Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture - 02
Stratified Flow in a Micro Channel: Velocity Profile

Welcome new to the second lecture of Multiphase Flows and what we are going to do today is basically look at a very specific problem in Microfluidics.

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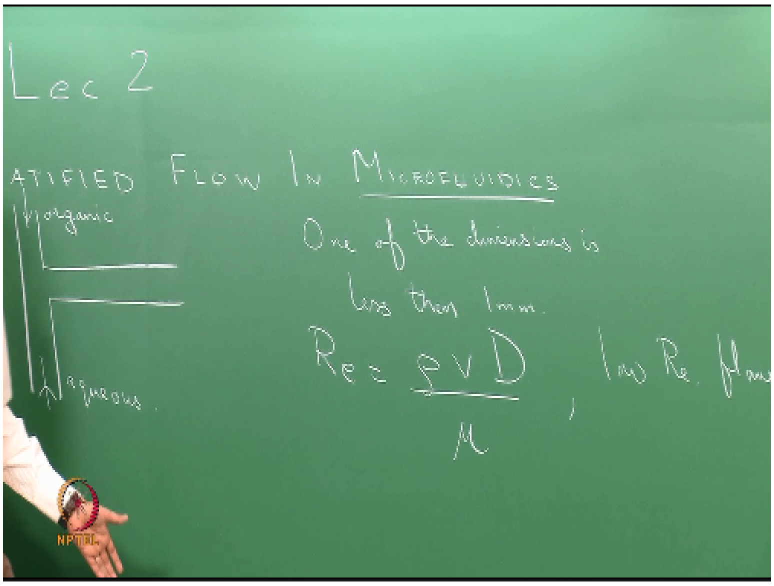

And that is the problem of Stratified Flow in Microfluidics. So like I have mentioned we will have one consider a system which has a T shaped and we have 2 inlets through these 2 limbs. What I want to be clear about is that the view I am drawing is the view of the channel as we see from the top. So the T junction is place horizontal on the table and we are viewing it from the top and so the 2 fluids are flowing side by side and you would have 2 different phases an aqueous phase and an organic phase which are flowing.

Now the objective today is going to be only to understand the hydrodynamics. What we are interested in is Microfluidics which means one of the dimensions is less than 1 millimeter. So one of the characteristics features of any flow in a Microfluidics channel is that the Reynolds number is very low because the Reynolds numbers as you know is going to be defined as the density multiplied by the velocity average velocity multiplied by the characteristics dimension divided by the viscosity.

So now we are looking at channels which have a very low diameter so we are talking about very low Reynolds number flows. So what does this mean? This means the inertial forces are negligible and viscous forces are dominant.

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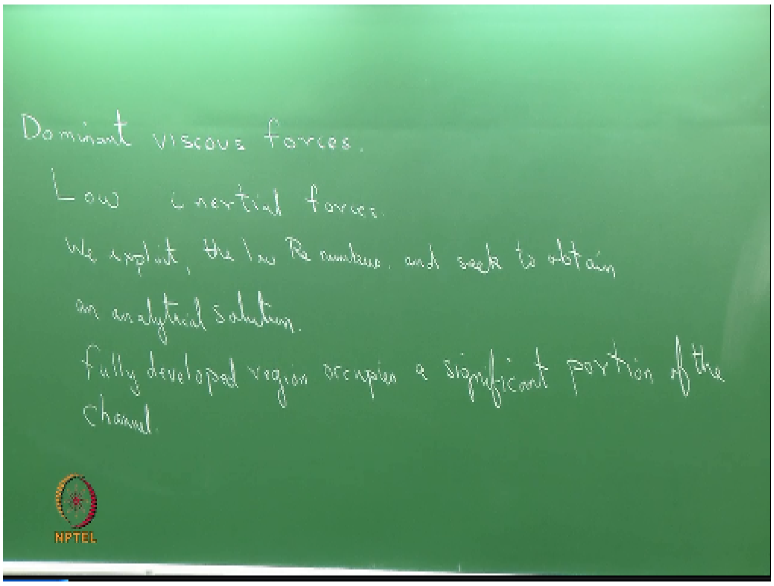

So we are looking at dominant viscous forces and very low inertial forces. So most specifically since we are looking at low Reynolds number we are looking at the lamina regime. So when you talk about lamina regime flows are very well defined and so that helps us go towards getting in analytical solutions. So whereas if you had large Reynolds number you would possibly go towards the turbulent regime and then you would possibly have to go to a numerical solution.

So our objective in this course is to try and get understanding of flow problems, transport processes analytically and to keep things simple we are basically going to exploit the fact that the Reynolds number is low. So we exploit the low Reynolds numbers and seek to obtain analytical solutions. So let us come back to this figure here. What do I expect? I have 2 liquids now which are actually immiscible.

The aqueous phase and the organic phase they do not mix with each other and let us say it is completely immiscible. So we are going to be looking at stratified flow. The aqueous phase is going to flow through this inlet channel is going to come and flow along this main channel here the organic phase comes and turns around. So this is going to be the path of the aqueous liquid, this is the path of the organic liquid.

Now if your channel is sufficiently long one of the flow regimes that is possible is what we have seen is the stratified flow which means that there is a very well defined interface between the 2 liquids and that is aqueous phase here and you have the organic phase here. So what we want to focus on is how is the flow field in this particular portion which is far away from the inlet here.

This particular portion close to the inlet is where you have what are called entrance effects because the fluid has to actually negotiate the bend and so you could have some mixing, you could have some lateral interchange of mass and momentum, but we are not focusing on this because in a typical situation what happens is the entrance length is just going to be a small fraction of the entire length.

So whatever you are really going to be actually visualizing actually you are going to seeing is going to occur over this part of the channel which is going to occupy a significant portion of the length. So as engineers you may be interested in understanding what is happening in this region which we will call the fully developed region. So typically the fully developed region occupies a significant portion of the channel.

So if you are interested in how a chemical reaction is taking place between these 2 liquids or how mass transfer is taking place between these 2 liquids. It is more sensible for you to focus on the fully developed region because that is when the actual process is going to take place. So what we are going to do today is mainly look at the flow because the flow is going to decide what the mass transfer behavior is going to be.

So today we will just focus on how to determine the flow behavior, how to determine the velocity profile when you have 2 liquids flowing in a fully developed manner in a micro channel. Now since we are interested in developing analytical solutions our focus is on getting not only mathematical solution but also getting physical insight. So we want to make sure that we simplify the problem in such a way that we retain the important physics.

So we want to make simplifications which will make it mathematically easy for us to solve a problem. So the solution is going to be mathematically tractable, but then we do not want to lose information about what is actually happening inside the system.

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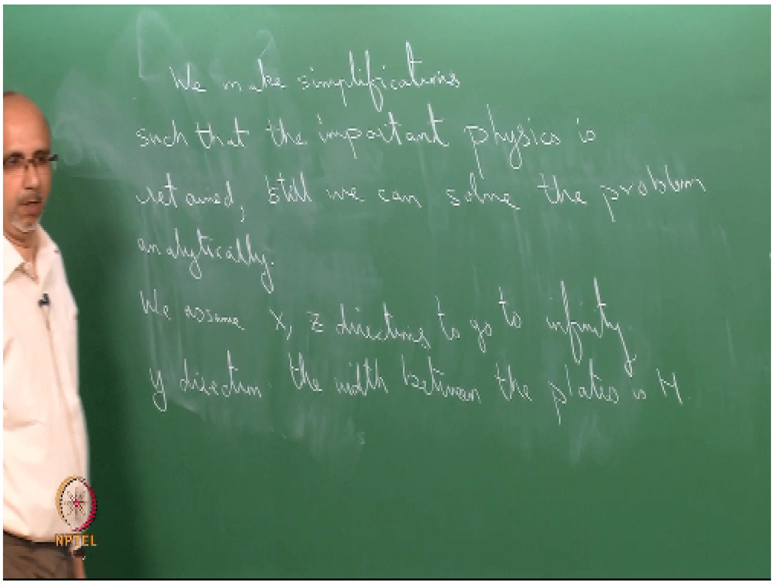

So our objective is make simplifications our strategy is as follows. We make simplifications such that the important physics is retained or still we can solve the problem analytically and so the simplifications are mainly mathematical. So now coming back to the fully developed flow now what I am going to assume is all of you have done a course in fluid mechanics. This is the PG level course.

So we will just go with our assumption later on we will actually derive some of the fundamental equations again. So let us look at just he fully developed region now which is x let say this is the flow direction which is the x direction and this is the y direction and have a z direction which is coming out of the board. Now I talked about making simplifications. So one of the simplifications I am going to make now is I am going to assume that instead of having a channel which has a finite cross section which is rectangular.

I am going to assume that these plates are actually going to extend to infinity in the z direction. So what does basically tell me that things are basically not going to be changing in a z direction. I am also going to assume that the things are going to be sufficiently long in the x direction. This direction remember I am going to have to keep it small and finite because I am looking at a micro channel.

So the simplification that we are looking at is we assume the x and z directions to go to infinity to be infinitely long and in the y direction the width between the plates is capital H.

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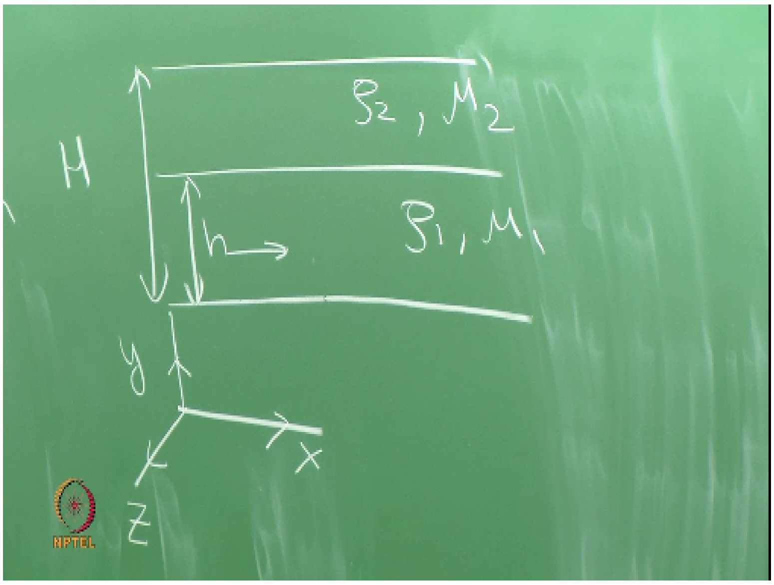

In order for me to specify the problem what I also need to know is the location of this interface. So I need to know so this gap between the 2 plates is capital H and the lower fluid is occupying h region of the entire space between the walls. So this is liquid 1, this is liquid 1 which has properties rho 1 and mu 1 this is the density of the first fluid, viscosity of the first fluid, this is the second liquid which has properties rho 2 and mu 2.

Now clearly the first fluid is occupied the distance $0<y<h$ and the second fluid occupies $h<y<H$. So one of the things that we are interested in is trying to understand how the velocity is going to change inside this because the velocity profile is going to decide for example the time spent by the fluids inside the channels and it is also going to decide any other transport processes which are going to possibly occur.

Now by simplifying the z direction going to infinity basically make sure that I can neglect the velocity component in the z direction and (()) (14:18) also neglect changes in the z direction. So that is basically what whether the flow direction is also going to infinity, but what I am going to do now is only focus only fully developed profile because that is what I am interested in.

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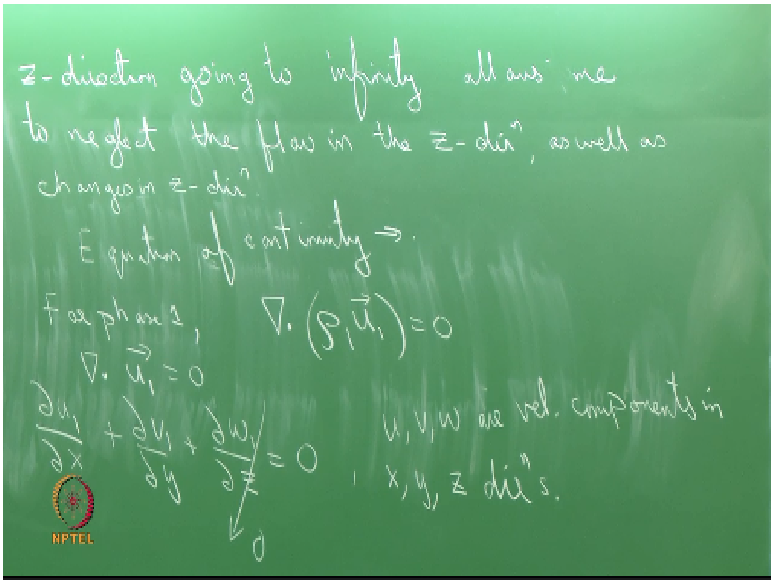

So z direction going to infinity allows me to neglect the flow in the z direction as well as changes in the z direction because the z direction I do not have any confinement or walls. So if the walls were there then the walls would actually affect the flow in the z direction. Now what I want to do is go back to understanding which velocity component is going to be important.

Okay so what we have done is we have simplified things by just saying that z direction things do not change. So we are going to focus only x and y direction now. So in order to focus on the x and y directions we are going to look at first the equation of continuity which is since we are looking at liquids. We are going to say that the liquids are incompressible no changes in the density and remember I am going to write this equation of continuity for each phase.

So I am going to write the equation of continuity for this liquid. I am going to write the equation of continuity of this liquid. So I am going to just say that the equation of continuity for phase 1 is the divergence of rho 1 u1=0. Density is at constant so I can actually take out the density and it basically bowls down to divergence of u1=0 since the liquids is incompressible and this essentially means if I want to write it in the scalar form where u, v, w are the velocity components in the x, y and z directions.

U, v, w are the velocity components in the x, y and z directions. What I want to do is I want to look at simplifying this equation. So the first thing we do is we observe that we have assume things will be infinitely long in the z direction so that is going to be no changes in the z direction besides the velocity is also 0 on the z direction. So I can essentially drop of this term so this term essentially goes to 0.

We have also assumed that the flow is fully is fully developed. So when I say things are fully developed what I mean is that there is no variation in the direction of the flow when direction of the flow is x. So when I am saying that the flow is fully developed I basically mean du1/dx is 0. So because of my fully developed through assumption and that is what I am focusing on du1/dx is 0 because it is fully developed.

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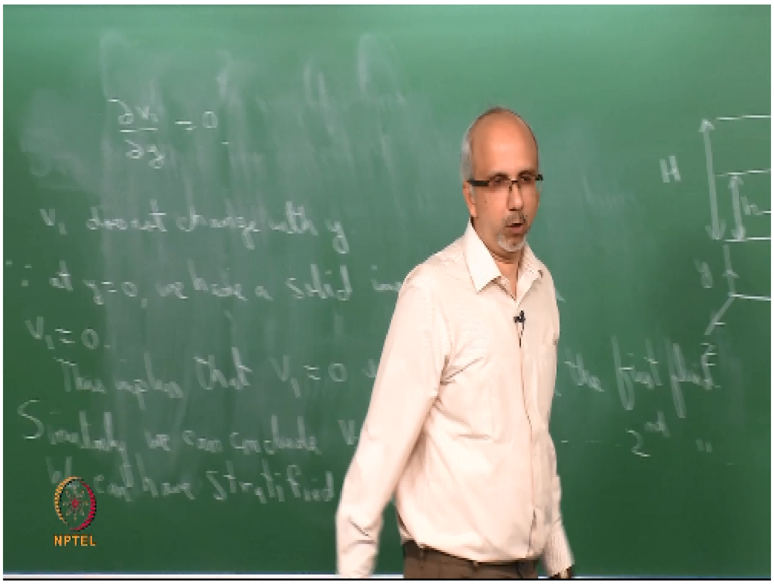

So what that leaves me with is that I have dv1/dy=0. So my equation of continuity for one of the liquids basically simplifying to this. I could have written the same thing for the second liquid and I would have got a similar expression dv2/dy=0/ Now from this what do I conclude I conclude that v1 does not change with y that all this means. Now what have here is at y=0 is a solid wall.

So the liquid cannot penetrate the solid wall. So the normal component of velocity which is the y component of velocity is going to be 0 at this wall. And this is going to be 0 everywhere. So what we know is since at y=0 we have a solid impermeable wall v1=0 at this boundary and we also know that v1 does not change with y. So from this I conclude that v1 is 0 everywhere inside the first fluid.

So this implies that v1=0 everywhere inside the first fluid. I could do a similar argument for the second fluid. I would use the fact that the top wall is a flat plate impermeable the velocity here is 0 and I would eventually come with dv2/dy being 0 use a boundary condition here and come to the same conclusion that v2 is 0 everywhere inside the second fluid. Similarly, we can conclude that v2=0 everywhere inside the second fluid.

So I want to emphasize here I have not assumed that the velocity in the y direction is 0. I only assume that the flow is fully developed. I only assume that the flow was fully developed and that told me basically where there is no velocity in the y direction. Now the fact that there is no velocity in the y direction the velocity is 0 here of this liquid velocity is 0 here in this liquid.

Basically tells me that the interface is going to remain flat. W will take a look at what the pressure is we will also find that there is no change in the pressure across this interface. So basically the interface is going to be flat. What this means is that the stratified flow is a possible solution that is we are being internally consistently. We assume that a stratified flow can possibly exist.

What I am showing here is a stratified flow can exist under fully developed conditions. So basically what this means is we can have a stratified flow. Sometime later on we will show you some videos wherein we have examples of stratified flow being seen. So experimentally you can see this, but that also consistent with the mathematical theory. So in particular what I am saying is you can have fully developed stratified flow where the interface remains flat interface does not get deformed.

So everything is coupled fully developed implies interface remains flat. Interface flat in turn implies there is no vertical component of velocity because if the interface had reflected that means there was a vertical component of velocity which actually cause them deflection. The fact that v1 is 0 follows from fully developed, follows from the fact that the interface is flat. And I am doing this in a rectangular condition coordinates because that is really simple and easy for you to possibly visualize.

Later on we will possibly look at curved coordinate systems. So having taken care of the equation of continuity and what that has told me is at the velocity component in the y direction is 0.

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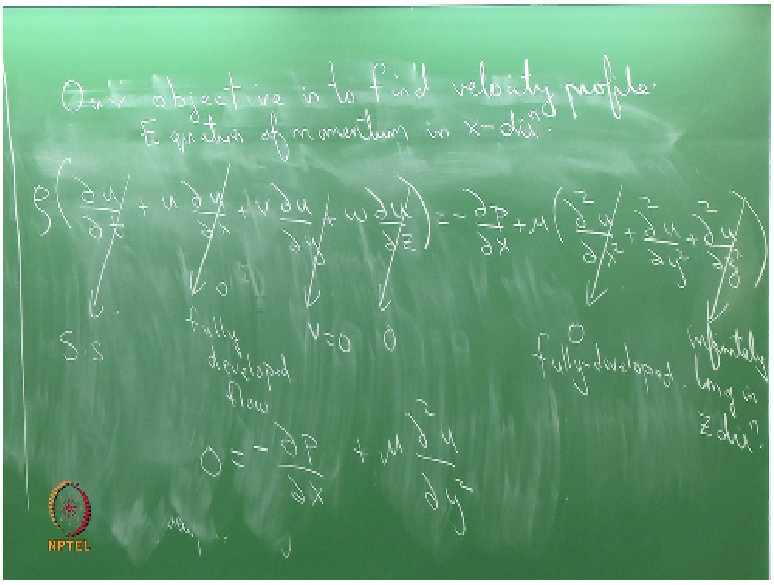

And I need to now understand our objective is what is to find the velocity profile. So the only component of velocity we need to worry about is the x component of velocity. So what you want to do is write down the Navier Stokes Equations in the x direction because that is what is going to decide what the x component of velocity is and simplify that with whatever we have already found out.

So that is what we are going to do. So let us write out the Navier Stokes Equation and I am just doing it for one of the fluids so I am not putting the subscript one it is the same thing for both the liquids. So basically what I have done is written down the Navier Stokes Equation or the equation of momentum. This is the equation of momentum in the x direction. I am writing this equation here and I want to use the information that I already have to simplify this.

So the fact that it is fully developed tells me that u does not change with x and that allows me to drop off this term because it is a fully developed flow. Remember that is what I have assumed fully developed. I am also assuming that the flow is at steady state. I am assuming that the flow is at steady state this goes off because at steady state this goes to 0. What about this term?

This term is going to drop off because we have already established that v0 everywhere inside the liquid. So this is 0 because v=0 everywhere inside the liquid and we have assumed things are not changing in z we assume that the velocity component in the z component is 0 so this drops off. Let us come to the right hand side. What about dp/dx. The flow is going to be driven by a pressure gradient.

So dp/dx has to exist. The only thing which is driving the flow is a pressure gradient so that is something which I am going to be externally imposing on the system by means of a pump so that is going to be there. There is something which I am controlling as an experimentalist. What about this term here? This term is going to be 0 because of fully developed flow. And this term is 0 because things are not changing in the z direction because it is infinitely long in the z direction.

This is u is this is infinitely long in the z direction. So things do not change in the z direction. I am having a flow only which is 1 dimensional I have the u component of velocity, the x component of velocity which is flowing in the x direction. So basically what this means is the equation of motion in the x direction simplifies to the following equation-dp/dx+ mu d square u/ dy square=0.

I want you to pause for a minute and think about the what I said earlier about flow being fully developed in low Reynolds number. Now if you remember these terms on the left basically represents inertial terms. This is a pressure forces this has a viscous force. Now low Reynolds number flows basically corresponds to very low inertial terms. So basically what I have done when I am simplifying. I could have done this earlier I could have just told you look we are looking at low Reynolds number flows.

We are just going to drop off this left hand side and then we are going to just retain the right hand side an done the analysis that would have been one approach which you can take because there are the inertial terms which are negligible in the regime of Microfluidics. So basically these are inertial terms so which we have neglected.

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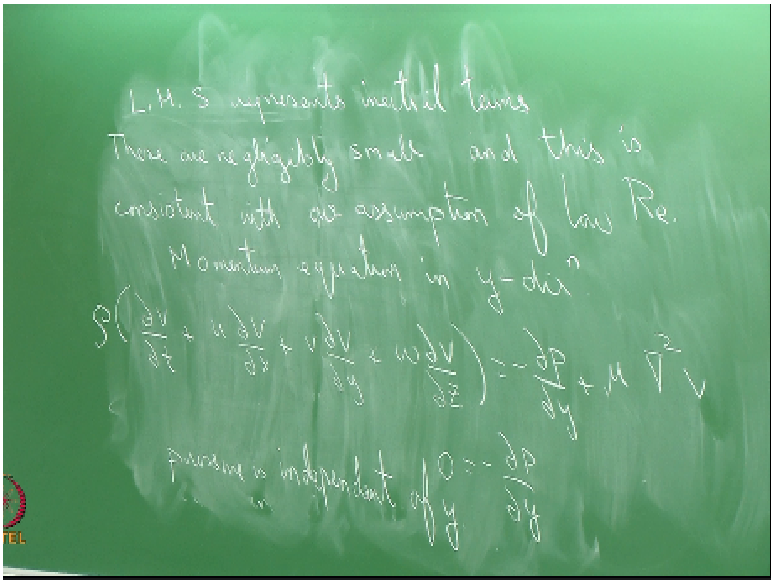

The left hand side represents inertial terms and these are negligibly small and this is consistent with our assumption of low Reynolds number. So I just want to tell you that what we have done now is basically neglecting inertial terms and remember I started off with low Reynolds numbers which means low inertial forces, large viscous forces they are fine. So basically the momentum equation tells me that the viscous forces are balanced by the pressure forces that is basically what it is.

So I need to also look at the momentum equation in the y direction. In the z direction I am not particularly worried about the z direction because nothing is changing in the z direction, but I want to simplify the momentum equation in the y direction. So let us do that. So look at the momentum equation in the y direction. What do we get? I have just written the Laplacian as it is.

So now I like to know because clearly there is some information which is present in the momentum equation and I like to extract that. So let us see what information is present here. You know that that we already established the vertical component of velocity the v component in the y direction that is v is 0 everywhere inside the domain. So essentially what this simplifies to is that the pressure is independent of y.

So what this basically means is that the pressure does not change in the y direction. So that the information which you are getting from the momentum balance in the y direction. So whenever you are doing a problem you need to make sure you get a result which is actually consistent. So remember if the pressure does not change in the y direction there is going to be no pressure discontinuity across this interface, there is no velocity component.

As a result, the interface does not deform because interface can deform also from a pressure change. Now we will look at what the regular boundary condition is at the interface the (()) (32:55) boundary conditions, but I just want to tell you here that the pressure is independent of y and which means the pressure can only be a function of x.

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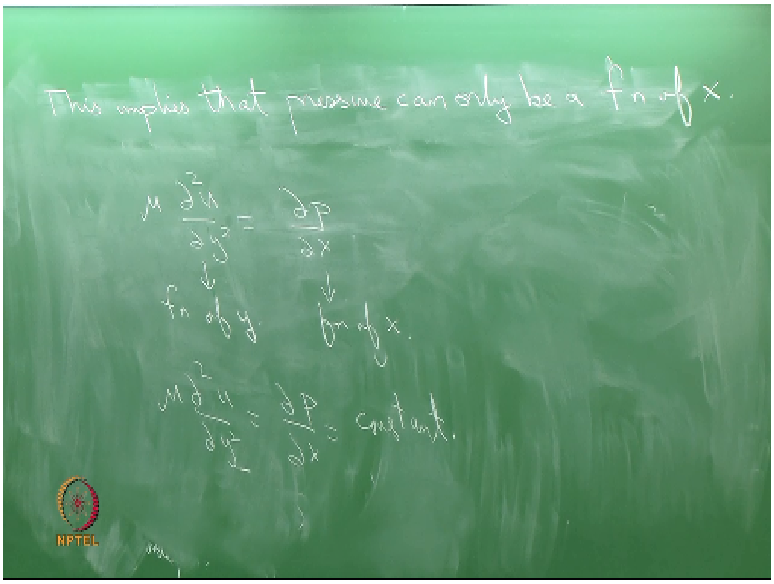

This implies that pressure can only be a function of x. And clearly that is what you expect we expect the pressure to decrease as you go along the flow. So pressure will change with x, but what we have found is that there is going to be no change in the y direction. So the pressure is uniform in the y direction So if you had a particular pressure at the inlet if you have a particular pressure this is going to vary along x, but there is going to be no variation induce in the y direction.

So there is going to be no pressure change across the interface so there is no velocity component across the interface, there is no pressure change and interface remains flat so far things are consistent. Now what we do is we need to find out how velocity is changing in the y direction and that tells me mu d square mu/d y square equals dp/dx. This is my differential equation which I am going to work with.

And now I am going to go back to the fact remember what about this what about dp/dx. It is a function of only x. This is a function of x. What about d square u/dy square. It is a function only of y. It does not depend on x because it is fully developed velocity is fully developed.

This is a function of y. So the only way a function of x can be equal to function of y is if both of them are equal to constant.

So what this means is that the dp/dx that you have is actually a constant. So you have constant pressure gradient along the flow which is driving a flow. So this implies that mu d square u/dy square=dp/dx which is equal to constant. Now what do you want to do is we want to get the velocity profile. I am going to go back and put my subscripts 1 and 2 because I need to find out what the velocity is for liquid 1 what is the velocity is for liquid 2.

What I have done is gotten rid of subscripts now because I just want to tell you that this is generic for both the fluids.

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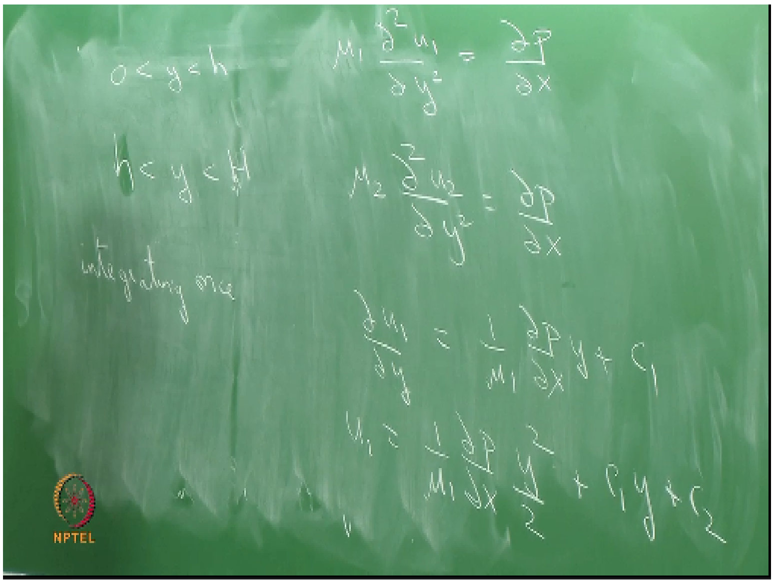

So in the domain $0 <y<h$. We have mu 1 d square u 1/d y square= dp/dx. In the domain mu 2 now see I have the same pressure gradient driving both the fluids. I have a particular pressure at the inlet and I already told you that I have pressure only varying in x direction. So I do not use at different constant for this pressure gradient and a different value for this both of them are the same.

The pressure gradient in both the liquids are the same and that is one of the simplification we have because our flow is parallel and straight. So if the interface has been bent then there would be a pressure difference between the 2 liquids which I would have to incorporate we will worry about that later right now I will do the simple problem. So the fact that this is a constant allows you to actually integrate this out and u1 depends only on Y so you can actually integrate this out.

And this would on integrating twice would give you u 1 maybe I should do this step by step. I do not want to make a mistake here. Let us just integrate the first guide du 1/du y integrating once du 1/du y gives me 1/mu 1 dp/dx+ c1 and u1 gives me 1/ mu1 dp/dx y square by 2+ c1 y+c2. So this is how my u 1 is going to be varying with y. C 1 and C 2 are arbitrary constant because I need to determine. So all I have done is integrated this differential equation.

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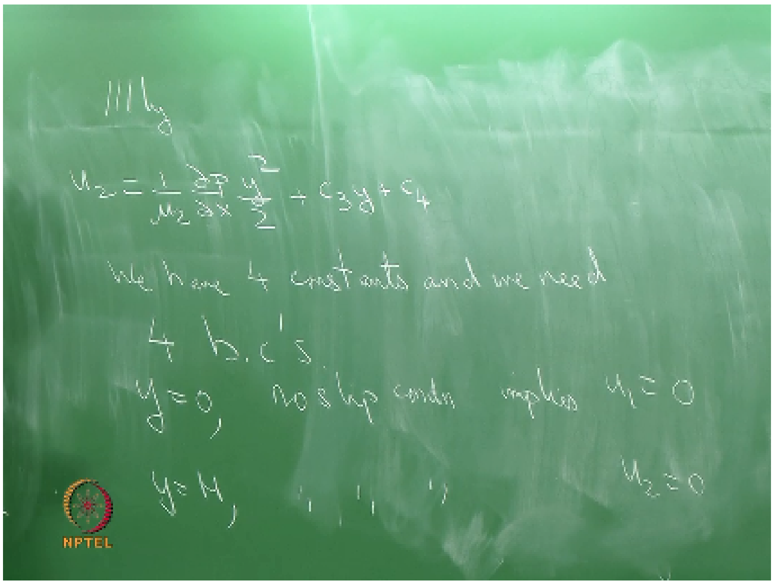

Similarly, I can conclude when u 2 is going to be 1/ mu 2 dp/dx y square/2 because I have mu 2 here+ c3 y + c4. I need to now find out these constant and for that I need boundary conditions and what is the boundary condition I am going to use. So we have 4 constant and we need 4 boundary conditions. So the boundary conditions are going to be clearly applied at the walls and at the interface.

So at y=0 or the 4 boundary conditions y=0 this is the lower wall. We have the no slip boundary condition. The no slip condition implies u1 is 0. Now at the upper walls and y equals capital h again the no slip boundary conditions because the wall is flat stationary the wall is not moving. The flow is being driven by pressure. This implies u2 is 0 it is the second liquid because the second liquid is going from small h to capital H.

So I got my two condition from my no slip boundary condition and I need 2 more. Now what I am going to do is these 2 conditions are going to be at the interface.

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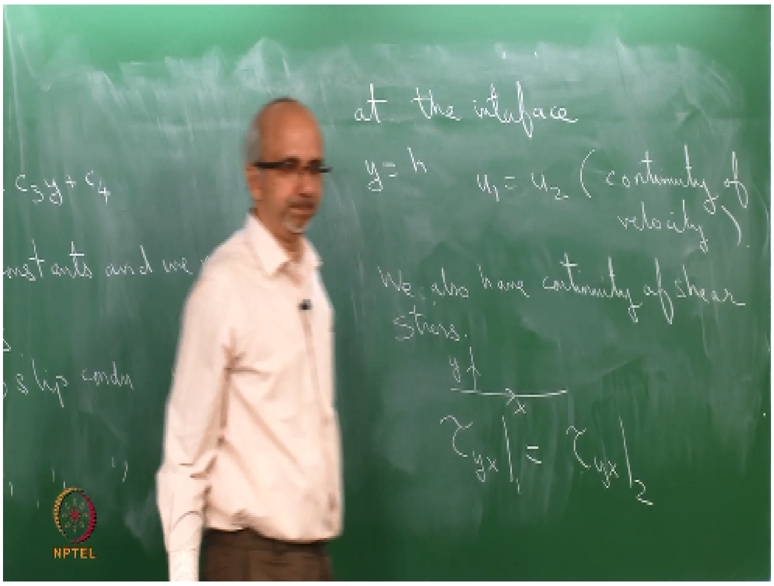

So at the interface where y= h we have the continuity of velocity. I want the physically admissible profile. I expect the velocity field to be continuous. I do not expect that to be discontinuity in the velocity field. So this means u1 must be= u2 and the other thing which I expect is where the shear stress the tangential stress exerted by the first fluid on the second must to equal to that exerted by the second fluid on the first.

So we want this is coming from the continuity of the velocity and we also have continuity of the shear stress along the interface. So we will derive the formal equation of how to go about calculating the shear stress, but right now what I will do is okay this is the x direction and this is the y direction. The shear stress at this interface is going to be given by tau y x the first subscript tells you the direction of the normal and the second subscript tells you the direction along which you are trying to find the stress.

So I am interested in finding the stress on a surface which is perpendicular to the y direction and along the x direction because that is where the flow is. So I want continuity of shear stress so I want tau y x in the first fluid must be= tau y x of the second fluid and if you are going to assume your liquid is going to be Newtonian the two liquids are Newtonian.

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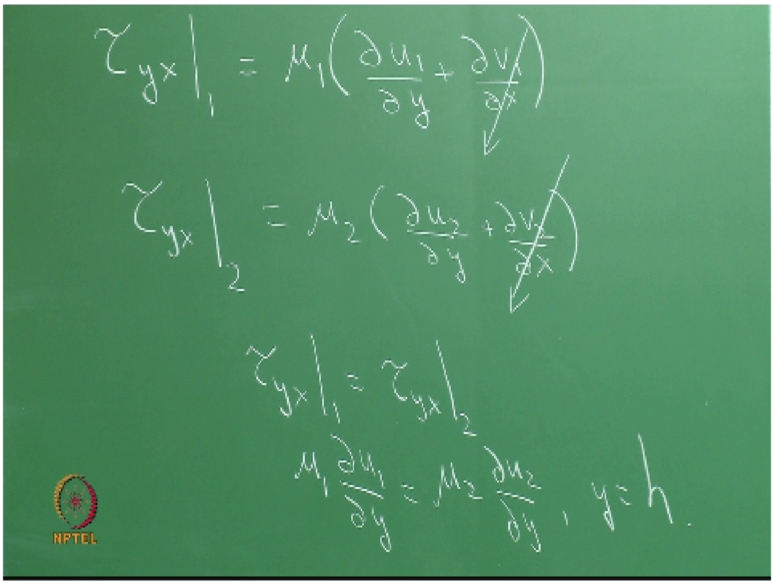

So the shear stress tau y x the first fluid is given by mu 1 du1/dy+ dv 1/dx. This is the shear stress at the interface and the shear stress at the interface given by the second fluid is mu 2 times du2/dy+ dv 2/dx. Remember these are evaluated at the interface y equals small h. Now since the interface is not reflecting v1 is 0 everywhere at the interface as a result of which dv1/dx is 0 and we have similarly dv2/dx is also 0 because of the fact that the interface is not reflecting and the vertical component of velocity is 0 for all x.

What this means is the equality of the shear stresses gives me for a Newtonian fluid. Remember this has been written for a Newtonian fluid mu 1du 1/dy= mu 2 times du 2/dy at y= small h of the interface. So these are the 4 boundary conditions this is the fourth one where we have equality of the shear stresses at the interface then we have the continuity of the velocity at the interface and we have the no slip boundary conditions at the 2 walls.

These 4 boundary conditions are used to find out the 4 constant of integration which arise when you integrate the second order differential equation of the velocity in each phase. And that gives you the velocity profile so these are the 4 conditions which I have to use to get my velocity field that is to determine my constant c1 c2 and they are just algebraic equation which you will use and you will find out what the constant are c1 and c2.

And then for particular values once you have found out c1 and c2 you can actually plot these velocity fields. So what I am going to do is I am going to ask you a couple of questions for you to think about and then we will stop.

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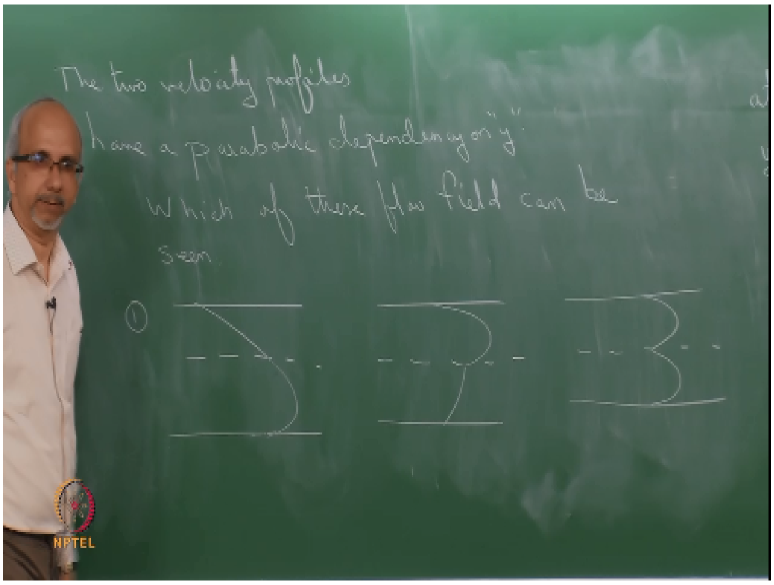

So now the two velocity profiles have a parabolic dependency on y in the y direction because it is quadric in y. So I am going to ask you a question which of these flow fields can be seen? One this is my interface. So what I want to emphasize is I have drawn these 3 flow fields keeping in mind 2 things. The velocity is continuous remember what I have is that the derivatives do not have to be continuous, the derivatives do not have to be equal.

There can be a difference in the slopes at the interface that is what it tells you the slope in the one liquid is going to be the ratio of the viscosity is multiplied by the slope in the other liquid. So that is discontinuity in the slopes that we are going to observe. So this is a possible flow field whereas one maxima in this liquid and there is one maxima here there could be a maxima in this liquid there is no maxima here or there could be a maxima in both the liquids.

So what I want you to do is think about whether these are possible if they are possible under what conditions will you actually see this kind of a behavior or this kind of a behavior or that kind of a behavior or if we are not going to able to see any of them. We will discuss this in the next time. Thanks.