Multiphase Flows: Analytical solutions and Stability Analysis
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Lecture - 18
Domain perturbation methods: Flow between wavy walls

So let us start with the lecture 18. What we did in the previous 2 classes actually talking about regular perturbation methods for finding approximate solutions to equations which had a small parameter and in both the systems the small parameter was in the differential equation itself. Now so far you are actually comfortable with solving problems where the boundary conditions applied along surfaces which are surfaces of some coordinated being = a constant.

For example, if it is a spherical coordinate system you would say r=constant that is the surface of a sphere. If it is a rectangular carnation coordinate system you would say Y=constant constant and you know how to apply the boundary conditions, right. But sometimes you could have a situation where the boundary the surface is not going to be defined as Y being = a constant or equal R being = a constant. It could be Y is a function of X, okay.

And when you write the boundary as Y equals function of X maybe there is a small parameter occurring in this form of a surface the definition of the surface, okay. So that means this is actually occurring on the boundary and what we are going to do now is trying to find solutions to problems of this kind using the method from Domain Perturbation. Again we are going to follow Gary Leal very closely.

So this is also explained in Gary Leal the earlier 2 examples were also worked out in Gary Leal, okay. This is just for your reference.

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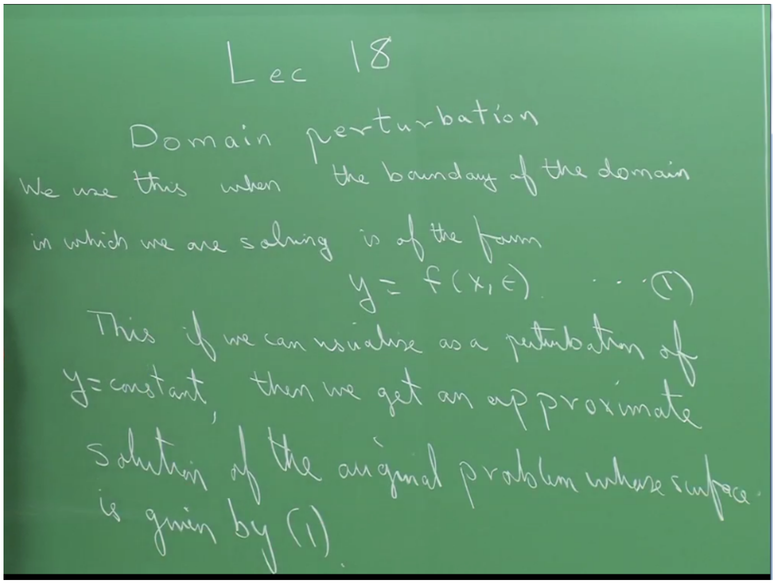

So I know we are going to focus on Domain Perturbation and we use this when the boundary of the domain in which we are solving is of the form—now y= let us say f of x, epsilon, okay. That is the surface of the actual problem. I will give you an example to illustrate this. What we are going to do is look at this surface. This if we can visualize as a perturbation of y=constant okay, because you know how to solve problems is at y is constant.

That is if this can be view as the perturbation of this kind of surface then we can possibly try to do a perturbation of a domain in terms of this and then seek a solution where the idea. Okay. So then we get an approximate solution, solution of the original problem whose surface is given by 1. Okay. So whenever you – so the earlier problems that we talked about they were what we called the regular perturbation problems.

Because you have the difference the small parameter occurring in the differential equation. Here the small parameter is actually occurring is the definition of the surface which is actually defining my boundary okay. So to give you an idea of this let us look at.

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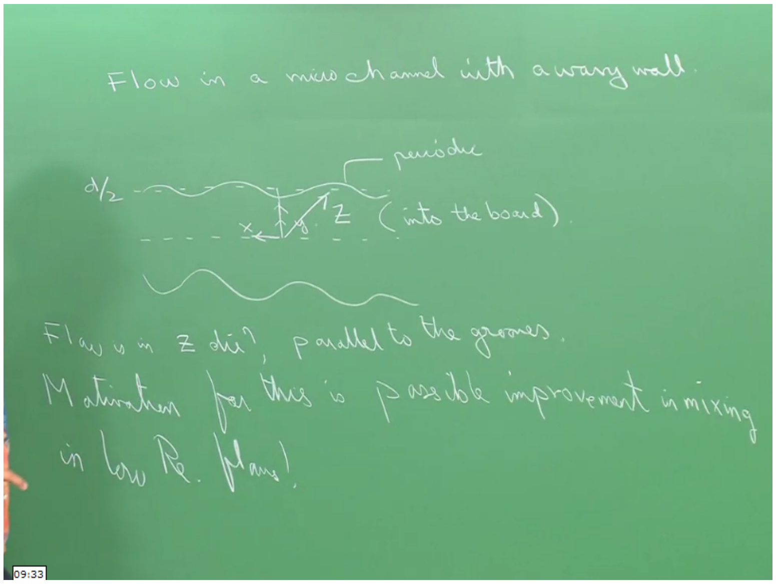

The flow in micro channel with a wavy wall. No we are use to pipes whose surfaces are rough. But supposing if there is a promiscuity in the roughness then is now just saying that random roughness which we have let us say that you have a periodically rough surface so again to keep life simple what we are going to do is we are going to look at a system like this like this-- so that is basically my wall with a maybe surface, okay.

Now let me get my get my coordinate system straight. This is Y and this is X and into the plane of the board is Z Z. Okay, so Z is into plane of the board. Looking at a problem were we have flow in a Z direction which is basically flow along the parallel to the grooves that we have. It is not in the X direction it is the Z direction. So these grooves that you have they are actually in the Z direction and the flow is in the Z direction which is parallel to the grooves

Now why would anybody be interested in a problem of this kind? Sometimes what happens when you are talking about flows in micro channels the Reynolds numbers are so low that the mixing is very poor the flow is laminar okay, so when you possibly give some induce some surface roughness then you can possibly induce not turbulence because your Reynolds number is low but mixing because you can possibly have velocities because induced in directions perpendicular to slow direction.

So you would not have velocities induced in the X and Y directions and so you can get better mixing. So basically the surface is actually inducing the 2 other components of velocity or that is the idea. So now what we are going to do is we are going to-- so motivation is motivation for this is possible improvement in mixing in low Reynold number flow, okay. So because surface being rough I may have some water stress induced here and so maybe better mixing.

The other thing now explain what is happening here, how would, I define the wall, these are my walls actually rigid walls. So this is supposing since this is periodic so I am assuming that this is periodic, and the first trigonometry function which comes to my mind when I say periodic is sin. So I am going to say that this guy the mean value is let us say at a distance of y=b/2, this b/2 mean value.

So the way I am going to look at this actual surface is why is b/2 multiplied by 1+ epsilon sign, 2 pi x/l where L is the wavelength in the x direction of this periodicity.

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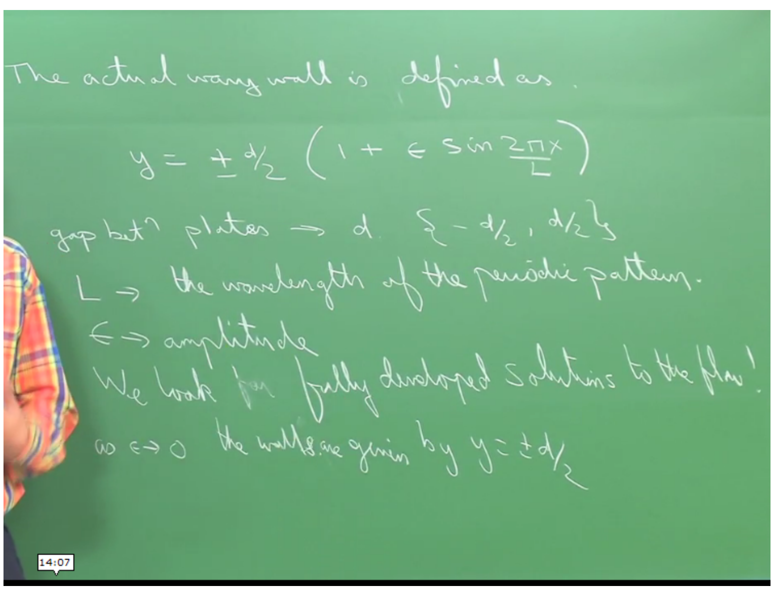

So the actual wavy wall is defined as y= +d/2 is the top surface the bottom surface will be –d/2 then the entire channel of the thickness b okay. So epsilon basically gives you some control over what is the amplitude of the periodic nature which is imposed. So the way I am looking at this problem that is a constant value d/2 so if this have been actually this is like if epsilon is 0 I reduces to the problem of flow between 2 flat plates which we have seen earlier. Okay.

The top surface is given by +d/2, the bottom surface is given by –d/2. So the channel in the y direction the gap between the plates is actually d deduction from –d/2 to +d/2 and L is what is L? The wavelength of the periodic pattern, epsilon is amplitude if you want that very large amplitude epsilon controls amplitude. And of course what we going to do is to apply simple and look for fully developed solutions that is we have very long plate extent in the Z direction and we look for fully developed solutions to the flow.

So as you can see now if epsilon is 0 my surface it becomes y= +d/2; y=-d/2 those are the walls and in y=constant you know how to solve the problem, I mean you will do just whatever separation of variable and then and then find those constants come out by neutral integration so you put y=d/2 and then you are able to find the constant. What is the problem now? when you do the you can possibly solve your differential equation.

But when you are trying to find the arbitrary constant which are going to come you cannot put y= this surface here because you want a constant and what you are going to get is a function of x. Okay. So that is what the problem is going to arise. So I just want to show you that you just cannot just substitute y= this because you trying to find a constant arbitrary constant of integration by applying the boundary conditions and since this is not a clean surface you got a problem. Okay.

So I just want to show here that as epsilon times to 0 the wall is given by 1= + or – d/2 I mean a 2 walls one is +d/2 and other is –d/2. Okay. The walls, I should write the walls are given by, right yeah. And we know how to solve that. So basically what I am going to do is I am going to do a perturbation analysis but now I am going to do a perturbation analysis keeping the fact that this guy is going to be perturb about this, okay that is the idea. And since that we are going doing it at a boundary so we have a domain which is actually periodic.

I am going to look at this boundary as a perturbation of the constant wall, that is the reason the domain perturbation problem the domain is being perturb, the domain of the solution is being perturb. Okay.

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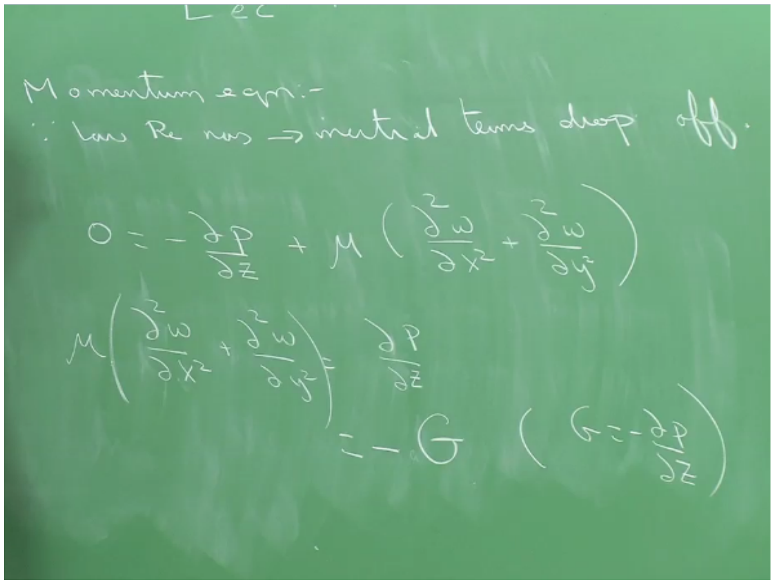

So we are looking at steady state lower Reynolds number flow. And what is going to be the Navier-Stokes the momentum equation, this is going to be lower Reynolds number flows so inertial terms are going off okay so we just go back to what we did earlier. So since the low Reynolds numbers the inertial terms drop off and what I have is 0=-dp/dz + Mu multiplied by d square w/d x square + d square w/dy square regard does not depend on z it is fully developed.

It depends on Y because clearly the boundary is there and depends on X because clearly there is a periodic nature of the surface in the X direction. So w1 is going to now function of x and y so what I am doing is I am basically looking at a 2 dimensional analog or extension or what we did earlier in the class where we had only a one dimensional flow, okay. So we are going to write this as d square w/d z square sorry x square + d square w/dy square = dp/dz.

And clearly the pressure gradient is negative so what I am going to write do write this as –G as G is positive. Okay. So here G=-dp/dz. And the--I am going to keep the Mu here. Okay. We do the usual stuff which is try to make things dimension less. Clearly what are the importance scales that we need to look at. We have a lane scale in the X direction so what are the characteristic lane scale in X direction? That is going to be L because that is the wavelength of your variation to x direction.

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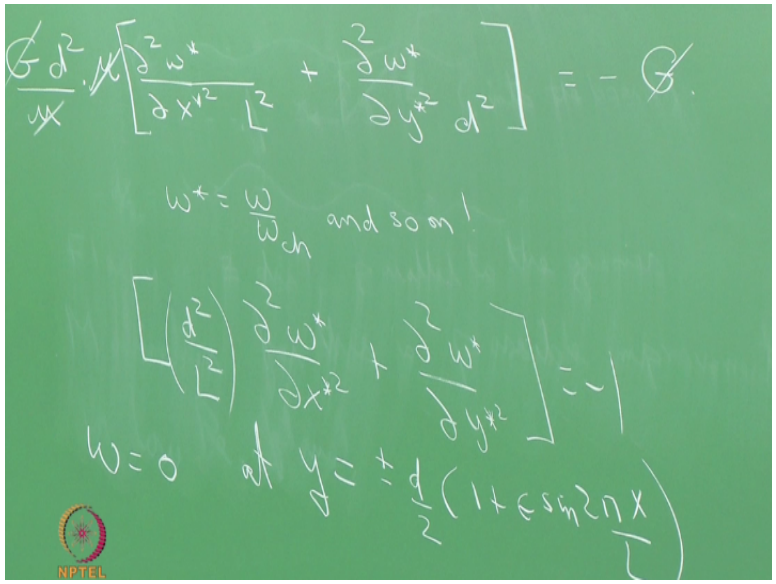

So x characteristic is L. what about y characteristic? That is D, there is a gap between the plates, and the other thing that we need to do is worry about the velocity and that is what we are trying to do to find out and we need to have the characteristic velocity which is wch. And since the flow now is driven by the pressure so we need to include the pressure gradient or G in this case, G is possible in the definition of our characteristic velocity and that is going to give me G multiplied by d square divided by Mu.

The gap between the plates is being used to define because that is the one which is going to decide the average gap between the plates (()) (18:36) define your viscous resistance to the flow. So let us use all these and make this equation dimension less. So in that I have d square w, w characteristic comes out I have G times d square/Mu okay and d square w */the x starsquare and x characteristic is L square + d square w star/dy star square and y characteristic remember is d square. Okay.

And this must be = –G and already a Mu there which I like to keep here. Okay. So I have just make my equations dimension less at the * variable defines my dimension less quantity so w star and w/w characteristic and so on. Similarly, for x star and y star, so all these * fellow are dimension less. Okay. So you can clearly cancel all these Mu's, clearly cancel all the Gs and what you have is. And multiply throughout by d square you get d square/L square times the partial derivative/ d x square + d square w square/d y star square = -1. Okay, that is what I get. So this is my differential equation.

What I need to do is look at the boundary conditions. The boundary condition is that on the surface is I have a rigid wall the velocity has to be 0, the No-slip boundary—that is not moving. The No-slip boundary condition basically tells me the W=0 at Y=+- d/2 times 1+ epsilon sin 2 pi x / L. I can define my characteristic, use this definition of Ych and Xch and write this equation in terms of a * variables which basically tells me that at w=0 I mean w is = 0 at, this tells me.

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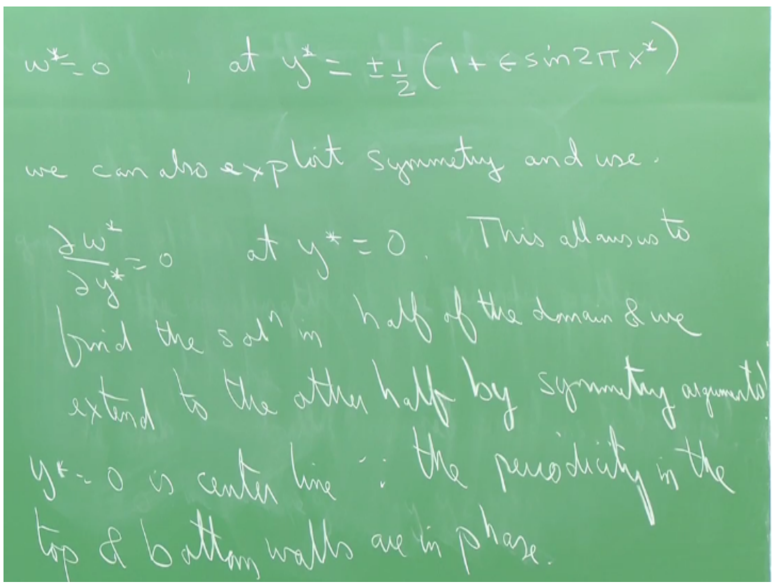

This tells me w=0 or w star=0 because I want to make w also dimension less at y star= + or – half 1+epsilon sin 2pi x star. Okay. That is my No-slip boundary condition. So either I can solve the problem in the full domain that is from + 1/2 to – 1/2 or I can just say that look I am only interest the solution is going to be symmetric and I am going to be looking at only at one half of the solution because by looking at one half of the solution I can get the other half.

So if you do we can use these 2 boundary conditions and solve perfectly fine or we can also exploit the symmetry and use that at w star dw star / dy star=0 at y star=0. Okay. So that means you are only solving half domain and then you can extend what is happening in the other half domain by just extending it, okay. This allows us to find the solution in half of the domain and we extend to the half by symmetry arguments. Okay.

“Professor - student conversation starts” So yeah. (()) (24:12) how can we say? Oh the centerline, unfortunately I drop this thing. **“Professor - student conversation ends”** So basically I am saying that the centerline is the center of the domain everywhere. So at every point y star=0 is going to the center point of my channel. So when you have the 2 turfs are in phase so y star=0 is always my centerline, see the way I have written this the top wall and the bottom wall both are in phase, the wavy pattern is in phase.

So y star=0 is always the centerline, and along this it is basically the pattern is symmetric across this. If they were out of phase, then you have—you cannot be able to use this. So this basically tells me that y star=0 is my centerline throughout x for – as I go along x. Okay. So what this means is y star=0 is the centerline since the periodicity in the top and bottom walls are in phase, that means 2 picks are coinciding I mean this guy pick and this guy bottom thing will be coinciding and then the turfs is going inside. Okay.

So now what we want to do is just extend what we have done earlier and which is seek W star at.

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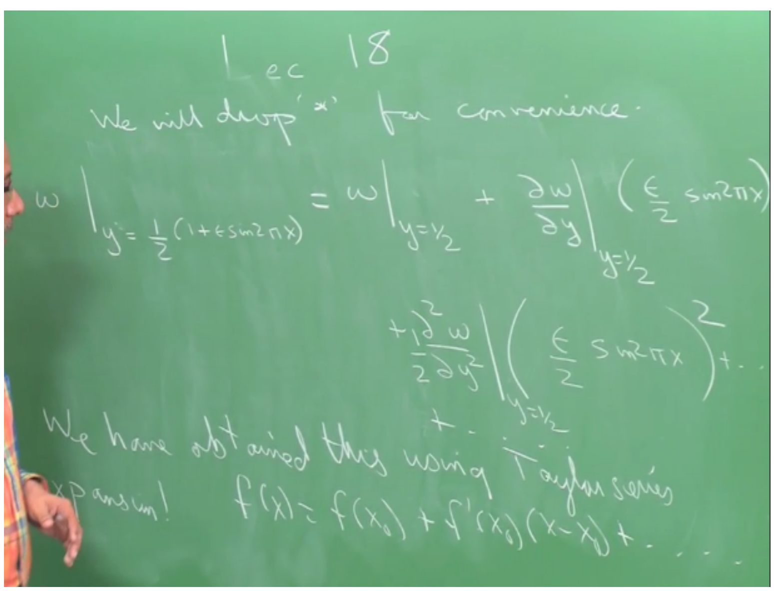

Okay we will now drop the start. We will drop the star for continence for my convenience, so I do not have to keep writing this stars. So w at y= 1/2 and 1+epsilon sin 2pi x. So this is the function which I want to evaluate, I want to evaluate w at along this surface at y=1/2 of 1+sin 2pi

x. I like to write this as w at y=1/2 just—I am going to use the definition of the Taylor series of a function via f of x I am going to write it as f of x = f of x naught + f – of x.

Evaluate at x naught multiplied by x – x naught, right. So that is the whole idea that is what we are going to—using Taylor series expansion I am going to write this as f of x evaluate x naught, x naught x corresponding to epsilon = 0, okay. dw/dy evaluated at y=1/2 multiple by x-x naught which is epsilon/2 sin 2pi x, okay. This is x, so x naught is x naught is epsilon/2 sin 2pi x + d square w/dy square times epsilon/by times 2pi x the whole square / 2 + higher order terms.

So all I am doing is we have obtained this using a Taylor series expansion, which is f of x = f of x naught + f dash of x naught multiplied by x naught – x naught + etcetera. The higher order terms which I am just not going to worry about. So what we going to do now is – now what I have done, see what I have done is made a transformation, this particular Taylor series expansion has allowed me to evaluate the w at y=1/2.

And that is basically something which I am comfort table with that is my comfort zone because when it tells me y=1/2 I know how to plug-in the boundary condition and get my arbitrary constant of integration because a boundary condition is basically used for getting my constant integration, okay. When I have to evaluate w at y=1/2 I am fine. I know how to do this.

So basically what I have done is taken this w evaluate on this surface which is not one of the regular surfaces is y= constant. And (()) (30:10) this is just a small deviation from a y= constant surface y=1/2 and doing a Taylor series expansion. Okay. So I have w and y=1/2 plus the first derivative + the second derivative term. And of course all the derivatives are evaluated at y=1/2. Okay.

So now the problem that I am solving is the differential equation which is right here and that symmetric boundary condition which I am comfortable with because there is no epsilon there. Okay. And this fellow here which has the epsilon in it, so what I am going to do is I am going to problem is now (()) (30:59).

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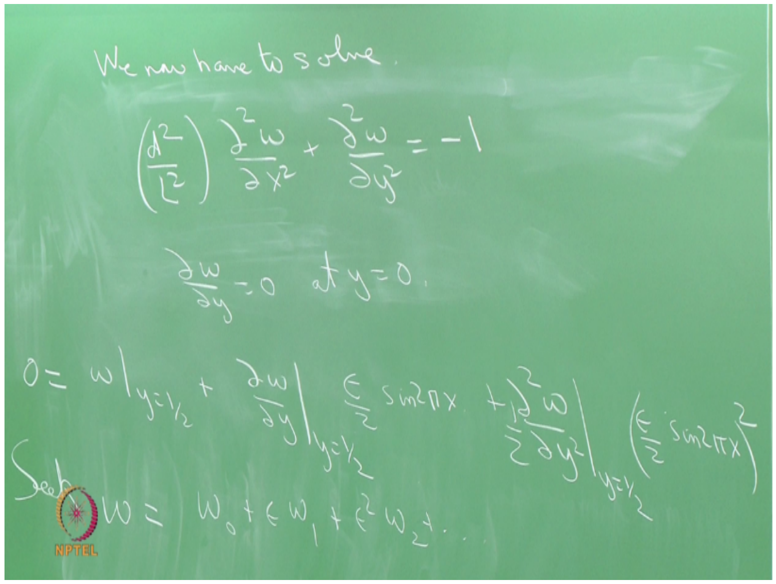

So we now have to solve this problem, equals -1. And let me write the easy boundary condition first which is the symmetric boundary condition 0 and the w=0 on that surface I am going to write as 0=w at y=1/2 + dw/dy at y=1/2 times epsilon/2 sin square 2pi x, sin 2pi; d square w/d y square evaluated at y=1/2 times epsilon/2 sin 2pi x the whole square. Okay. So w along the surface—I am just using this w – this w is 0 so 0 must satisfy this.

So basically what I have done is I am trying to show you now just transform the problem into something where the boundary condition is evaluated at y=1/2, the parameter epsilon is occurring and this a small parameter assuming that this guy this wave is having a small amplitude. And I want to seek a solution to this problem. So now to seek a solution to this problem we know how to do this because my earlier irritant was my surface was having the function of x in it.

So I use this Taylor series expansion and (()) (32:55) that problem, I have all these condition being evaluated y=1/2. So now I can hope to proceed further, in the sense that use the same idea what we had earlier, seek w as w0 + epsilon w1 + epsilon square w2 etcetera. Because now w this guy is fine does not have epsilon in it, this does not epsilon in it but this boundary condition has epsilon on in it, okay. And so if epsilon above the 0, I know what the solution is.

I mean this is flat wall. If epsilon is 0 that means the wall is flat and I can find the solution w at y=1/2 is 0 and so epsilon = 0 I can possibly find the solution. And for epsilon non-zero I have to make corrections. So the same idea what we had earlier, I am just going to seek in this form. And now I am going to substitute this everywhere in the differential equation, in this boundary condition and in this boundary condition and group terms of order epsilon to the power 0 and epsilon to the power 1.

Let us do a one which is challenging which is this and in the sense slightly more challenging than the other 2, (()) (34:13) challenging I would have not done in the class, right. So what we going to is just substitute this, so shall I just say this is sum number 3 and this is sum number 2.

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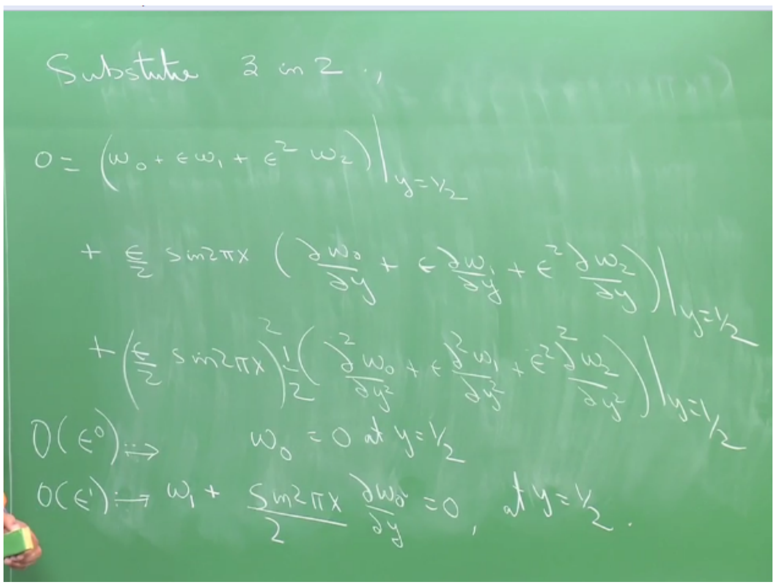

Substitute 3 in 2, and what do I get 0 = w0+epsilon w1 + epsilon square w2 evaluated at y=1/2. That is my first term. I am just substituting this guy there, okay. + epsilon/2 sin 2pi x that is this multiplied by derivatives half + whole square and there is a /2 also which I need to careful about, okay. All I have done is substituted this in this expressions here. So basically I am saying just look at this problem the first 3 lines that is my problem.

It has a small parameter epsilon in it and that is what that motivate me to seek my solution in the form of this power series, okay in epsilon and I am just substituting this to find out w0, w1, w2 like we have done before, only thing is that now the epsilon was onto to the differential equation but this was in the domain. So what we do, we need to group terms of order epsilon to the power, epsilon power to the 1 etcetera.

So what about order epsilon power to the 0 which is—so this guy is clearly of order epsilon because there is always epsilon here it multiples everything, this has epsilon square because it multiples everything, only thing which is the order epsilon to the power 0 is this w0 term. Okay. So this tells me at order epsilon, at this (()) (36:53) remember for all epsilon any arbitrary epsilon that means each and every coefficient has to be 0.

This means this implies w0 must be = 0 at y=1/2. Because this is the only term which is of order epsilon power to the 0 or order 1, all these guys are whatever. What about order epsilon to the power 1. This contributes, okay so I have w1 + this will contribute, only this term will contribute and that is w1+sin 2pi x/2 multiplied by dw0/dy, this equals 0 at y=1/2, okay, I will consider = this. So it gives me this. Okay.

So in a (()) (38:11) sequence emerging I would have solved it for w0 first and when I know the solution I come back and I find w1 and so on and so forth. And I am going to just do order epsilon square and then we will stop, okay.

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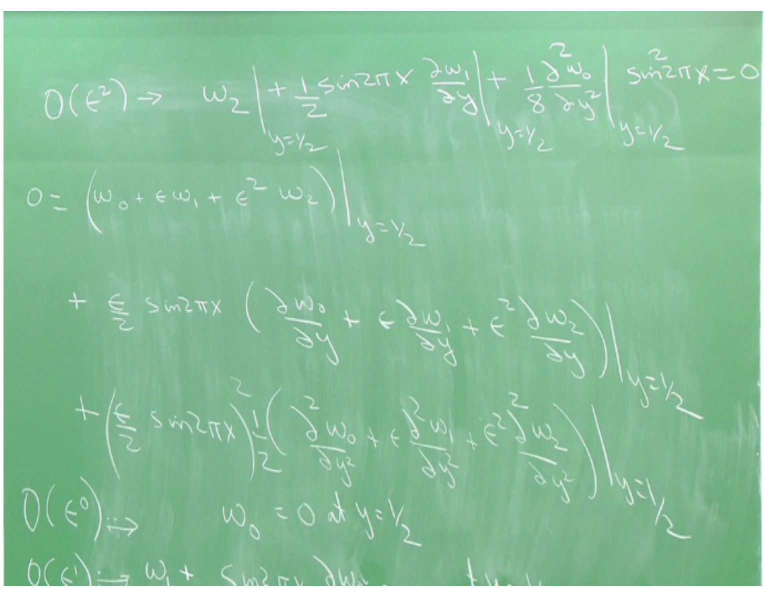

So order of epsilon square gives me w2 at this term, this multiply by this gives me order epsilon square which his +1/2 of sin 2pi x times dw1/dy, okay. And this multiply by this gives me epsilon square again all the other guys give me higher order terms, so I have +1/8 of d square w0/dy square multiplied by sin square 2pi x and this must be 0. Okay. And remember this is all evaluated at y=1/2, all these guys are evaluated at y=1/2.

So now I am in a good position because my boundary is, are all being evaluated at y=1/2 and then seek to proceed with finding my solution for w0, w1, w2 okay. Now that is the differential equations, so these are the 3 boundary conditions which I have, I mean it is the same boundary condition of different order terms. Let us do the same analysis for the differential equation as well as the other boundary condition that is I have a substitute w in terms of this here as well as here. I got to do everywhere.

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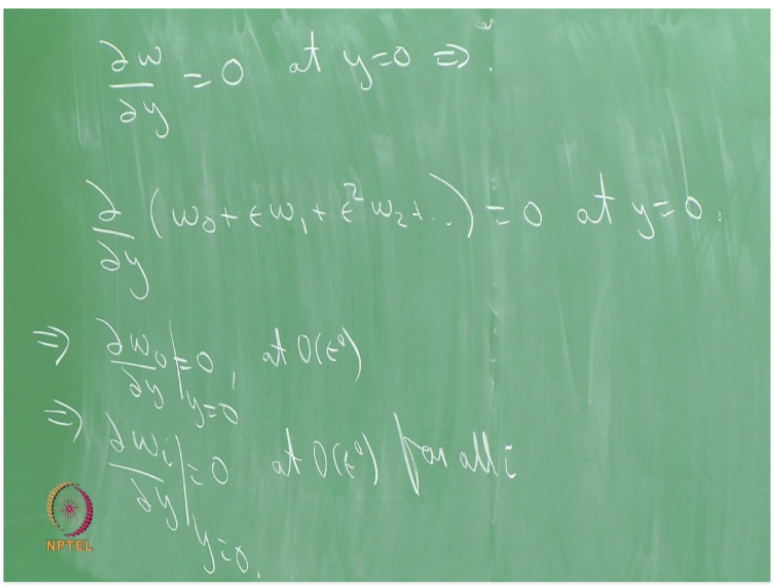

Let us do this easy fellow first dw/dy=0 at y=0 implies d/dy of w0 + epsilon w1 + epsilon square w2 + etcetera = 0 at y=0, okay. And here there is no expansion business because I have already y=0, so I am perfectly fine with just using it as it is. So this implies d w0/dy 0 and order epsilon to the power 0 implies dwi/dy = 0 at order epsilon power to the 0 for all i, that is what we are going to get because epsilon, or the epsilon will give me dw1/dy 0 and so on.

Professor to Student conversation starts ”What about the differential equation? That is going to give me, yeah, that is fine?”

Student to Professor conversation starts “That is y=0.”

Professor to Student conversation starts “Yeah, this all evaluated at y=0, this is all at y=0, correct.”

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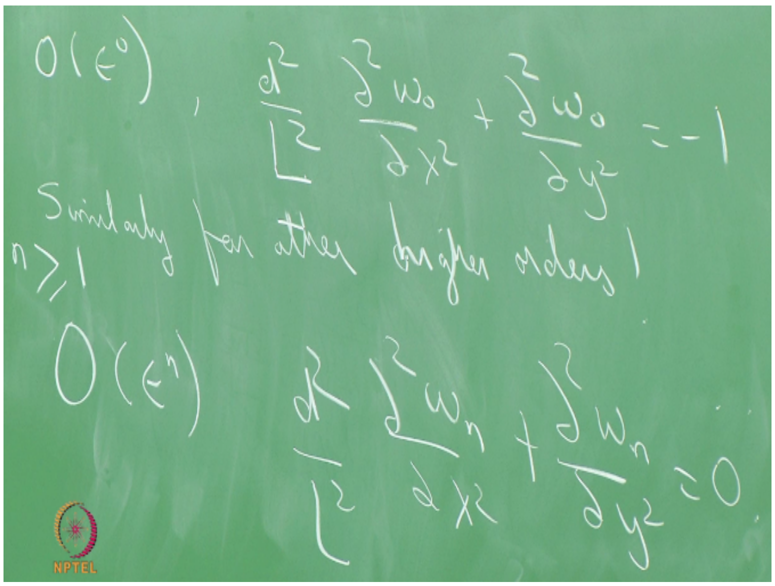

And substituting in the differential equation what do I get? And order epsilon to the power 0 I would get d square/ L square, d square w 0/d x square + d square w0/dy square = -1 and so on and so forth. For all (()) (42:23) it would be the same thing. Similarly, for other orders, other higher orders, yeah.

Student to Professor conversation starts “other orders is = 0”

Professor to Student conversation starts “other orders would be = 0, you are right, absolutely right, thank you otherwise that would have been a bigger problem tomorrow.”

So for n > or = 1 order f1 to the power n d square/L square d square wn/x square + d square wn/dy square will be = 0, that is absolutely right, because this is what order epsilon to the power

0 or the other higher order terms do not exists, okay. So this would be 0 and that is true for all higher order terms. Okay, good. So we all set to find out. So we do the usual stuff find w0 or find w1, w2. Okay. So how to go about finding w0?

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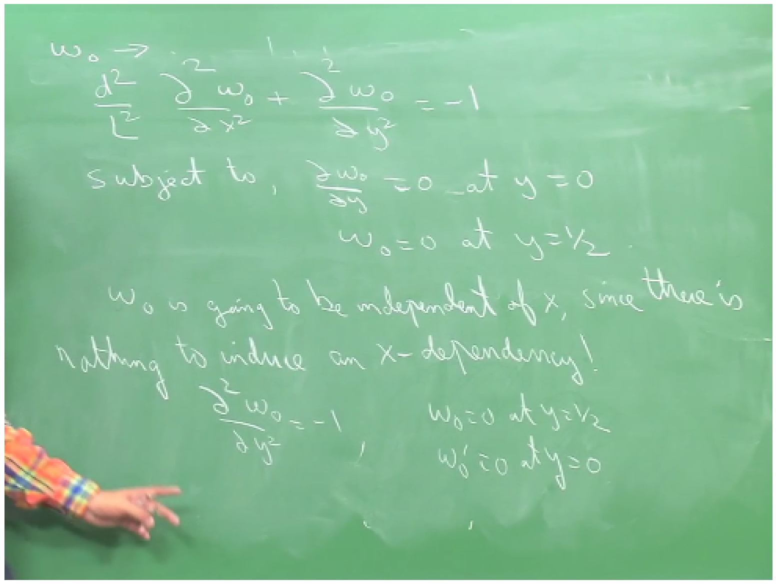

So w=0, a problem for w0 is d square/L square; d square w/d x square + 1, okay subject to dw0/dy=0 at y=0 and w0=0 at y=1/2. So now I want you to just look at the differential equations which tells me that w0 is a function of x and y, okay. And but when you look at the boundary condition you do not have anything in the boundary condition which is going to actually induce a variation in the x direction. Okay.

If at all – the reason why we put a variation in x direction was that the original problem had a variation x direction in the boundary but here I am saying look at the boundary condition, there is nothing in the boundary condition to impose a flow the dp/dx is independent of dp/dz is independent of x. So basically what I am saying is w0 is going to be independent of x since there is nothing to induce an x dependency, and therefore I can come here and I can just neglect the dependency on x.

And I have my classical problem -1 and w0 = 0 and y=1/2 and w0 derivative = 0 at y=0. So that is basically saying that when epsilon is 0 you have flat plates okay and my w0 solves-- remember these all consecution because w0 solves the problem with epsilon being = 0. If epsilon is 0 my periodic perturbation that I have on the wall is not there I have flat plates, and a flat plate with a pressure drop impose in the z direction and is sufficiently long in the x direction I can neglect the variation in x direction and I just get my parabolic profile in the y direction with these boundary condition.

So this of course you can solve and if you be able to get your parabolic profile. Okay. So once you get the solution for w0 you go back and find the solution to problem for w1 then find the solution for the problem w2, okay. And one of the thing which we would also like to ask is what is the effect of this perturbations on the wall on the flow rate which is through the channel okay. If I am going to keep my pressure drop the same the gradient the same for a fix lane.

And I have a flat channel and I have a wavy channel is it going to result in an equal flow rate or a higher flow rate or a lower flow rate. So that way you can I can idea about whether you can process more chemicals in your channel or not.