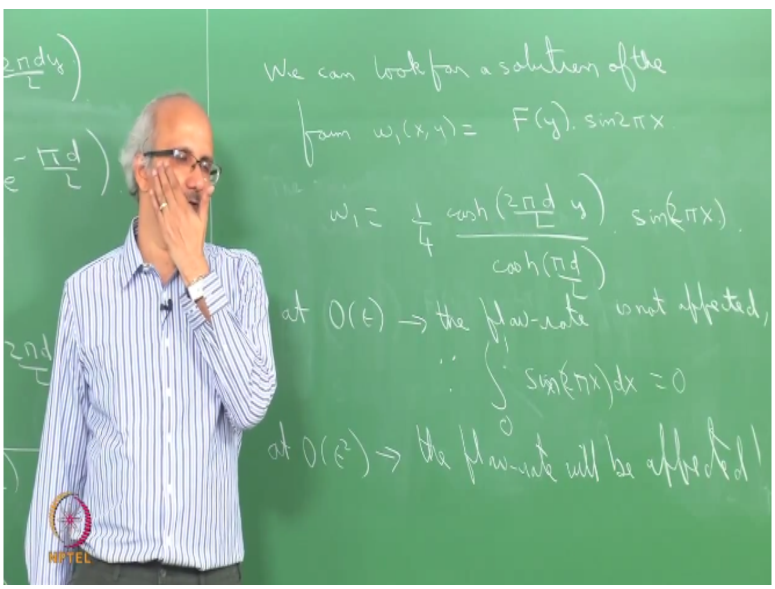


Just one last observation before we proceed further. I want you to realise that w1 is going to be of the form 1/4 cos hyperbolic 2pi d/ly/cos hyperbolic pi d/l*sin2 pi x. Like I had mentioned earlier, one of the things we are interested in finding out not only is actual velocity profile but to find out if there is by having these kinds of corrugations, is there any change in the flow rate which is passing through the channel, okay.

You are going to look at flow rate per unit wavelength because you have an infinite channel which is extending to infinity in the x direction in this periodicity. So, it makes sense to concentrate on one wavelength here through the gap of D and see what the flow rate is. So, how do we find the flow rate. You would find the flow rate by taking this velocity and integrating this out in x and y direction, okay.

Now, what you are going to observe is that since you have the dependency and the solution is of variable separable form, some function of x multiplied by function of y. So, when you going to integrate this out in the x and y directions, you would be able to integrate this out in the y direction, you would be able to integrate this out in the x direction separately, okay. When you do the integrate out in the x direction, you are going to see that this is periodic and therefore sin2 pi x when you are integrating from 0 to 1/2 or -1/2 to +1/2 because you are looking at the whole channel remember, okay.

So, in the x direction from 0 to l, in the y direction is from -1/2 to +1/2. Here it is from 0 to l, you will find that this guy is not going to make any contribution. So, what this means is that because this is integral will go to 0. So, what this means is the first order affect that is going to be no change in the flow rate that we are going to see. If you want to go to the second order affect which you have to go because there is a homework problem, you will find that it is going to make a difference to the flow rate.

So, the flow rate is going to be affected only at the second order, okay. So, point I am going to make here is at order epsilon the flow rate is not affected since sin2 pi x and if I am working in dimensionless coordinates is from 0 to 1, okay equals 0. At order epsilon square, the flow rate will be affected and that is something I am telling you I want you to proof to yourself that this is indeed true. So, all you need to do is find what w2 is.

What we will find is that the x dependency is in the form of a square, may be sin square or cos square. So, when you integrate it out, you are not going to get a 0 value, okay, you just need to solve again the same process. So, this is a part which has been wrapped up and this portion you guys have the wrap up. I think I just want to make one more remark that the (()) (23:24) perturbation method is extremely important when it comes to solving stability problems.

I think it is kind of the basis for solving stability problems because remember now I have a flat surface and I gave a small perturbation which was periodic and then I was able to actually construct solutions, okay. So, when we are talking about multiphase flow problems, like the problem of the jet breaking up into drops.

What do you have, you have a cylindrical surface which is your base solution without any epsilon in it which is defined as r=r0 which is constant and now you give a small perturbation which could be periodic and then you are going to ask the question is this perturbation going to make the jet breakup or not.

So, you can see now, we are actually interested, whenever you are talking about multiphase flow problems where there is an interface and the base state is normally going to correspond to boundary which is like y=1/2 but when I give a perturbation, it is going to be y=1/2 plus some perturbation, okay. So, now when you want to find out whether something is stable or unstable, you need to therefore use the domain perturbation method that we just spoke about to actually do the calculation. So, that is where all these things are fitting in.

I am just trying to tell you this to show you that whatever we are doing here is actually fitting in. So, the first part of the course, we just did a small revision of some of the concepts you have learned and we extended it to the boundary conditions when you have actually an interface which is not necessarily corresponding to a coordinate access, okay and I told you how to find the normal stress and the tangential stress when you have some surface of the form y=F of x.

That is because when I am going to actually solve the problem I am going to be using normal stresses for balancing, tangential stresses for balancing, okay, not stresses in x direction, not stresses in y direction is a normal stress and the tangential stress so that is what we use. Then we came to a perturbation because this is going to be basically the starting point for stability. So, let me discuss a little bit about stability and then what we will do is we will start solving some problems on stability, okay.

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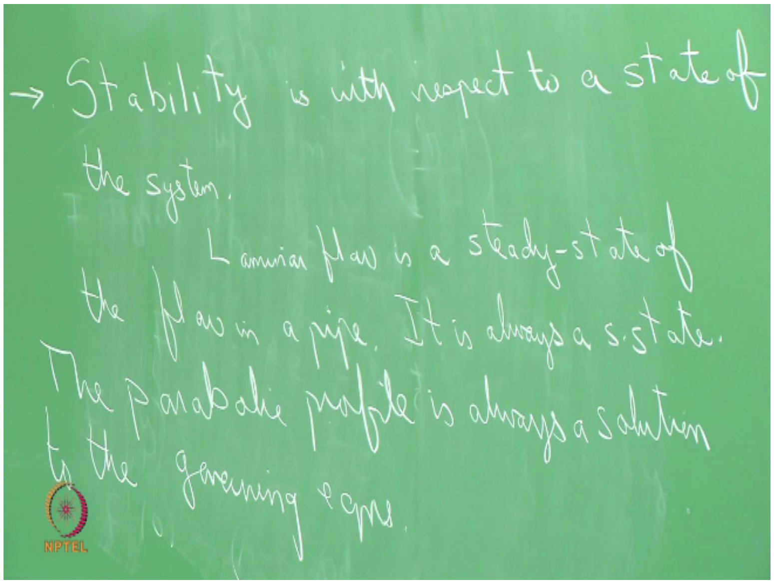


So, the perturbation method is to just give you some insight about how you can convert a non-linear problem into a bunch of linear problems. So, you remember that viscosity problem was actually non-linear because you have du/dy whole square term. So, then what we did was we converted it into a bunch of linear problems and then we got analytical solutions, okay. So, that is one of the things we are going to do when you are talking about stability.

So, let us quickly jump into stability. So, first of all I want to clarify that when we talk about stability, we are talking about stability is with respect to a state of the system. What does this mean. People normally talk about stability of a system. This system is stable, this system is unstable, okay but what I am saying is you should talk about stability of a particular state, may be a steady state. So, if you have a steady state, let us say had a steady state like you have laminar flow in a pipe, that might be the easiest example, okay.

So, if you have a laminar flow in a pipe and because you are doing multiphase flows, we start with single phase and then we go to multiphase. So, laminar flows are a steady state of the flow in a pipe, okay. It is always a steady state. What does this mean. You can calculate like today we found out this parabolic velocity profile, okay. This parabolic velocity profile was dimensionless, so it did not have this G and all that as pressure drop but otherwise it would have had a pressure drop.

So, no matter what the pressure drop is that is a solution to the equation if you have a flat plate, you understand. So, what I am saying is the parabolic profile is always a solution to the governing equations. Do you all agree with me? The parabolic profile earlier did not have the pressure drop because it was scaled inside my velocity but if I actually write it in terms of an actual dimensional velocity, I would have had a pressure drop just like your Hagen-Poiseuille equation.

Hagen-Poiseuille equation has some dp/dz. So, if I give a very small dp/dz, I have very low velocity. If I give a large dp/dz, I have a large velocity, right. So, basically what this means is there is always a solution. However, you all know that for Reynolds number < 2100 only you will have the laminar flow which is going to be experimentally absorbed. When Reynolds number is more than 2100, the flow becomes turbulent, okay.

So, what is this critical thing about 2100. So, that means that what I am doing is just visualize an experiment where you are actually increasing the pressure drop. So, there is a parameter in your problem as an experimentalist which you can actually change, okay. So, you are doing an experiment where you keep increasing the pressure drop. So, let us say that you have a pressure drop which gives you a flow which corresponds to a Reynolds number of 100, okay.

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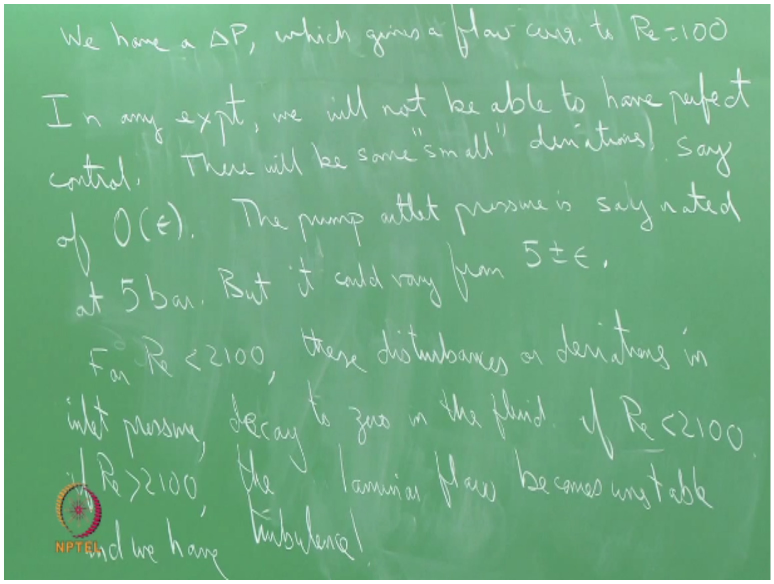


So, we have delta P which gives a flow corresponding to Reynolds number=100. What does this mean. Given delta P I can find my velocity profile. I can find my average velocity, right. I can go back, I know my properties of the liquid, I know the diameter of the channel. I can get rho vd/mu and you do this calculation, you find Reynolds number is 100. So, what does this mean. You can actually when you are doing this experiment, you will actually see a laminar profile.

So, whenever you are doing an experiment. Remember that your experiments are always going to have some disturbances, okay. I mean when you do an experiment, it is not possible for you to perfectly control you pressure drop. It is not possible for you to perfectly control your flow rate. There will always be some small disturbances. If you are able to observe this laminar velocity profile, then it means in spite of these disturbances that are existing, in some kind of time average sense, you are able to get your laminar velocity profile, okay.

So, what I am trying to tell you is that any system, there will always be some small disturbances present, okay. But it turns out that these disturbances are decaying to zero, they do not get amplified as long as the Reynolds number is < 100. But suppose you know, you slowly increase the pressure drop and let us say Reynolds number is 1500. Again, you will see laminar flow. Again, there will be disturbances that are present, okay and again your system is stable.

But supposing you change the pressure drop a little bit more and you cross the threshold of 2100, your flow becomes turbulent, what is the difference. Your laminar profile is still theoretically steady state, okay, but there are some disturbances which are going to be present but now the disturbances are getting amplified. The disturbances do not decay to zero and because of disturbances will decay to zero, you would get a turbulent state, you understand.

So, what I say is the laminar profile is stable, that is a steady state solution, the laminar profile is stable for Reynolds number < 2100. The laminar profile is unstable for Reynolds number > 2100. So, I am talking about stability in the context of steady state. So, that is how you should always talk. You know you cannot say, the system is unstable for more than 2100 or < 2100 is stable. You are talking about stability in the context of the steady state, okay.

So, what I have done is I have introduced the concept of stability in the form of small disturbances which are always going to be present in experiment, okay. So, now let me write down a few things. We have delta P which gives a flow which corresponds to Reynolds number of 100. In any experiment, we will not be able to have perfect control, okay. There will be some small deviations, okay. I am talking order epsilon.

So, very small deviations, say of order epsilon where epsilon is very much < 1. The question is how does the system responds to these disturbances. So, you have a pump which is pumping a liquid, okay. There is going to be some small fluctuation coming into the pump. So, it is not that the pressure and outlet is always to be whatever 5 bar, 10 bar. Although, it has the rating of 5 bar at the outlet pressure, may be actually it is going to be slightly varying 5.0001 to 4.999, okay.

So, that is my perturbation. See for example, the pump outlet pressure is say rated at 5 bar but it could vary from 5 plus or minus epsilon. Clearly, it is not going to be exactly 5 bar and if somebody is telling you it is going to be exactly 5 bar and you believe him you are crazy. So, there is going to be some fluctuation and depending upon how much money you have paid, how accurate the pump is, epsilon is going to be smaller and smaller, okay.

So, I am saying that this epsilon is my disturbance, the inlet pressure but for Reynolds number < 2100, these disturbances are deviations in the inlet pressure decay to zero in the fluid, okay. That is how the dynamics is when you actually solve the partial differential equations and you are actually trying to find out whether this kind of deviation of pressure from 5 to 5+epsilon, okay. Whether it is going to make any difference or not, decay to zero in the fluid if Reynolds number is < 2100.

But if the Reynolds number is > 2100, the laminar flow becomes unstable and we have turbulence, this is something you all know. I mean, you guys have possibly month something up and said oh Reynolds number < 2100 laminar but actually what is happening is when you are actually crossing, you are having a stability problem, okay. What was stable has become unstable?

And the way, I want you to visualize the stability problem is there are the small fluctuations that all is going to be present and what happens is for < 2100, these deviations decay; more than 2100 the deviations get amplified and you have turbulence. Remember, the laminar flow is the possible solution for Reynolds number > 2100, okay. The parabolic velocity profile is possible.

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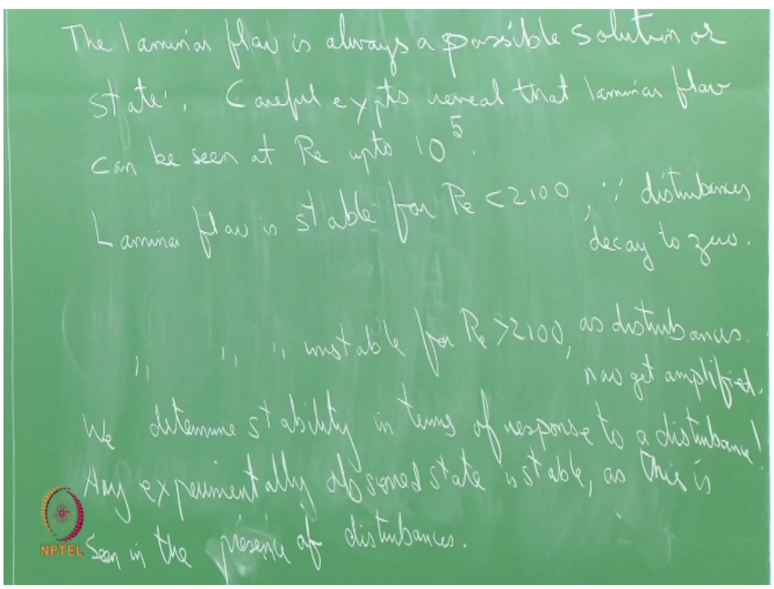


So, the point I am trying to make here is the laminar flow is always a possible solution or is a possible steady state. In fact, the people who have done experiments very, very carefully and they have been able to observe laminar flow up till Reynolds number of 10 par 5, okay. So, if you really interested in this, you could do very, very careful experiments and see that when you have Reynolds number of as high as 100,000, you would get laminar flow.

But then, normally you are not very careful. You just have a regular pump. You are doing undergraduate lab. I mean people are possibly trying to cut out on cost. So, we will get this transition at 2100, okay. So, I am just trying to tell you, how do I know this. Careful experiments reveal that laminar flow can be seen at Reynolds number up to 10 to the 5, okay. So, all this means is that we are talking about a steady state now. Steady state is my laminar flow profile.

So, this laminar flow profile is stable, I am going to call it stable for Reynolds number < 2100. It is unstable for Reynolds number > 2100, okay. So, the laminar flow is stable for Reynolds number < 2100 since the disturbances decay to zero, okay and the laminar flow is unstable for Reynolds number > 2100 as the disturbances now get amplified. So, this is something we are always going to do when you talk about the stability problem.

When you talk about the stability problem, what we are going to do is we are going to worry about changing some parameter which you can control as an experimentalist and you are going to find out how does the system behaviour change, how does the response to disturbances change when you have one parameter which is being varied. So, just like we have done now. When Reynolds number is low, when disturbances are present, they are going to decay, everything is fine and stable.

When Reynolds number is higher and that is the parameter I am changing, okay. The parameter I am changing is the flow rate or the pressure drop. So, the pipe is of course the same pipe, so diameter is constant. Fluid is the same, density and viscosity are the same, the velocity has to change for changing the Reynolds number. I have changed the velocity by changing the pressure drop, okay and I will keep changing the parameter which is the inlet pressure.

You will find that above a critical threshold value which in this case happens to be 2100 for Reynolds number, you have a change in the behaviour. So, what was stable has become unstable. So, that is what I have written down here. So, the stability is always talked in terms of responses to disturbances. So, what you do is you have a particular steady state and this steady state would normally correspond to what in my perturbation method I had as a 0th order solution, okay.

We did perturbation series now. Usually, I have a zero the order solution for which I know the solution. Like I have the parabolic profile for the laminar. Then, I do is (()) (42:38) giving disturbances and whatever I am talking about experimentally, I have to do with using my model to find out what this threshold is. So, what we are going to do is we determine stability in terms of the response to disturbance, okay.

Experimentally, if you are actually able to observe a particular state experimentally, you can conclude that that particular state is stable, I mean without doing anything special. Why, because disturbances are always going to be present when you are doing an experiment and if you still see this particular steady state that means the system is ensuring that the disturbances are decaying to zero, okay. So, any experimentally observed state, as long as you have not done something to control it, okay.

To make an unstable system stable, if you have not done anything special, if you are just letting a system run and if you observe it experimentally, then that particular state is stable. So, any experimentally observed state is stable as this is seen in the presence of disturbances. So, what we are going to do in this course is try and transform this simple experiment which you are comfortable with into a mathematical language, so that we can make a prediction of these threshold values, okay.

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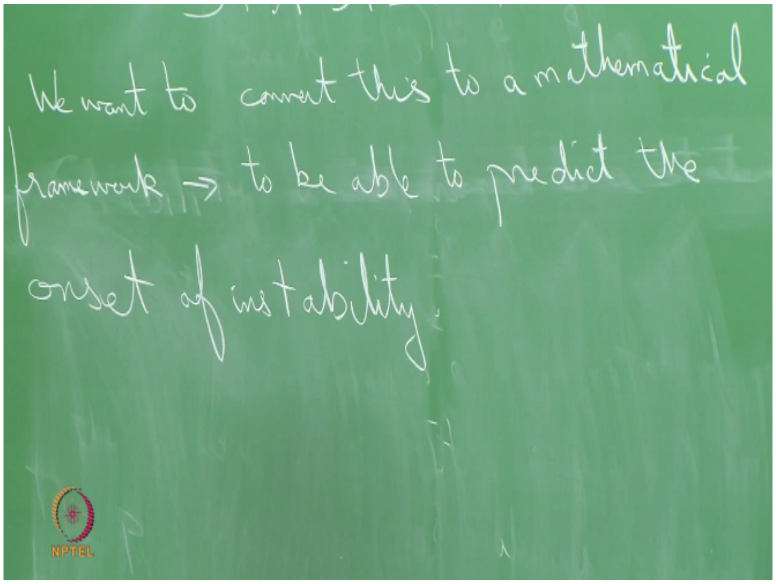


So, what we want to do is we want to convert this to a mathematical framework to be able to predict the onset of instability. In different context especially in the context of multiphase flows, okay, that is the whole idea.