# Multiphase Flows: Analytical Solutions and Stability Analysis

**Prof. S. Pushpavanam**
**Department of Chemical Engineering**
**Indian Institute of Technology - Madras**

## Lecture - 16
## Pulsatile flow: Perturbation solution for Rw >> 1

So welcome to the lecture 16 of multiphase flows.

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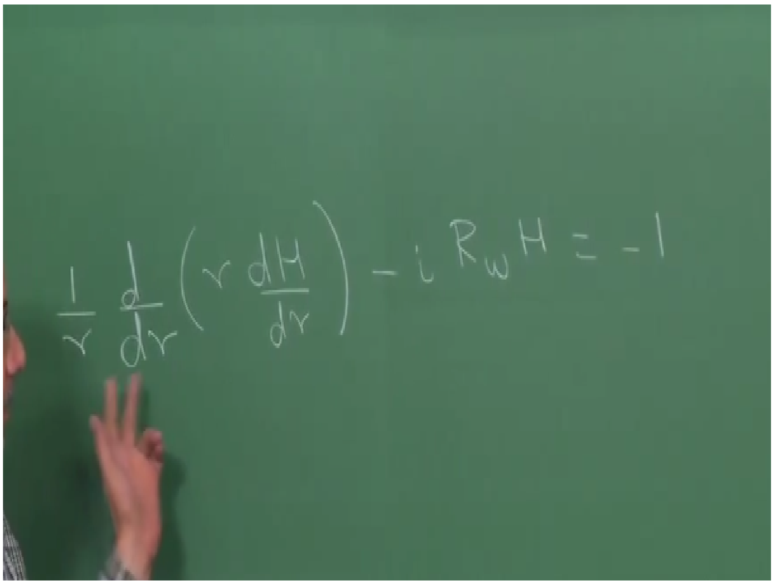


We were looking at this particular equation last class where this equation arises in the context of a pulsatile flow in a circular channel okay.

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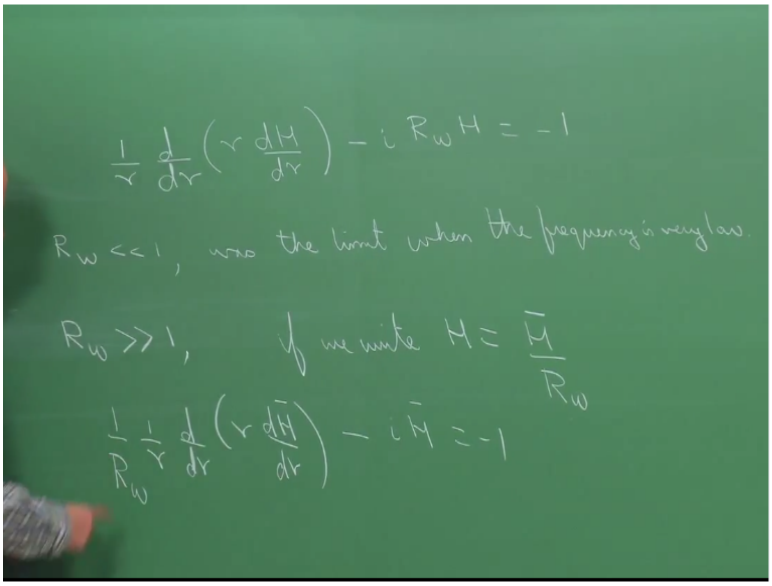


And what we did was we concentrated on the limit of Rw must lower than 1 was the limit when the frequency is very low okay. So now what I want to talk about a little bit before I go on to the next problem is talk about the other limit of Rw being much>>1 because you know you have 2 time scales and we looked at one the limit wherein we said when Rw is very much lower than 1 then the inertial terms are not significant.

What you have is a balance between the viscous forces and the pressure term and we found that things were in phase. The velocity was going to be in phase with the pressure. So now the question is what happens when the frequency is very large when Rw is>>1 the other extreme? And is it possible for us to take a look at the system. So clearly when Rw is very large the one thing which we need to include we need to include the pressure term.

Because the pressure is the one which is going to be driving the flow and remember the pressure term is coming from here okay. This comes from the inertial term the time derivative term and this is called viscous term. So now in this limit we need to retain the pressure term. The inertial term is also going to be important because the inertial term is negligible in the limit of Rw being very much lower than 1.

So when Rw is>>1 this has to be important. So essentially what we want is that we want to be able to simplify this equation so that these 2 terms become dominant and this becomes negligible okay, but the way this equation is written it looks like when Rw is>>1 you do not have balance between the inertial forces and the pressure forces okay. So how can you possibly get a balance between these 2 terms?

One way in which you can get a balance is if we write H as let us say H bar/Rw that is as Rw increases H becomes smaller. Then H Rw becomes H bar okay and this would have magnitude of which is comparable with this whereas as I increase Rw H has to decrease and if it decreases inversely as I have written then this will have a magnitude, which is going to be comparable with this term.

And that I get the balance between the inertial term and the pressure term and what is going to happen when I substitute this here, I get H bar/Rw okay. So then I have 1/Rw here which makes this term very negligible okay. So when I substitute this here what I get is multiplied by 1/Rw I need to put this bar here-i H bar=-1. So now I have a situation what I have done is basically hypothesizing that in the limit of Rw being>>1 H is going to be very small H per se.

And the way I am going to capture that is by seeking H as H bar/Rw okay. Now whether it is inversely with Rw or it is power 2 or something it is going to be decided by the form of the equation. So here this equation and what this means is in the limit of Rw being>>1 this guy is now 0, I can knock off this term the viscous term and now I have a balance between the inertial term and the pressure term okay.

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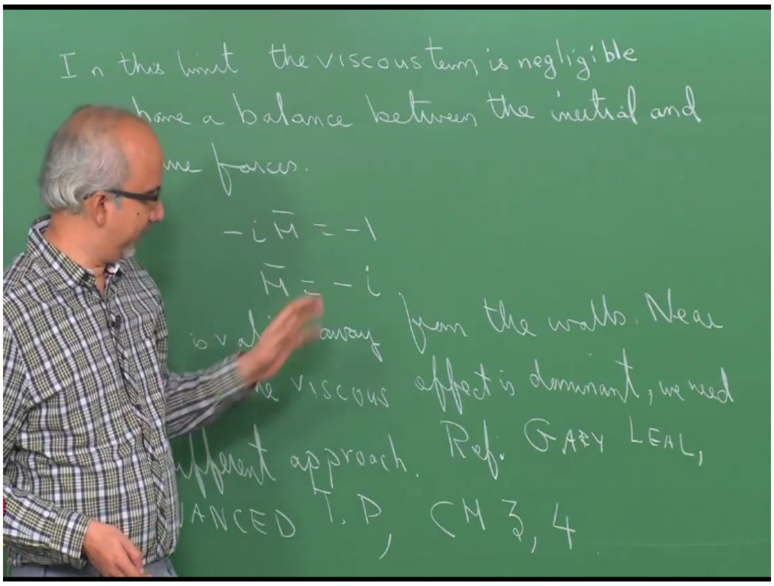


So here in this limit the viscous term is negligible and we have a balance between the inertial and pressure forces okay. So basically what I will do is I will knock off that and I have –i H bar=-1 or H bar=-i okay. So that is my H bar and once I have no H bar at least to some approximation of Rw tending to infinity this is H bar I can go back and calculate H. I can go back and substitute and get back my velocity by taking the imaginary part okay.

The thing which I want to mention here is that we have neglected the viscous terms. Remember this is therefore going to be valid as long as you are away from the walls of the channel. Near the wall of the channel, this business of neglecting the viscous term is not a good idea physically because you have the no-slip boundary condition. The liquid is going to at rest at the wall and immediately next to it you have the liquid moving.

So you are going to have velocity gradients, viscosity is going to be in effect. So point I am trying to make here is that this is valid away from the walls, near the walls the viscous effect is dominant and we need a different approach okay. So basically this particular problem of this pulsatile flow in a circular channel is discussed in Gary Leal's book on Advanced Transport Phenomena.

I have pretty much try to stick to the notation, which he has used okay so that when you guys go and read the book, it will be easy for you to follow okay. So I am suggesting that you go and look at reference to this problem is Gary Leal, Advanced Transport Phenomena okay and I believe you need to look at chapters 3 towards the end and chapter 4 towards the beginning. So chapter 3 basically talks about the complete solution, which we spoke about.

And chapter 4 talks about the asymptotic solution okay, which we have discussed and here he will also talk about how to take into account this viscous effect and where this is. So if time permits towards the end, we may come and revisit this problem of how to do matched asymptotic expansions. Right now, I just want to talk about this perturbation theory and then move on okay.

So this is one problem which you solved. The next problem that I am going to talk about is also a workout problem in Gary Leal okay. So again that is on chapter 4.

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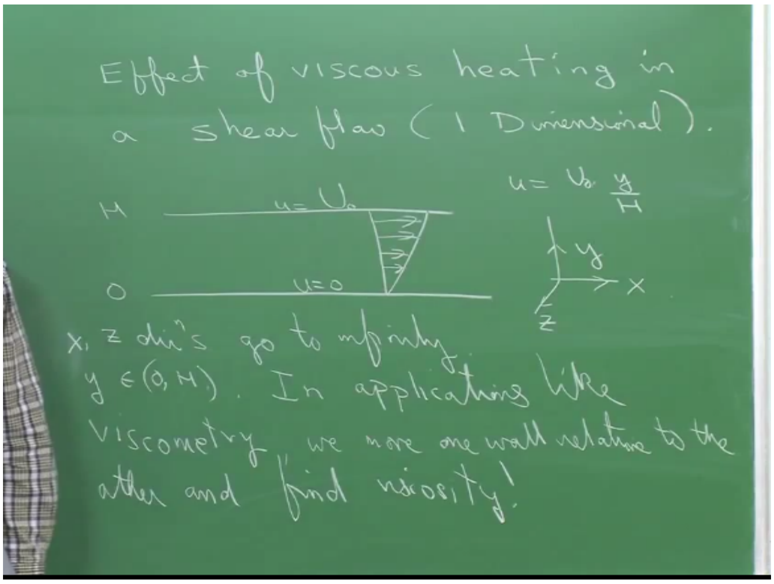


And this problem has to do with taking into account the effect of viscous heating in shear flow, which is let us say 1 dimensional. So our objective is just to illustrate the idea. So we are going to keep the math as simple as possible okay. So we just restricting our analysis focus to 1 dimensional problem and the applications of this are in basically in viscometry okay.

So for example how do you measure viscosity of a fluid, you put it in a viscometer, then you have a wall at the bottom, which is stationary and usually you have a cylindrical surface of the top, which is actually rotating and then you know the shear rate, you measure the shear stress and you find the viscosity of the fluid right. So there are many applications where you have one wall which is stationary, another wall which is moving and there is a liquid below.

So here the motion is going to be induced by the motion of the one of the walls. So one of the walls moves and that induces the motion in the liquid, so rather than worry right now about the circular geometry what I am going to do is worry about the simplified things at least geometrically, which is just talk about 2 flat parallel plates, which are infinitely long okay. So keep life simple, which is take the x direction going to infinity.

Keep the y direction going to infinity and have the z direction also going to infinity okay. So y and z directions go to infinity, oops y and z do not go to infinity, x and z go to infinity and y is bounded between 0 and H and in applications like viscometry where we are focusing on determining viscosity of fluids okay, one wall is moving relative to the other wall. So the idea is the same, so I am just using the same idea to try and explain the concept to you.

Tomorrow if you really want to go and do a cylindrical geometry, you should be able to go and do it okay. We move one wall relative to the other and find the viscosity. That is the idea okay. So objective is to find viscosity and then we do this by doing the simple experiment of moving one wall relative to the other. Now so what I am going to do is I am going to keep this lower wall stationary, u is 0 here.

And I am moving it in the x direction and I am going to move this with the velocity capital U okay and just to make sure that there is a difference between the small u and the capital U, I am going to put a subscript 0 there okay. Now you know this particular classical problem what is this called? This is the Couette flow right and you know that if you have a liquid, which is between 2 plates and if the upper plate is moving, you have flow profile velocity which is actually linear varying from 0 to u0 okay.

And what is this profile going to be given by? This profile is going to be given by u=u0*y/H okay and y=0 it is 0 and y=H it becomes u0 and it satisfies your momentum equation. There is no externally imposed pressure gradient. What we want to do now is supposing your fluid happens to be very viscous, you are trying to measure the viscosity of oil let say okay.

And it is very viscous, viscosity being something like friction is going to end up generating a lot of heat okay. So when you are doing this movement, there is going to be heat which is generated and this heat is going to be generated in the form of viscous dissipation term. Normally, what we have done in the past is we would end up neglecting this viscous dissipation term.

As a result of this viscous dissipation term, heat is generated and although the temperature of the lower wall and the temperature of the upper wall would be constant let say at room temperature because of the heat generation there is going to be a temperature profile which is going to be induced inside okay. Now if your properties, your viscosity and your thermal conductivity were actually dependent on temperature okay.

Then what you need to do is you need to include this effect, you will say I am not worried about the dependence of the viscosity on temperature because the actual temperature prevailing inside your viscometer is going to be different from what you think it is because and therefore what you would be measuring is possibly an inaccurate value of viscosity okay.

So the point I am trying to make here is there can be liquids when you can have a significant amount of heat generation. If there is significant amount of heat generation, temperature will go into rise. If the temperature rises, you would actually be measuring some kind of an apparent viscosity or an average viscosity over the temperature range okay because viscosity is a function of temperature; thermal conductivity is a function of temperature okay.

So what we want to do is see how we can get an estimate of the velocity field and the temperature field by in this problem where viscous dissipation is the cause of the heat generation okay. So that is the idea and clearly you can already see what I am going to do. I am going in the limit of the viscous dissipation not existing. I know the solution; I know this linear profile.

Temperature is going to be uniform and I know the velocity is linear. So what I am going to do is include viscous dissipation and treat the dimensionless parameter, which is going to be associated with it as my small parameter and do a perturbation series to find how the actual velocity profile and temperature profile is. So that is the strategy that is the plan okay. So now let us go back and write some equations and proceed.

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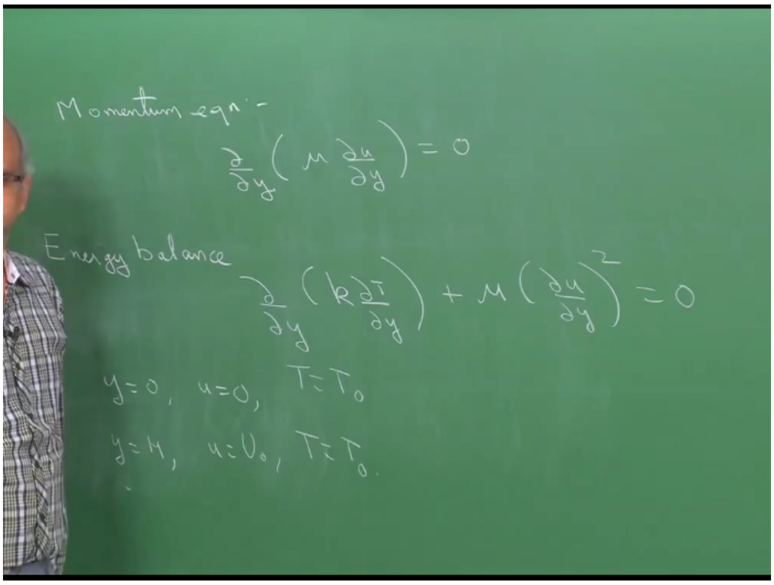


So the momentum equation, look I am going keep things 1-dimensional, remember the actual momentum equation will have d/dy of mu du/dy=0 okay. I am keeping the viscosity inside here because normally what we are used to is saying viscosity is the constant when you take it outside the derivative but in general remember what I am interested in is viscosity can be a function of y implicitly because viscosity is the function of temperature and temperature can be a function of y and therefore viscosity comes inside.

For my energy balance, my differential equation is going to be d/dy of k dT/dy okay+mu times du/dy the whole squared=0. So this is my thermal conductivity and this is my viscous dissipation term. Remember this viscous dissipation term is a source of heat, therefore the square is always positive, is always going to be generating heat okay and whenever you are going to have flow there is going to be.

Only thing is depending upon the value of the viscosity, this can be small or high. Now I need to have some boundary conditions for both velocity and this and you already know the boundary conditions at y=0, u is 0 and temperature=t0 and y=H, u=u0 and t=t0. So since we want to talk in terms of doing a perturbation analysis, what I want to do is rather in talking terms of the actual magnitudes of k and mu, we need to make things dimensionless.

And then talk in terms of the relative role of time scales etc. So you know when you make things dimensionless, you get dimensionless numbers, which tells you the magnitude of inertial over viscous or this time scale over the other time scale. So it is relative magnitude is what we interested in, so next thing to do is to make this dimensionless right.

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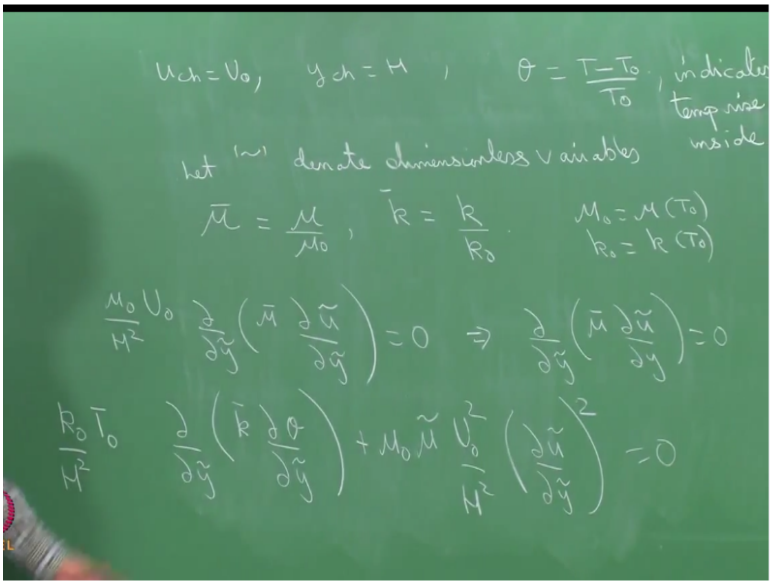


So I am going to make this dimensionless by choosing my u characteristic as U0 because that is going to decide my velocity, y characteristic is clearly H okay and rather than talk in terms of T0, I am going to define temperature scale will be t0, but I want to do it in terms of getting my boundary conditions homogenous. So what I am going to do is I am going to define theta as T-T0/T0.

Clearly, T0 is my reference temperature, whatever is the temperature of the walls okay but rather than talking about the absolute temperature, I want to talk about the thing in a relative way. So I am just going to define this as T-T0/T0, this will basically give me an idea of how much does the temperature rise inside my viscometer okay? This indicates the temperature rise inside okay.

So let us make this equation dimensionless. What do I get? Let tilde denote dimensionless variables and clearly you would get okay then I am also going to define mu tilde or mu average as big mu/mu 0 and k=k/k0. What is mu 0? Mu 0 is nothing but the viscosity evaluated at T0, k0 is the thermal conductivity evaluated at T0 okay. So mu 0 is mu of T0, k0 is k of T0 okay.

Now I am going to substitute all this here and I am going to get d/dy tilde mu is written as mu bar times du tilde/dy tilde=0. What I am going to get out here is mu0/H squared times u0 right, mu is mu0 times mu bar, this is U0 times that H squared. So this basically this is a constant and this cannot be 0, therefore this has to be 0 okay. This implies d/dy tilde of mu bar du tilde/dy=0.

We need to make the energy balance dimensionless okay and let us do that, I am going to get k0/H squared times T I am going replace in terms of theta we get a T0 here, d/dy tilde of k bar. So that is basically what I have and this clearly must=0. I am going to make the coefficient of this one, move these terms to the next to the second term and basically what I will get is the dimensionless number.

This dimensionless number is called the Brinkman number okay. So I will just write that final equation maybe here, I am going to write that equation here.

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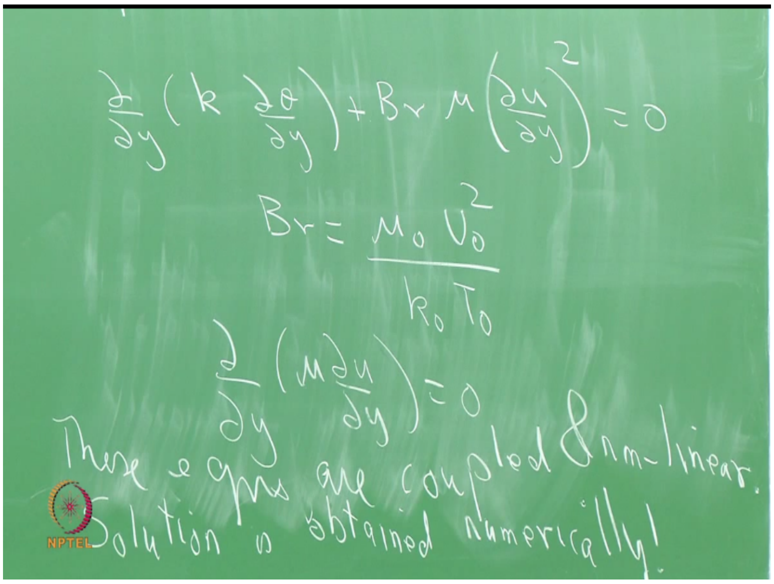


And I am going to draw the tilde's but for convenience okay but remember my equations are dimensionless just to keep life simple. So what do I do? I am going to write this as d/dy of k d theta/dy+mu du/dy the whole squared. This is going to be multiplied by this Brinkman number=0 and let me define what is this Brinkman number is. It is going to be mu0 times U0 squared the H squared does not show up and have a k0 and have a T0 okay.

So what I have done is moved those coefficients there to the second term and this is my dimensionless group okay. So I have a 1 dimensional problem because I just want to keep my life simple and I have this equation, which relates the temperature to the viscous dissipation of the velocity gradient. I also have the momentum equation, which is this one which is d/dy of mu du/dy=0.

I want to tell you a couple of things here that these 2 equations are actually coupled to each other okay. That is how is the coupling? The coupling is occurring because clearly the theta depends upon u and this equation is nonlinear. Mu depends upon theta the temperature, so you need to actually solve both these equations simultaneously. It is not that I can solve this equation for velocity and I can substitute it here and solve for temperature.

The 2 equations have to be solved together. So one way to do this is to do a numerical solution, do a finite difference scheme and you know try to get a solution but the other thing is what we are going to do which is exploit the fact that for small values of Brinkman number we can possibly get an analytical solution.

What motivates me to do that is when Brinkman number is 0 when there is no viscous dissipation, I know the solution. I know that the temperature is uniform, there is no temperature gradient. The velocity is linear, so when the Brinkman number is going to be small I expect only a small deviation from these 2 solutions, some small correction and what is that correction that is what perturbation theory helps me get to okay.

So that is the idea. Remember these 2 equations are coupled and nonlinear and we can do a numerical solution or we can do an approximate analytical solution okay. So the solution is obtained numerically or approximately by using some perturbation method.

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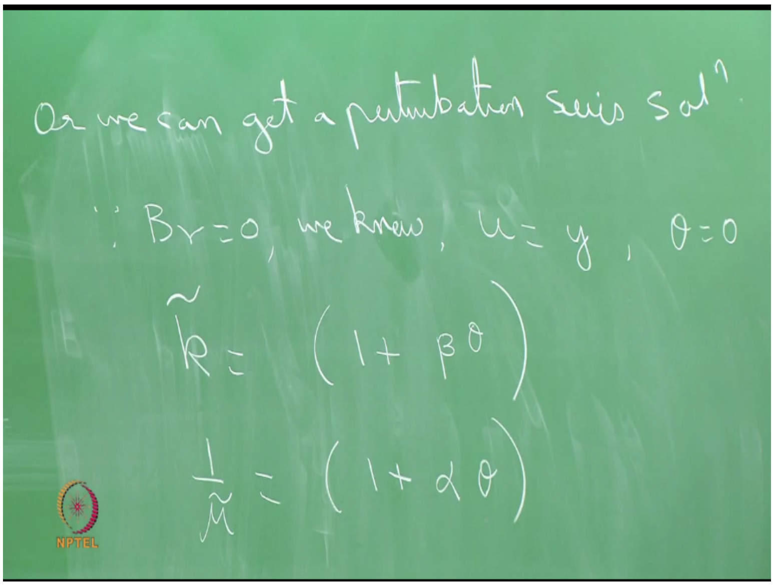


Or we can get a perturbation series solution okay. Why? Because for Brinkman number=0 we know u will be=y when I have scaled everything u will be y is going to be going from 0 to 1 okay and theta will be 0 because temperature will be=T0 everywhere, so in terms of the dimensionless numbers theta will be 0 because T=T0. So Brinkman number=0, this is a solution.

So for small amounts of Brinkman number is going to be different from this and so we need to find out what is going to happen right okay. Couple of things but before I proceed I need to make an assumption of how does the thermal conductivity and how does the viscosity actually change with temperature. So clearly the easiest thing to do is to assume a linear variation depending upon how crazy you are you want to make a higher order approximation maybe second order, third order, but will keep things simple.

I will not be too crazy today then we will just keep k as k0 times 1+just going to think beta theta. So basically the idea is or maybe I do not want k0 here because we already normalize right. So it is just 1+ because this is remembered k/k0 which is my k tilde so this is my k tilde although I will not have my tilde in the dimensionless equation now but this is the linear form okay.

When theta is 0, when T=T0, k tilde is 1 but k tilde is k/k0 remember that. Similarly, I am going to do something here, which is slightly different rather than say the viscosity is linear function I am going to say the reciprocal of the viscosity the linear function because it helps me with the algebra okay. So I am just going to assume that the reciprocal of the viscosity is 1+alpha times theta.

**“Professor - student conversation starts.”** No, I mean you can always say mu tilde as 1/1+alpha theta, which is 1-alpha theta in a binomial expansion. So I mean I think it does not really matter because alpha the only thing which is going to decide is whether it is increasing or decreasing function that is going to be decided by the sign of alpha okay. So the sign of alpha will tell you whether increasing or decreasing.

You can choose mu tilde as 1+alpha theta as well but my point is the alpha star let us say that alpha star will be negative of this alpha. So that is the only thing, so this is only to make my life simple when I saw substituting this back inside the differential equation okay. I mean you could have chosen mu tilde as 1+alpha theta only thing is you have to do some more differentiation and some more algebra okay. **“Professor - student conversation ends.”**

I possibly need a bigger black board. So we all set.

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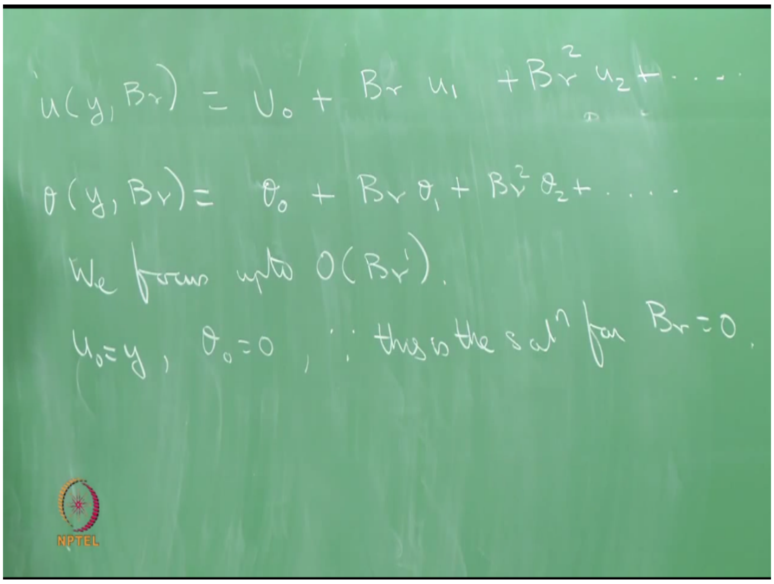


Now clearly u is a function of y and Brinkman number okay and what I am going to do is I am going to seek this as U0+brinkman number times u1 that is my correction. So I mean you can go further as a power series+higher order terms, but we are going to stop only at Brinkman number to the power 1 today okay and as far as theta is concerned that is also a function of y and Brinkman number.

And that is going to be=theta 0+Brinkman number theta 1+Brinkman number squared theta 2 okay. Restrict we focus up to order of Brinkman number to the power 1 that is I am interested in getting only the corrections u1 and theta 1, you already know what is U0 and theta 0 because U0 and theta 0 correspond to Brinkman number=0 okay. So I mean and that should come out when you substitute this inside your differential equation okay.

So U0 is clearly=y and theta 0 is clearly=0 okay. I am going to sense this is the solution for Brinkman number=0 okay. So now we are all set. We will look at the momentum equation.

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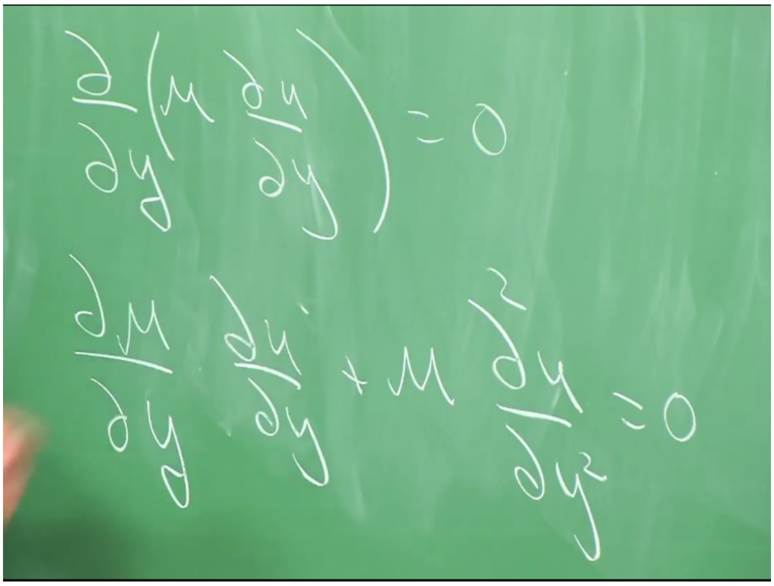

The momentum equation tells me that d/dy of mu du/dy=0 and I am going to say this is therefore d mu/dy, which is nothing but d mu/d theta, sorry let me write it this way, let me do this step one step at a time d mu/dy times du/dy+mu times d squared u/dy squared=0. That is just expanding this out okay. All I have done is just expanded this out and now I am going to write substitute for u in terms of U0 and Brinkman number times u1.

And I am going to write d mu/dy as d mu/d theta times d theta/dy because mu is the function of theta okay and this I am going to leave it as it is and I am going to u is the fact that it is the linear dependency of mu and things like that so let us do that.

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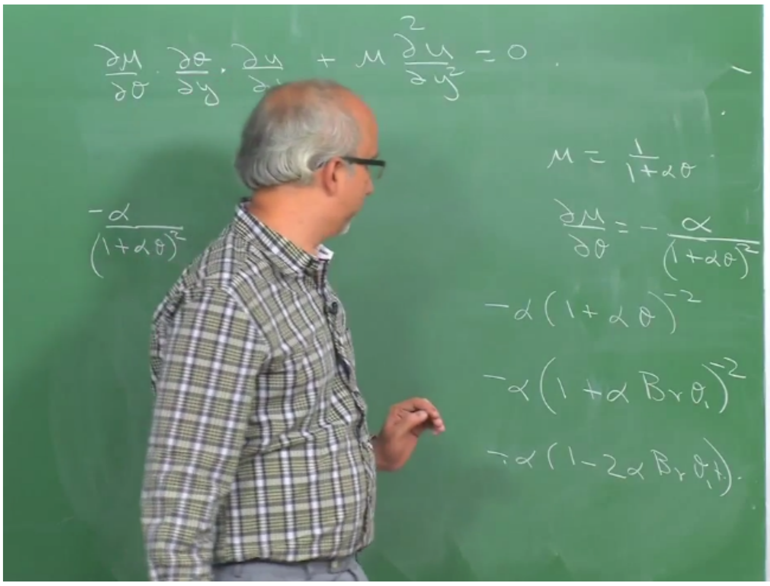

So what is d mu/dy and is d mu/d theta times d theta/dy times du/dy+mu times d squared u/dy squared=0. What is d mu/d theta? Now of course remember this all in dimensionless terms, so somebody has to tell me what is d mu/d theta. So this is the only one place where I have to do this in one place alpha theta, so d mu/d theta turns out to be -1/1+ alpha theta squared-alpha okay. Is this right?

So d mu/d theta is that theta squared okay and maybe I will do something beyond this since I am only interested in powers of Brinkman number to the power 1 what I want to do is I am going to write this using a binomial form -alpha times 1+alpha theta to the power -2 okay and I am going to substitute this as –alpha times 1+alpha times theta. Remember is theta 0+brinkman number times theta 1.

Theta 0 is 0 so I am going to write this as 1+alpha times Brinkman number to the power theta 1 to the power -2 okay. So what I am going to do is if I did a binomial series expansion this is going to be –alpha times 1-2 alpha Brinkman number theta 1+higher order terms, this I am not going to worry about because that is going to involve Brinkman number squared and I am not interested in Brinkman number squared okay.

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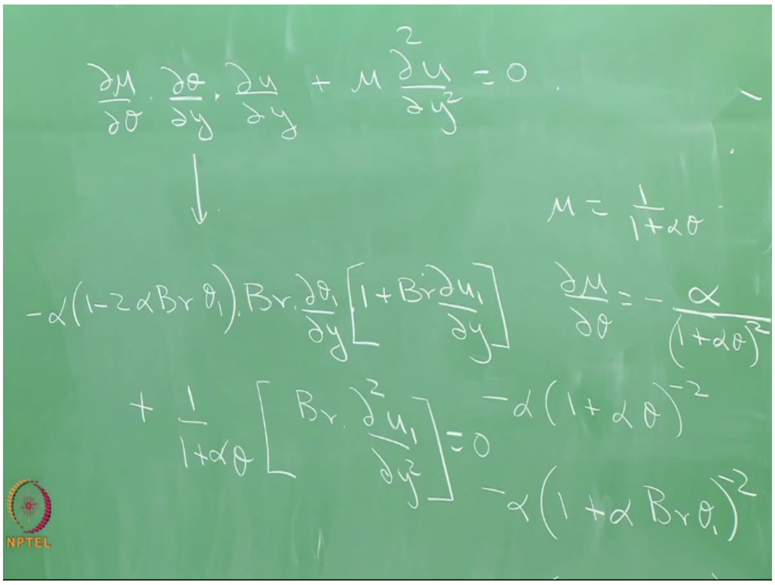


So what I am going to do is I am going to write d mu/d theta as –alpha times 1-2 alpha times Brinkman number*theta 1 okay. What is d theta/dy? d theta/dy is I am going to substitute in terms of theta 0 is going to be Brinkman number times d theta 1/dy okay. That is Brinkman number times d theta 1/dy and what is du/dy? du/dy will have 2 terms, which is the U0 term and the Brinkman number term du0/dy is 1 we know that and+brinkman number times du1/dy.

That is this is this term here+the viscosity term, which is now 1/1+alpha theta times d squared U0/dy squared is 0 because we already know U0 is y and what I am left with is Brinkman number times d squared u1/dy squared=0 okay. Why I am interested is? Taking the terms will have Brinkman number to the power 1 right that is the thing I am interested in. So what are the terms you have Brinkman number to the power 1?

I have this term here and here this I have a Brinkman number here so when I multiply this with this and this I get Brinkman number squared, this with this is going to give me Brinkman number squared. So the only thing I need to worry about is the contribution of this term and this term because there is already a Brinkman number outside okay.

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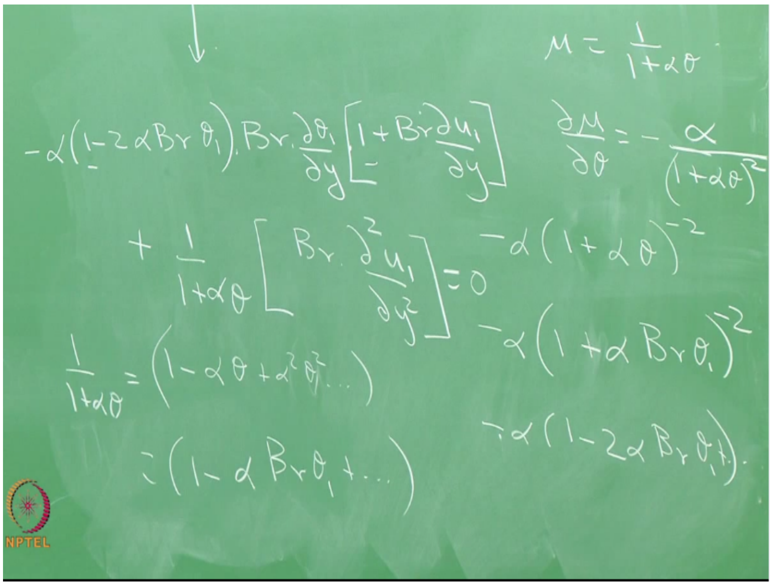

So this is at order Brinkman number to the power 1 what do I have? This gives me –alpha d theta 1/dy+1+alpha theta and I should actually substitute this here again in terms of which is important. I should do okay. Give me a minute, let me write this later. I am going to write 1/1+alpha theta is what? 1-alpha theta+alpha squared theta squared binomial series. Remember theta is theta 0+Brinkman number theta 1.

So this becomes 1-alpha times theta 0 is 0, so I am just going to directly write Brinkman number theta 1 okay. So theta I am just going to substitute in terms of this. Clearly, now when I substitute this here the only term which is going to contribute is going to be the one multiplied by this term because this already has a Brinkman number. So this multiplied by this is Brinkman number squared, so I am not interested in that okay.

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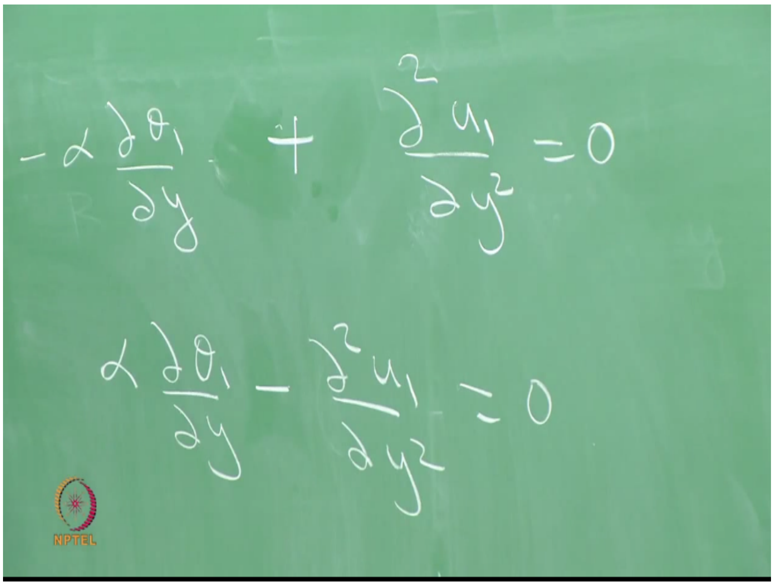

So what we do is get this equation –alpha times d theta 1/dy times right+again there is –alpha d squared u1/dy squared=0. Do you guys think you agree with me on this? **"Professor - student conversation starts."** Yeah, the alpha is going to cancel off in both the terms. Yeah, it is 1 multiplied by I think you are right. There is no alpha here yeah okay. **"Professor - student conversation ends."**

So I believe this is what I am getting dy squared or +this=0. I need to go back to my energy equation. **"Professor - student conversation starts."** Yeah both are plus right, yeah this guy is – and the second term is + yeah you are right. So now this is – okay. That is what we get okay. So now I need to go back to my energy equation and work on that. Yeah, in both the terms, yeah, yeah you could have possibly done it.

I should do yeah okay. See the order 0 has disappeared because I am directly substituting the solution which I have. See I am not trying to find out U0 and theta 0. If I had left the U0 and theta 0 as it is then the Brinkman number to the power 0 would have contributed. I would have gotten U0 and theta 0 okay. That would have corresponded to Brinkman number to the power 0 that is order 1.

Brinkman number will be the first order term. This is what I have got, so what I have done is basically use the final that already know the solution for the isothermal system okay. So what I am getting is the first order. This is not the 0th order; you are suspecting this is 0th order. No, this is not 0th order. I have already substituted the solution okay **"Professor - student conversation ends."**

Yeah because it is just going to minimize my algebra, so that is the reason I wanted to get rid of the theta 0 otherwise I keep it and then at the end I have to throw it out. Since we already knew the solution, I just wanted to use that directly. You know I thought we would be done today, but I guess we would not be done today.

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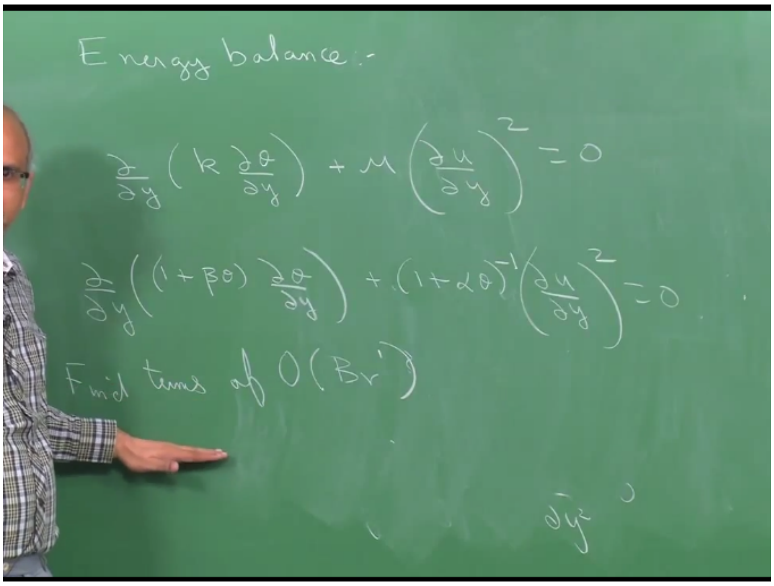


So let us look at the energy balance and this is possibly a good time to stop because it gives me an opportunity to ask you people to do the energy balance and I will do it in the class next time and then we can verify if indeed is right okay. So energy balance is d/dy of k d theta/dy+mu du/dy the whole squared=0 okay and remember this is actually mu bar mu/mu0 and this is k/k0 and what I want to do now is substitute this as I had a alpha there so 1+beta theta*d theta/dy+1+alpha theta*du/dy whole squared=0 okay.

So all I am doing is substituting the assumed dependency on temperature and now I am going to substitute things in terms of my Brinkman number, the expansion that I had and you should be able to equate terms of order Brinkman number to the power 1 that gives you your first order term okay. Brinkman number to the power 0 is implicitly been already built in because I assumed mu 0 as y and theta 0 as 0.

So when you do that you should be able to so find terms of order Brinkman number to the power 1. I think there is a small problem here. What is the problem? mu is not 1+alpha theta, it is 1+alpha theta to the power -1 the way assumed it and when I bring the mu down here I will get 1+alpha theta and that simplifies my calculation, that is the advantage of choosing the reciprocal to be 1+alpha theta.

So you guys were asking me you know saying why is this guy choosing the reciprocal. I am choosing the reciprocal because now mu is the reciprocal of that when I bring it to this side becomes the numerator; it simplifies my algebra little bit okay. Of course, I have to differentiate over there a little bit. So anyway you cannot have everything our way. So I want you to find the solution to this equation to the Brinkman number to the power 1.

In fact, you would find that you will get something for theta 1. You solve for that and then you can substitute this and then get the velocity so you should be able to actually decouple. So you will find out is other 2 equations are getting decoupled okay and then you will be able to find a solution.