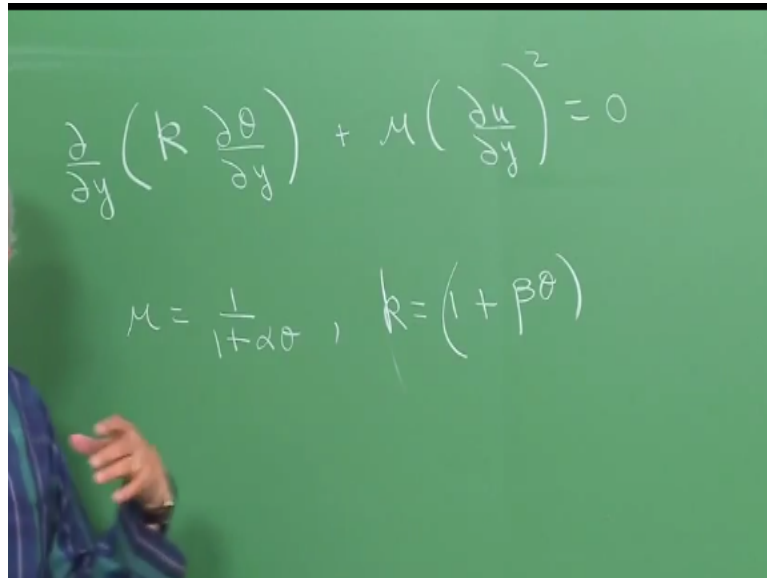


Multiphase Flows: Analytical Solutions and Stability Analysis
Prof. S. Pushpavanam
Department of Chemical Engineering
Indian Institute of Technology - Madras

Lecture - 17
Viscous heating: Apparent viscosity in a viscometer

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$$\frac{\partial}{\partial y} \left(k \frac{\partial \theta}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 = 0$$
$$\mu = \frac{1}{1 + \alpha \theta}, \quad k = (1 + \beta \theta)$$

So we will continue today working on the problem of incorporating the effect of viscous heating in a viscometer okay and one of the things we are going to see is that the estimation of the viscosity, if you neglect viscous heating what is it going to be and when include the effect of viscous heating how is it going to be affected? So that is the whole idea because viscosity is the function of temperature okay.

Now in the last class what we did was we worked on the problem of the momentum equation and we simplified it. So today we are going to take the energy equation, which we have derived last time and we are going to simplify this again and do a perturbation series analysis right. So we had assumed the viscosity to be varying as $1/1 + \alpha \theta$ and k as varying linearly with θ okay.

Now, it is perfectly fine this has been done just to keep the algebra simple in some sense but you know $1/1 + \alpha \theta$ can also be approximated as a linear relationship if you are interested in a small interval of θ for example okay.

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$$\frac{d}{dy} \left(k \frac{\partial \theta}{\partial y} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2 = 0$$

$$\mu = \frac{1}{1 + \alpha \theta}, \quad k = (1 + \beta \theta)$$

$$\frac{\partial k}{\partial y} \frac{\partial \theta}{\partial y} + k \frac{\partial^2 \theta}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 = 0$$

$$\frac{\partial k}{\partial \theta} \left(\frac{\partial \theta}{\partial y} \right)^2 + k \frac{\partial^2 \theta}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 = 0$$

So what we will do is we will proceed with simplifying the equation given as $dk/dy \cdot d\theta/dy + k \cdot d^2\theta/dy^2 + \mu \cdot (du/dy)^2 = 0$ okay. Now I am going to find dk/dy as k is a function of θ and θ is a function of y so the first term dk/dy becomes $dk/d\theta \cdot d\theta/dy$ times $d\theta/dy$ so this is just $d\theta/dy$ the whole squared $+ k \cdot d^2\theta/dy^2 + \mu \cdot (du/dy)^2 = 0$.

Now there was some confusion in the last time because I am not here into account the 0th order term.

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We seek

$$\theta = \theta_0 + Br \theta_1 + Br^2 \theta_2 + \dots$$

$$u = U_0 + Br u_1 + Br^2 u_2 + \dots$$

$\theta_0 = 0, \quad U_0 = y,$ from the cond $Br = 0$.

But I am just going to say that we seek θ as being $\theta_0 + Br$ times $\theta_1 +$ higher order terms θ_2 , u as $U_0 + Br$ times $u_1 + Br^2$ times $u_2 +$ higher order terms. These are the corrections, which we need to find out.

Now basically what this means is when the Brinkman number is 0 and that is the small parameter about which I am doing the expansion.

The solution to the governing momentum and the energy equation is going to be $\theta = 0$ and $u = y$ okay. Physically what is it? When there is no viscous heating, the temperature is going to be isothermal, temperature is going to be T_0 everywhere or $\theta = 0$ okay and the velocity profile is going to be linear and when which is we made dimensionless, we know that $\theta = 0$ and $U = y$.

Let us just say from the condition that Brinkman number is 0. When Brinkman number is 0, then basically there is no viscous heating. The momentum equation there is no temperature change. Momentum equation, temperature equation become decoupled and you can solve separately okay and physically that is what you expect. No temperature change, linear velocity profile.

I am going to substitute all the things that I know, $dk/d\theta$ is nothing but β okay.

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The image shows a green chalkboard with handwritten mathematical derivations. At the top, it says "We seek" followed by two expansion equations: $\theta = \theta_0 + Br \theta_1 + Br^2 \theta_2 + \dots$ and $u = u_0 + Br u_1 + Br^2 u_2 + \dots$. Below these, it states $\theta_0 = 0, u_0 = y$, from the cond $Br = 0$. The main part of the derivation is the energy equation:
$$\beta \left(\frac{d}{dy} (\theta_0 + Br \theta_1 + \dots) \right)^2 + (1 + \beta \theta) \frac{d^2 \theta}{dy^2} + \frac{1}{1 + \alpha \theta} \left(\frac{du}{dy} \right)^2 = 0$$
 The next line shows the expansion of the first term:
$$\beta \left(\frac{d}{dy} (Br \theta_1 + \dots) \right)^2 + (1 + \beta (\theta_0 + Br \theta_1 + \dots)) \left(Br \frac{d^2 \theta_1}{dy^2} + \dots \right) + (1 + \alpha \theta)^{-1} \left(1 + Br \frac{du_1}{dy} + \dots \right)^2 = 0$$

So I am going to write this as $\beta \cdot d\theta/dy$ whole squared $d\theta/dy$ whole squared is nothing but d/dy of $\theta_0 + Br$ number times $\theta_1 + \dots$ whole squared $+ k$, which is nothing but $1 + \beta \theta$ times $d^2 \theta/dy^2$ squared $+ \mu$ is $1/1 + \alpha \theta$ times du/dy the whole squared $= 0$. I am just doing this possibly in a slightly inefficient way, but then I hope to minimize mistakes that I may end up making.

So this is beta and now what I need to do is substitute for theta in terms of this expansion because my job is to find out theta 0, theta 1, theta 2 and if I find out this then I can substitute it back here and find out what theta is right. So I know theta 0 is 0 and what does this give me? d/dy of I am just inserting this. So it simplifies my life otherwise if I do not put theta 0=0 here, I will get 0th order term and then I have to calculate it.

I am just using the fact that I have already calculated the 0th order term. So I am just going to use that to directly get the first order term okay and you can clearly see that this term is not going to be first order because the lowest power that this is going to contribute to is the Brinkman number squared. So this actually going to give me a second order term, so for the first order thing this is basically 0 okay.

This is not a first order term. Let us come to this term here, $1+\beta$ and theta is written as theta 0, which is $0+\theta_1$ multiplied by Brinkman number+higher order terms, which I do not want to write because that is going to give me Brinkman number squared okay and $d^2\theta/dy^2$ is going to be when I substitute this back inside here I am going to get theta 0 is 0.

I get Brinkman number times $d^2\theta_1/dy^2$ okay+higher order terms, which we do not worry about and this term here represents what? $1+\alpha$ theta inverse times du/dy^2 okay du/dy^2 is going to be du_0/dy , which is $1+\text{Brinkman number}$ times du_1/dy that is when I substitute this back inside here du_0/dy is $1+\text{Brinkman number}$ times that+higher order terms whole squared=0 okay.

So I am telling you here that this term is of order Brinkman number squared, this term because the leading order term will be Brinkman number squared. So this is going to be drop off okay. So I do not worry about this. What about this term here? This term here gives me $1*$ this will give me a term of order Brinkman number. This*this is going to give me a term of order Brinkman number squared okay.

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Second term on LHS at $O(Br) = 1 \cdot \frac{d^2 \theta}{dy^2}$

Third Term on LHS $\rightarrow (1 - \alpha \theta) \left[1 + Br \left(\frac{du}{dy} \right) \right]^2$

$$= (1 - \alpha Br \theta + \dots) \left(1 + Br \left(\frac{du}{dy} \right) \right)^2$$

$$= (1 - \alpha Br \theta + \dots) \left(1 + 2 Br \left(\frac{du}{dy} \right) + Br^2 \left(\frac{du}{dy} \right)^2 \right)$$

$$= 1 + 2 Br \left(\frac{du}{dy} \right) - \alpha Br \theta + Br^2 \left(\frac{du}{dy} \right)^2 + \dots$$

So to order Brinkman number what I have is so the second term on the left hand side at order of Brinkman number is $1 \cdot d^2 \theta / dy^2$ okay. Because this term is order Brinkman number squared because Brinkman number * Brinkman number and what about the term on the right hand side? This I am going to do a binomial series expansion, so it is not on the right hand side, it is also on the left hand side. So the third term on the LHS.

Yeah, the third term on the LHS is doing a binomial series expansion like $1 - \alpha \theta$ okay etc * $1 + \text{Brinkman number} \times du/dy$ the whole squared. The whole thing is squared right and now you substitute this back inside this is $\theta = \theta_0 + \text{Brinkman number} \times \theta_1 + \text{higher order terms}$ squared okay. So the point is yeah what is the point here? This term here gives me $1 +$.

I keep making this mistake here right, this is Brinkman number times $\theta_1 +$ the higher order terms we neglect and this gives me the du/dy , which is $1 + \text{Brinkman number}$ this whole squared yeah let us keep going. So I get $1 - \alpha \text{Brinkman number} \times \theta_1$ times $1 + 2$ times $Br \times du/dy$ the whole squared + okay. So what am I left with? I have actually got a problem.

So you think I made a mistake here? Because this should simplify to 1 times Brinkman number. So these guys will not contribute, only this I have to worry about okay. Let us go ahead. I get $1 + 2$ times du/dy times Brinkman number okay and $-\alpha$ times Brinkman number times θ_1 . So clearly things are not as good as they should have been because I made a mistake somewhere here.

Actually, I could have taken the simplification in the algebra, I could have taken this in the denominator, I would have got $1 + \alpha \theta$ here and I would have got $1 + \alpha \theta$ here and then things are fine. So what is this $1 + \alpha \theta$? Yeah, so that is where you think I made a mistake? Because this should actually simplify to 1, this should simplify to Brinkman number.

And then I have the second term should be at order Brinkman number this should give me 1 okay. Now clearly I am not getting 1 here and I am unable to find my mistake. Let us do it like this. Yeah **“Professor - student conversation starts.”** This is the order 0 yeah, yeah the order 0 term is not a problem because I have not taken that in account but this is let us do like this. **“Professor - student conversation ends.”**

I am going to redo this thing, yeah, no problem. So we come back to this equation and we start here okay.

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The image shows a green chalkboard with several handwritten mathematical expressions. The top line is $(1 + \alpha \theta)(1 + \beta(\theta, Br + \dots)) Br \frac{d^2 \theta}{dy^2} + (1 + Br \frac{d^2 \theta}{dy^2})^2 = 0$. Below it is $(1 + \alpha \theta, Br)(1 + \beta(\theta, Br + \dots)) Br \frac{d^2 \theta}{dy^2} + (1 + 2 Br \frac{d^2 \theta}{dy^2} + Br \frac{d^2 \theta}{dy^2}) = 0$. At the bottom left, there is a simplified equation $\frac{d^2 \theta}{dy^2} + 1 = 0$.

$1 + \beta \theta$ and I am going to take this $1 + \alpha \theta$ to the other side. Yeah, we should get this thing, do not worry, we will figure this out. Hopefully soon. Right, so all I am doing is taking the $1 + \alpha \theta$ reciprocal here and neglecting that because that is going to contribute everything of order Br squared okay and let us see this helps if it does not help then we really do have a problem okay.

So now we come back to this one and say this $1 + \alpha \theta_1$ and this is $1 + \beta$ times θ_1 and θ_1 is $\theta_1 + \theta_1 \frac{d^2 \theta_1}{dy^2}$. So this there is already a Brinkman number here and the only term which will be of order Brinkman number which contributes to this is going to be when I take the 1 and 1 here because everything else is going to give me higher order term okay.

So that definitely makes my life simple. Then, alright I do not seem to have solved my problem because I really have not found my mistake right. So this guy here gives me $\frac{d^2 \theta_1}{dy^2} + 1 = 0$ and if you do it right what you should get is $+1$ here okay. So you guys have to do it right and get $+1$ and so now we move on assuming it was indeed $+1=0$. Let us not spent time on this.

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$$\frac{d^2 \theta_1}{dy^2} + 1 = 0$$

$$\frac{d^2 \theta_1}{dy^2} = -1$$

$$\frac{d \theta_1}{dy} = -y + c_1$$

$$\theta_1 = -\frac{y^2}{2} + c_1 y + c_2, \quad \theta_1 = 0 \text{ at } y=0, 1.$$

$$c_2 = 0.$$

$$\frac{1}{2} = c_1$$

The point is $\frac{d^2 \theta_1}{dy^2} + 1 = 0$ okay if you do it right and so now what I can do is I can solve this for θ_1 and that gives me $\frac{d^2 \theta_1}{dy^2} = -1$ and you can integrate this out, you get $\frac{d \theta_1}{dy} = -y + c_1$ okay. θ_1 is again integrated out one more time. We get $-\frac{y^2}{2} + c_1 y + c_2$ okay. The boundary conditions on θ_1 are the θ_1 has to be 0 at the 2 walls okay.

So every term in my perturbation expansion has to satisfy this. So boundary conditions are $\theta_1 = 0$ at $y=0$ and 1 okay. I use that at $y=0$, I should get θ_1 is 0 so I get $c_2 = 0$ okay and I get $\frac{1}{2} = c_1$.

(Refer Slide Time: 20:27)

The image shows three equations written on a green chalkboard:

$$\theta_1 = \frac{1}{2} (y - y^2)$$

$$\frac{d^2}{dy^2} = \alpha \frac{d\theta_1}{dy}$$

$$\frac{d^2}{dy^2} = \alpha \left(\frac{1}{2} - y \right)$$

So this implies that theta 1 is 1/2 of y-y squared that is theta 1. Now you go back to the equation, which we have derived yesterday which actually a related u1 to theta 1 okay and you can actually solve this equation for u1. Yesterday from your momentum balance into the first order, we derived an equation which states something like d squared u1/dy squared and you have to tell me what we got.

We got this equals alpha times d theta 1/dy is it? Can you check? Is a + or -? This is + okay. So now I already found out what theta 1 is okay. I can substitute this here and I can solve this equation for u1. So this way I am getting my correction for u1 for the velocity and the first order. So substituting the theta 1 which I have got here and that is what we want to do. Remember solve a bunch of equations sequentially is alpha*d theta 1/dy which is 1/2-2y/2, which is y okay. So again you can do the same thing, integrate this out and find u1.

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$$\frac{d^2 \theta_1}{dy^2} + 1 = 0$$

$$\frac{d^2 \theta_1}{dy^2} = -1$$

$$\frac{d\theta_1}{dy} = -y + c_1$$

$$\theta_1 = -\frac{y^2}{2} + c_1 y + c_2, \quad \theta_1 = 0 \text{ at } y=0, 1.$$

$$c_2 = 0.$$

$$\frac{1}{2} = c_1$$

So the point is $d^2 \theta_1 / dy^2 + 1 = 0$ okay if you do it right and so now what I can do is I can solve this for θ_1 and that gives me $d^2 \theta_1 / dy^2 = -1$ and you can integrate this out, you get $d\theta_1 / dy = -y + c_1$ okay. θ_1 is again integrated out one more time, we get $-y^2/2 + c_1 y + c_2$ okay. The boundary conditions on θ_1 are the θ_1 has to be 0 at the 2 walls okay.

So every term in my perturbation expansion has to satisfy this. So boundary conditions are $\theta_1 = 0$ at $y=0$ and 1 okay. I use that at $y=0$ I should get $\theta_1 = 0$. So I get $c_2 = 0$ okay and I get $1/2 = c_1$.

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$$\theta_1 = \frac{1}{2} (y - y^2)$$

$$\frac{d^2 \theta_2}{dy^2} = \alpha \frac{d\theta_1}{dy}$$

$$\frac{d^2 \theta_2}{dy^2} = \alpha \left(-\frac{1}{2} - c_1 \right)$$

$$c_1 = \alpha \left(-\frac{1}{2} - c_1 \right) + c_2$$

So this implies that θ_1 is $1/2$ of $y - y^2$ that is θ_1 . Now you go back to the equation, which we have derived yesterday which actually a related u_1 to θ_1 okay and

you can actually solve this equation for u_1 . Yesterday from the momentum balance to the first order, we derived an equation which state something like $d^2 u_1 / dy^2$ and you have to tell me what we got.

We got this equals α times $d \theta_1 / dy$ is it? Can you check? Is the + or -? This is + okay. So now I already found out what θ_1 is okay. I can substitute this here and I can solve this equation for u_1 . So this way I am getting my correction for u_1 for the velocity and the first order. So substituting the θ_1 which has got here and what we want to do? Remember solve a bunch of equations sequentially is $\alpha * d \theta_1 / dy$ which is $1/2 - 2y/2$ which is y okay.

So again you can do the same thing, integrate this out and find u_1 . So $du_1 / dy = \alpha$ times $y/2 - y^2/2 + a$ constant c_1 and you can integrate this one more time to get $u_1 = \alpha * \int (y/2 - y^2/2 + c_1) dy = \alpha (y^2/4 - y^3/6 + c_1 y + c_2)$ okay. So I think the chances of me making mistakes here is very minimal.

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The image shows a green chalkboard with handwritten mathematical work. At the top, it states boundary conditions: $B.C's, \text{ at } y=0, 1, u_1=0$ and $c_2=0$. Below this, it shows the equation $0 = \alpha \left(\frac{1}{4} - \frac{1}{6} \right) + c_1$. This is followed by the derivation of $c_1 = \alpha \left(\frac{1}{6} - \frac{1}{4} \right) = -\frac{\alpha}{12}$. Finally, the velocity profile is given as $u_1 = \alpha \left(\frac{y^2}{4} - \frac{y^3}{6} - \frac{y}{12} \right)$.

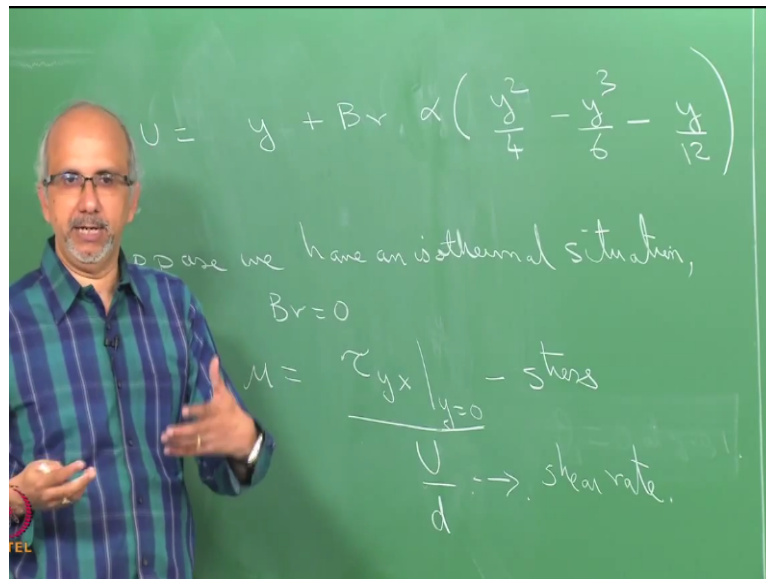
But you never know, the point I want to make here now is using the boundary conditions, what are the boundary conditions at $y=0$ and 1 u_1 is 0 , the no-slip boundary conditions okay. So this implies what? c_2 is 0 and what is c_1 $0 = \alpha * 1/4 - 1/6 + c_1$, which gives me c_1 is α times $1/6 - 1/4$, which is α times $-\alpha/12$ okay. So I have got u_1 as α times $y^2/4 - y^3/6 - y/12$ okay.

So this is my first order correction to the velocity profile u_1 and you already know the temperature equation. So basically what I am trying to tell you here is when you have viscous heating, there is going to be a modification in the linear velocity profile. Without the viscous heating when the thing is isothermal, your velocity profile is linear.

If you have actually a temperature dependency, heat being generated, viscosity being a function of temperature and let us say it does not vary too much a small temperature rise and you will have a difference in the velocity profile thus being caused by the variation of the viscosity with temperature okay. The variation of the viscosity with temperature makes the velocity deviate from the straight line okay.

So that is the idea, so what is the deviation that you get? What is the actual velocity profile? So what you have done is you have calculated the temperature first and you are substituting it and getting the velocity profile.

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So actual velocity is going to be U_0 , which is nothing but $y + \text{Brinkman number times } u_1$ and u_1 is what we have just calculated, which is $\alpha * y^2 / 4 - y^3 / 6$. I should have written this in a slightly better way, but then in increasing powers or something like that but let it be as it is. So that is the correction, which I have to the velocity profile. So remember if Brinkman number is 0, I get the Couette flow linear profile.

If α is 0 which means my viscosity is constant, viscosity does not change with temperature, then again there is no correction to the profile. So basically the modification of

the velocity profile is occurring only because of the Brinkman number or the alpha, if one of them is 0, you will have no change okay. So you may ask so what of course if you are interested we can go ahead and do higher order terms and get better corrections right.

Now the question is the objective of any viscometric study is we are trying to estimate the viscosity of the liquid okay and if you remember suppose we have an isothermal situation means Brinkman number is 0, no temperature rises and how do you estimate the viscosity? μ is τ_{yx} , the plane was x right, so τ_{yx} , yeah τ_{yx} is not it? y is in this direction, τ_{yx} at $y=0$ /the shear rate.

The shear rate is du/dy and what you know is you know the bottom plate is at velocity 0, upper plate is velocity capital U and U/d because it is a liner profile, you are going to assume that this is just U/d okay. This is the shear rate, so this is the stress at the wall and this is the shear rate okay. So this would be the actual viscosity, which you are going to measure because viscosity is not changing.

Now supposing it is actually a non-isothermal situation, what we are going to do is you can do 2 things, one is you can assume that there is still a linear velocity profile inside that is you will just know what that the velocity is at the bottom, you know what the velocity is at the top and assuming that the variation is linear, of course that is wrong, but what you are going to get is something like an apparent viscosity, something like an effective viscosity okay.

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$$\mu_{app} = \frac{\tau_{yx}|_{y=0}}{\frac{U}{d}}$$
 This is an apparent viscosity, since we are not correcting for the deviation from the linear profile.

The actual shear rate at $y=0$,

$$\frac{\partial u}{\partial y}|_{y=0} = \left(1 - \frac{\alpha Br}{12}\right)$$

NPTEL

So basically what I am saying is if we have a temperature rise then the effective viscosity or the apparent viscosity is going to be given by τ_{yx} at $y=0/U/d$, μ is an effective viscosity because I am assuming that the profile is linear, but it is actually not linear. I am not correcting for the nonlinear velocity profile here okay. So this is viscosity since we are not correcting for the deviation from the linear profile okay.

So in order for you to get the actual viscosity what would you have to do? You have to find out the shear stress at the wall and you have to find the shear rate also at the wall but to find the shear rate at the wall you have to include the correction terms as well okay. So when you actually use the entire velocity field and get the shear rate, then the viscosity that you are going to be calculating is the one which is the actual viscosity, not the effective viscosity okay.

So the reason I am calling this apparent is because I have assumed it is still linear and that is wrong okay, actually it is varying. So what about the actual shear rate at $y=0$ that is du/dy at $y=0$ that is going to be $1 + \text{sorry at } y=0 \text{ right only this guy is going to contribute } 1 - \alpha \text{ Brinkman number}/12$ because I am evaluating this at $y=0$ because that is what I want to find the viscosity right.

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$$M = \frac{\tau_{yx}|_{y=0}}{\frac{U}{d} \left(1 - \frac{\alpha Br}{12}\right)}$$

From these two expressions.

$$M_{app} = M \left(1 - \frac{\alpha Br}{12}\right)$$

actual viscosity where the actual shear rate at $y=0$ is used

So now the actual viscosity is going to be τ_{yx} at $y=0$ /the actual shear rate, remember this is the dimensionless shear rate, I want to convert it to dimensional, which means I am going to multiply this by the characteristic velocity and the characteristic length, I get my U/d again

okay. It is $U/d * (1 - \alpha Br/12)$. So this is my actual viscosity that I have. I want to basically relate the mu apparent to the actual mu okay.

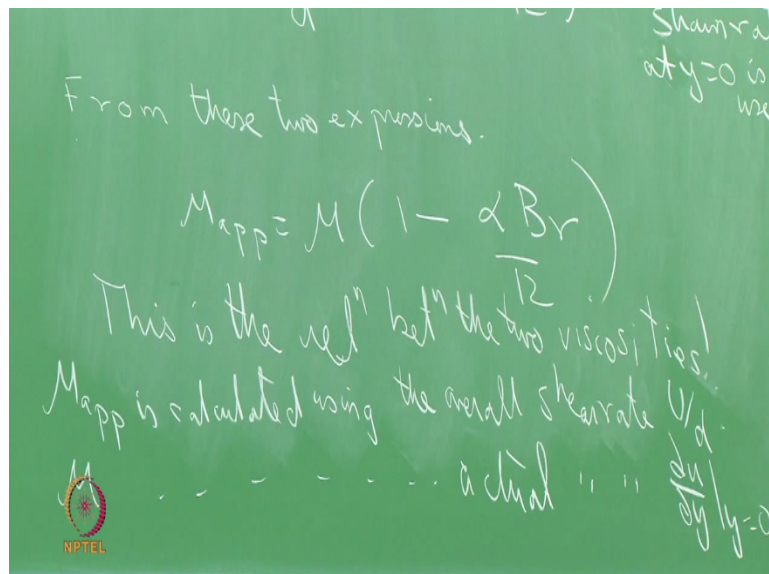
And what I am going to do is I am just going to eliminate the τ_{yx} the shear stress between these 2 equations and I am going to find that mu, so from these 2 expressions I find that mu apparent is going to be $\mu * (1 - \alpha Brinkman\ number/12)$ okay. So that is the correction which you have to do so if you do not want to worry about, see as an engineer you do not want to worry about what the axial velocity profile is.

You know my top rate velocity, you know your bottom rate velocity, so what we would do is we would just use that to find the shear rate and the instrument is going to give you the shear stress, you divide to get the apparent viscosity. So once you know the apparent viscosity and if you know the properties of a fluid, you know your alpha and you know your Brinkman number, you can use this to get the actual viscosity okay.

So this is the correction which you have to do if your fluid has a viscosity, has a thermal conductivity, which depends on temperature. So the point I am trying to make here is that this is the actual viscosity where the actual strain rate at the lower wall at $y=0$ is used okay. So that is the reason this is actual viscosity which is there and if you do not want to take into account the correction and assume that the velocity profile is still linear, then you get the apparent viscosity.

So that is the 2 different things that we have been doing here.

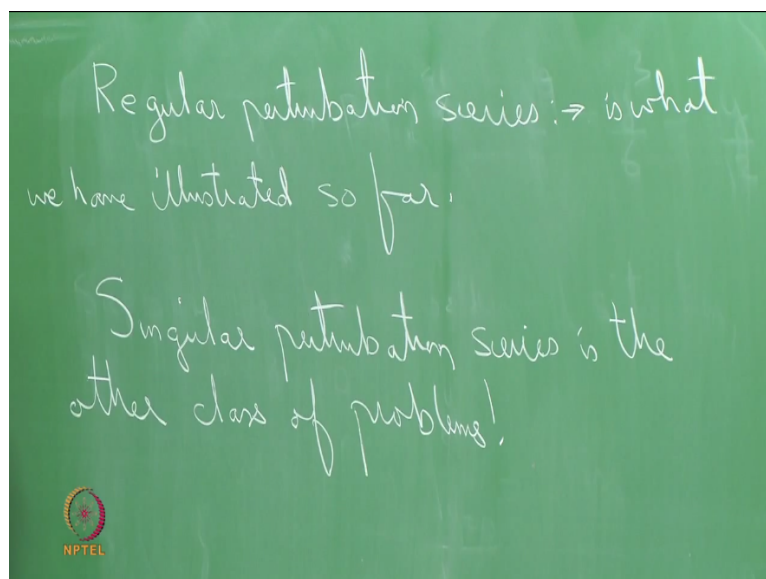
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So this is the correction or this is the relationship between the 2 viscosities okay. The apparent viscosity is calculated using the overall shear rate U/d . I do not remember what dimension I used last time but maybe it was d and μ is calculated using the actual shear rate du/dy at $y=0$. This is basically done in Gary Leal and we can take a look at it, but only thing I realized after the last class was I think I switch the alpha in the beta.

So I have the alpha with the viscosity, he has alpha with the thermal conductivity, so some small adjustment you have to make possibly okay. So what we have done is we have basically seen 2 problems where we just illustrated the idea of doing this perturbation series analysis okay where we can get analytical solutions for nonlinear equations. The first one was the pulsatile flow problem and the second one was problem with the viscous heating.

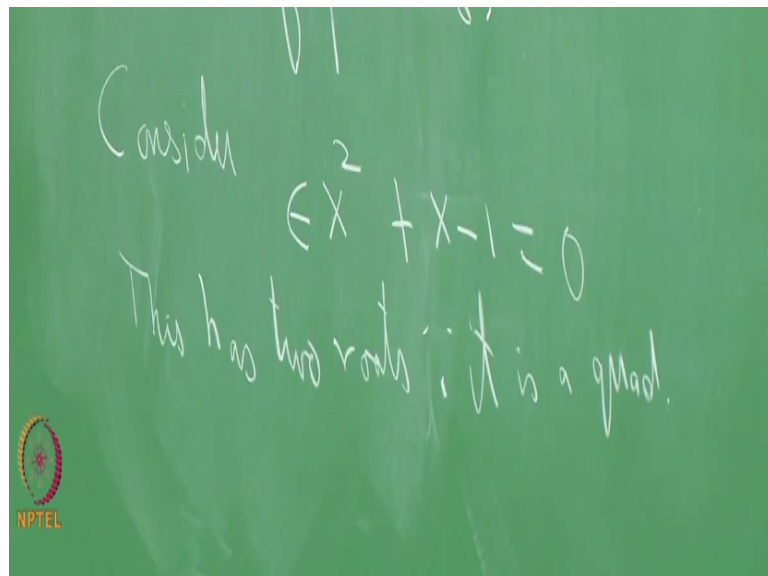
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Now the approach that we have been using is what is called a regular perturbation series is what we have illustrated so far okay. Now there is another class of problems where this regular perturbation series will not work. So I am just going to tell you what the name is first and the other classes what is called a singular perturbation series.

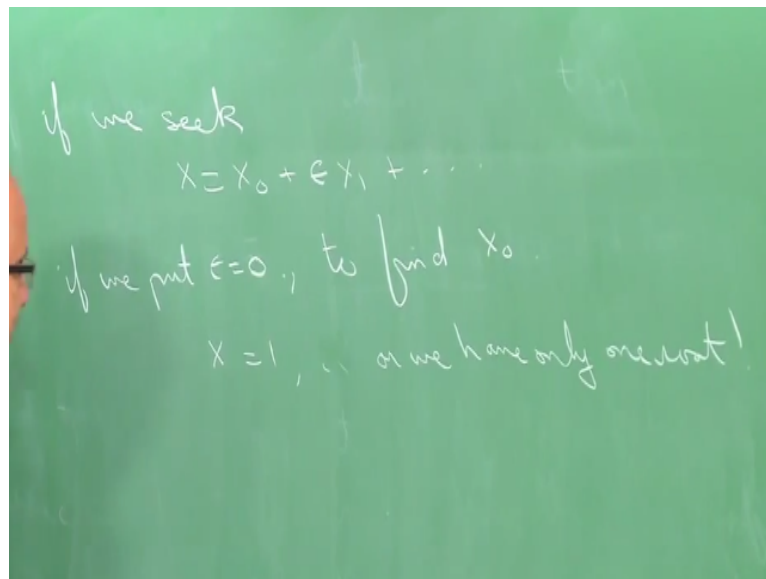
So to give you an idea just again to illustrate this idea in a very simple way, we will do what we always do we start with the polynomial and the simplest polynomial is a quadratic right and everybody understand what a quadratic is. So we have started off with the quadratic to begin with if you remember and then we came to a differential equation. So now we go back to our quadratic again because that is my comfort zone right.

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So consider this quadratic $\epsilon x^2 + x - 1 = 0$ okay. Clearly, because it is a quadratic I need to have 2 roots okay and so this has 2 roots since it is a quadratic and what you would do is you would say oh there is small parameter ϵ so let us do a perturbation series and let us get these roots right. Of course, you know what the roots okay let us get the roots right.

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So if we do a perturbation series, if we seek x like we did last time x_0 ϵx_1 + etc and now if you are going put $\epsilon = 0$ because that is the solution to that is what gives you x_0 okay. If we put $\epsilon = 0$ then what do I get? To find x_0 we have $x = 1$ or we have only one root, so the other root, the information is not even there.

So if you remember we need to have in the last earlier problem when we did we have 2 roots and then we got corrections to each of those roots okay but now what has happened the other root has disappeared because the parameter ϵ is now multiplying by leading order term. If my ϵ had been in one of the other order term my x to the power 1 term or my constant term, I could have proceeded it, even then there are situations when the perturbation series may not work okay.

So the point is let us look at why this problem is arising okay and one way to do that is see I already know the solution to this equation because it is a quadratic, you know the formula for the quadratic equation. So let us use that and see what happens to the 2 roots okay.

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exact solⁿ

$$x_{1,2} = \frac{-1 \pm \sqrt{1 + 4\epsilon}}{2\epsilon}$$

$$\approx \frac{-1 \pm (1 + 2\epsilon + \dots)}{2\epsilon}$$

So the exact solution is $-b$ which is $-1 +$ or $-$ square root of b squared $-4ac/2$ epsilon correct and we will do for small epsilons. We do our famous binomial series expansion and what do we get for small epsilon? This is approximately $-1/2$ epsilon $+$ or $- 1+2$ epsilon+etc okay. Yeah it is $+$, so what do I get? What are my 2 roots?

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The two roots are $1, \frac{-2 - 2\epsilon}{2\epsilon}$
 $1, -\frac{1}{\epsilon} - 1$
 Here as $\epsilon \rightarrow 0, x \rightarrow \infty$
 We cannot do a perturbation about this other root! This implies that we have multiple scales, and we define
 $y = x\epsilon$

My 2 roots are 2 epsilon/ 2 so the two roots are 1 2 epsilon/ 2 epsilon and then have $-$ and $-$ cancels I have -1 oh something is wrong, yeah -2 epsilon/ 2 epsilon correct, take $+$ I have 2 epsilon/ 2 epsilon, yeah, wonderful. So this gives me 1 and $-1/\text{epsilon}-1$. So in the limit of epsilon tending to 0 , this root is going to go to infinity okay and that is what it means is normally what we are doing is we are assuming that the x is going to vary smoothly.

Small changes in epsilon gives you small changes in x , but what we see is this guy is pushing off to infinity as epsilon tends to 0, so x is actually becoming unbounded. So the usual way in which you can actually solve this problem and these kinds of problem is usually arising when you have different scales in the problem okay when you have multiple length scales or multiple time scales in the problem.

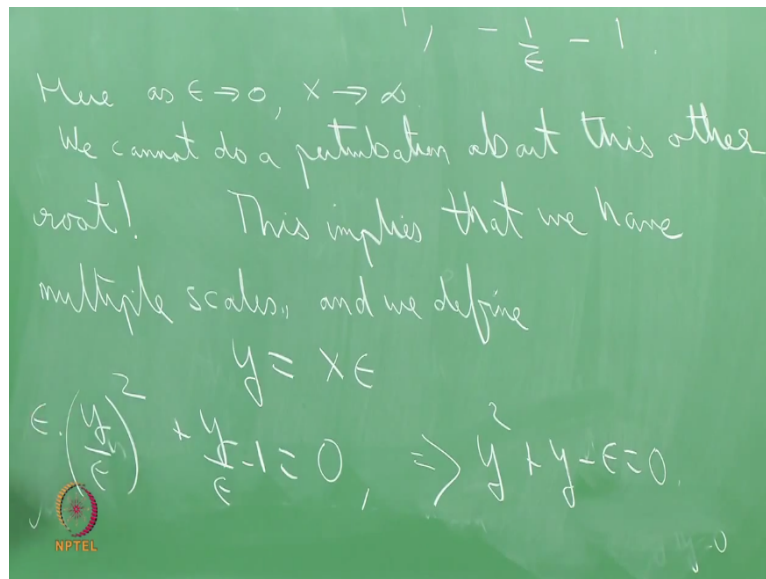
So what we have to do is we have to remember that there are multiple scales and in the limit of epsilon tending to 0 see x is actually tending to infinity you actually go about defining a new variable and then you calculate okay. So this here as epsilon tends to 0, x tends to infinity okay. So we cannot do a perturbation about this other root, it is actually infinity okay. So how do you overcome this problem?

By exploiting the fact that this implies that we have some multiple scales and we define that has a new variable y I want to make sure y does not push off to infinity, I want to make sure y is you know going to be bounded, x is going off to infinity that means x is you can see x is going as $1/\epsilon$ okay. The lower the epsilon is x is pushing off to infinity and there is kind of inversely proportional to epsilon.

So I am going to define y as $x \cdot \epsilon$ so as epsilon tends to 0, x goes to $1/\epsilon$ and so this guy is going to be bounded, finite in the limit of epsilon tending to 0. So this is a new variable which I am defining okay and basically what I am doing is I am scaling actually at the physical problem that means I am defining my length scale in a different way to actually get a bounded solution in that region okay.

Now of course you are looking and this is a mathematical problem but tomorrow when you ask me solving a physical problem, this would correspond to something like a length scale or dimensionless number, which means like we saw earlier, your time scale can be different okay. So now if you want to substitute this in your quadratic equation and you will get a quadratic equation in y .

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So let us do that so x is y/ϵ , so now this gives me $\epsilon \cdot y/\epsilon$ whole squared $+ y/\epsilon - 1 = 0$ okay and when I expand this out I get this implies $y^2 + y - \epsilon = 0$. So this of course has gotten rid of the ϵ from my second power, it is not the coefficient of my quadratic term my y squared term; it is now moved to my constant term. So now I can be bold enough to try and seek a perturbation series solution okay.

And I can get y as an approximate solution. Once I get y , I can scale it back and get x okay. So this is one way in which you can actually solve problems just singular perturbation problems. So singular perturbation problems occur in many areas. For example, if you have viscous flow in a channel or along a wall and you used to boundary layer flows right. In a boundary layer flow what you do?

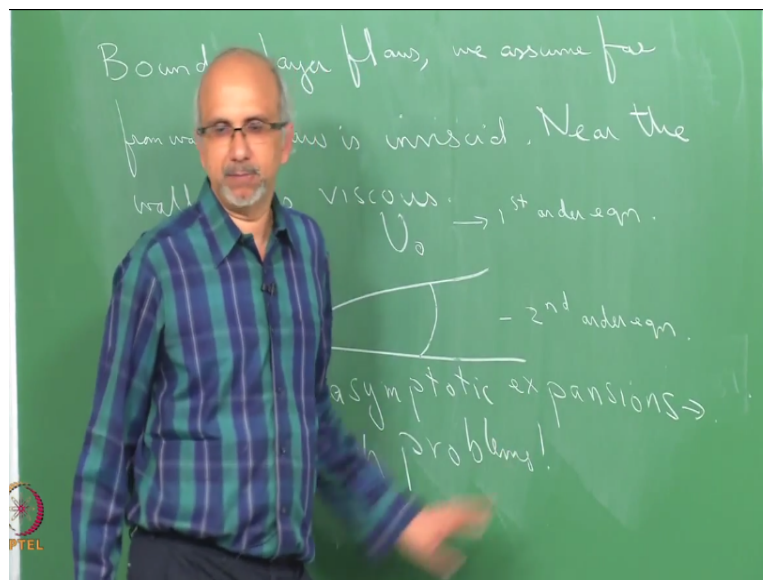
You assume that very close to the boundary layer, we have a small region where the effects of viscosity is important. Far away from the boundary layer, the effect of the viscosity is not important right. Remember in your Navier-Stokes equation, your viscous term occurs as a second order term, it is a differential equation now. So that means far away when the viscous terms are not important, it actually drop the viscosity effect.

Your second order equation has now become a first order equation because you are dropping off all your $\Delta^2 v$ terms okay. So effectively to solve the equation you have to have 2 boundary conditions but now since it is the first order equation, you need to satisfy only one boundary condition.

So the fact that you are unable to satisfy both the boundary conditions tells you that you will not be able to satisfy the boundary condition near the wall where the viscous effects are important. So what we have to do is just like we did here. For low epsilon, we did a rescaling. What you do in boundary layer theory is in near the boundary layer you rescale your variables to find the solution in the boundary layer.

Then you find the solution away from the boundary layer and then you try to match okay. So basically boundary layer flows is another example where singular perturbations are important so I just wanted to mention this because the perturbation series method has some maybe I should not call it limitations, it is not so simple and straightforward to apply all the time. It is not that every time you can seek you know a perturbation series solution and expect it to work, then at times when it will not work okay.

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And one example is this singular perturbation. So I will just say that when you do boundary layers flows okay we assume far from the wall the flow is inviscid, near the wall it is viscous. So if you remember you have this boundary layer over a flat plate, you have U_0 uniform, so effectively there is no velocity gradient far away from the wall but here it is varying parabolically or in some shape, immaturely it matches with this.

So the point is here I would actually have to retain the effect of viscosity, here I have to retain the effect of viscosity and here I have a second order equation and here I have a first order equation and I will not worry about trying to satisfy the no-slip boundary condition. This profile I will worry about satisfying the no-slip boundary condition because this is my second

order equation, I need to have 2 conditions, one is no slip here and $U=U_0$ at the boundary layer.

Here I am just not worry about satisfying this, so it looks like I am allowing it to slip okay. So these are problems, which are actually solved by the method of what is called matched asymptotic expansions help us solve such problems okay and I am just giving you some key words here for you to you know pick up and read those of you who are interested how this matched asymptomatic expansion done.

So once you get the idea you can possibly apply to some problem okay. What we want to do tomorrow is talk about what is called the domain perturbation method okay. So far what we have done is we always talked about the small parameter occurring in the differential equation and then we said how do you go about doing this perturbation series. We need to now worry about the situation when maybe the small parameter is occurring in the boundary condition.

And the boundary is not going to be coinciding with one of your coordinates $y=\text{constant}$, $z=\text{constant}$ and the boundary itself is actually having the small parameter. How do you go about solving that? We need this method to be able to actually solve the stability problems will be worrying about later okay.