Multiphase Flows: Analytical solutions and Stability Analysis
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Lecture - 15
Pulsatile flow: Analytical solution and perturbation solution for Rw<<1

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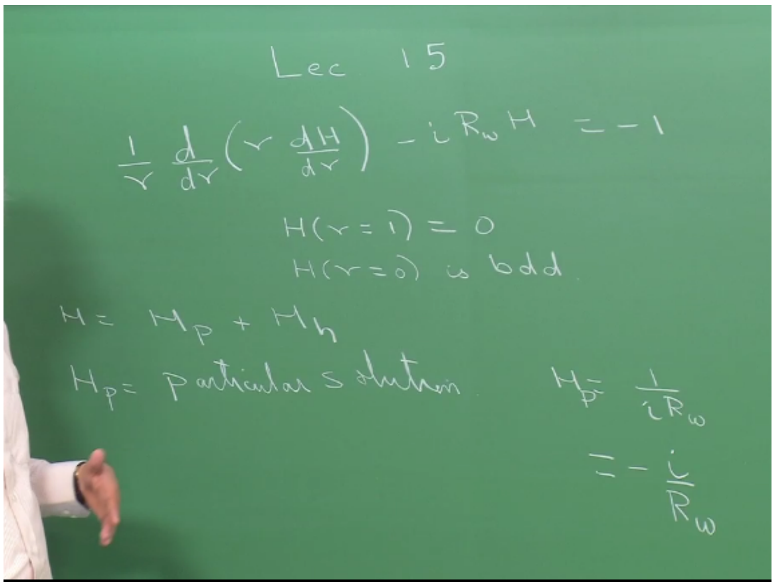

Yeah, welcome to lecture 15 and what we are going to do is continue the solution to the problem we were looking at yesterday, okay. And if you recall this was the differential equation which we had derived, okay. And remember this only represents the radial dependency of the velocity profile when you consider only the periodic part of the pressure gradient, okay. And this is in the limit of T tending to infinity.

So now this is of course subject with a boundary condition at r=1, we have H must be 0 and H at r=0 must is bounded. This equation is a non-homogenous equation because you have this –1 appearing here. And therefore the solution is going to be in the form of a complimentary function and a particular solution, okay. So clearly we have H has two parts H particular and H homogenous, okay.

And now we just have to recall some of those things we did in mathematics when we are trying to solve differential equations without possibly having a physical basis, you are doing it mathematically. Now you see an equation where you have a differential equation and then this has some physical meaning in the sense that it tells you something about a velocity, okay.

So let us go back and I think we will first calculate the particular integral this is the particular solution. And clearly the particular solution is H=1/iR omega because I put H as 1/iR omega, R omega is a constant when differentiated I get 0 and I get –1 and that gives me -1 = -1 everybody is happy, okay. But then I do not like to have this i in the denominator. I want to put it in the numerator.

So I am going to multiply the numerator and the denominator by i and that gives me –i/R omega, okay. So this tells me the particular integral that has a particular solution. Now we need to look at the homogenous version of the thing. And the homogenous version is

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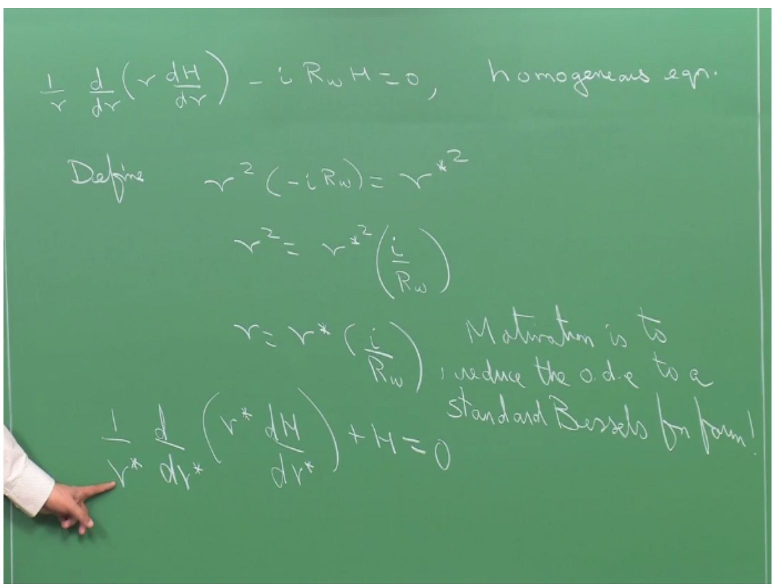

1/r, d/dr of r, dH/dr and this is the homogenous version – I R w H = 0, this is the homogeneous equation corresponding to which I will get my two solutions, my complimentary functions, okay. What I am going to do is I am just going to write, I am going to make a small transformation, r is my independent variable here. I am going to seek the solution in the form of I am going to define r squared multiplied by –i R w = r star squared, okay.

What am I trying to do? I am trying to define a new variable and get this equation in one of these standard forms that you have possibly come across earlier, okay. The idea is when I do this transformation you will get a differential equation whose solution is the classical Bessel's function, okay. I mean and just redefining of the scales. So how did I get this? I basically want a coefficient of +1 here.

If I get a coefficient of +1 here, this is going to collapse through my Bessel's function equation. So in order to get +1 here the denominator scale has R squared so I am just saying R squared multiplied by this is a new variable. So when I do the write the differential equation in terms of R* I would get a +1 that is the objective. So since I do not like this –1 I am going to multiply it throughout by i, I get r squared = r start squared times i/Rw, okay.

So I have just multiplied throughout by i. I get –I square as + so Rw comes here, i gets there. So basically what I am saying is my r, I am going to write it as R star times i/Rw. And r star is a new variable. What is the motivation? See, whenever you do something you need to have a motivation, motivation here is to reduce the ODE to a standard Bessel's function form, okay. And now instead of using r as the independent variable.

Now if I use r star as the independent variable what would I get. I would get 1/r star, d/dr star of r star dH/dr star + H = 0, okay. This gets transformed to this as you can work out the algebra it is not a problem. Now this has my classical solution, which is--

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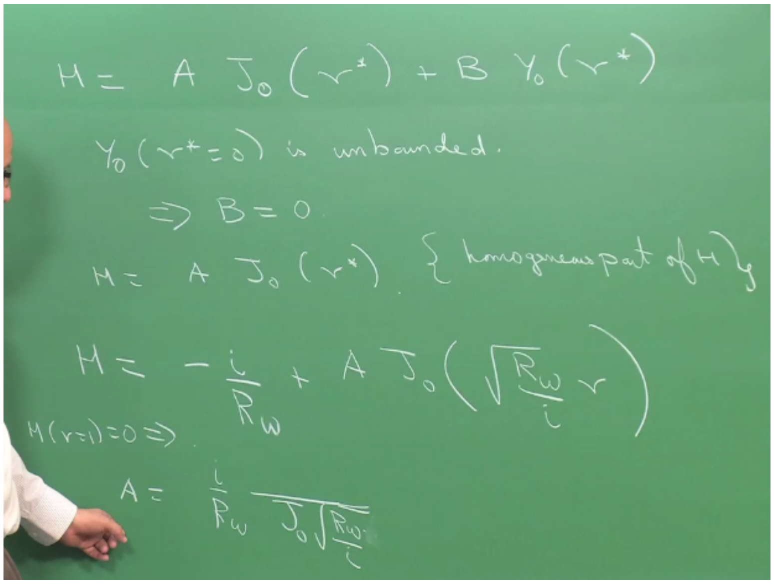

The solutions to H is basically going to be some constant A multiplied by J0 of the Bessel's function of r star + some constant B times the other Bessel's function r star, yeah. "**Professor-student Conversation starts"** Sir, there is an under root there. There is an under root, where…? r=r star and then…Yeah, absolutely correct, absolutely correct, there is an under root there. Yeah, thank you. "**Professor-student conversation ends".**

Now this is your solution for the homogenous part because I just reduced it to the Bessel's function. So I mean, you guys have done Bessel's functions and cylindrical coordinates in your math course. So that basically I am just trying to tell you that the solution is a Bessel's function. See, since you are working in radial coordinates in polar geometry your solution is in the form of Bessel's function.

If you have been working in the form of in a rectangular Cartesian coordinates you would have got a sine and cosine your trigonometric functions, okay. Okay, what we are going to do is we are going to look at this function y0, y0 of at r star=0 is unbounded. And what that means is if you want to retain Y0 that means your solution is going to become infinitely large. And since you have a physical problem you really cannot have an infinitely large solution velocity.

So this implies that the constant B must be 0 because if B is nonzero then your velocity is going to be unbounded at the center point r star=0. And you know that velocity has to be bounded in the middle, okay. So we use this bounded conditions whenever we are actually seeking solutions analytically. Supposing you are actually seeking a solution numerically then you would use something like a derivative = 0 from a symmetry, okay.

But if you are actually getting an analytical solution that I use this bounded argument, okay. So they are kind of equivalent but not exactly. But since when you are doing a numerical code you cannot say there cannot be infinity, right. You have to have some other condition. So what this means is I have H=A times and in fact this is remember the homogenous part, okay J0 of r star. And so what is my actual, this is the homogenous part.

So what is my actual H? My actual H is going to be the particular solution + this solution, which is -i/Rw+A times J0 of r star. I am just going to go back to r now because I know r is going from 0 to 1, okay. And instead of r star I am going to write this as square root of Rw/i times r. Going back to this definition of this because I mean, r is my physical quantity which I know goes from 0 to 1. Okay, our job is to evaluate A now and remember I have not yet used by other boundary condition.

I have already used up one boundary condition, which is the bounded boundary condition and I have got instead of B. I have got to use the other boundary condition which is the fact that H at r=1 is 0 and that implies H at r=1=0 implies A=i/R omega/J0 of square root of R omega/i r, okay. I have just moved it to-- I put i=0, I put r=1 and therefore thus going to be it. Okay, because this is evaluated at r=1 and that gives me what my constant A is.

So I can now substitute the A value back here and get the H that I was actually interested in.

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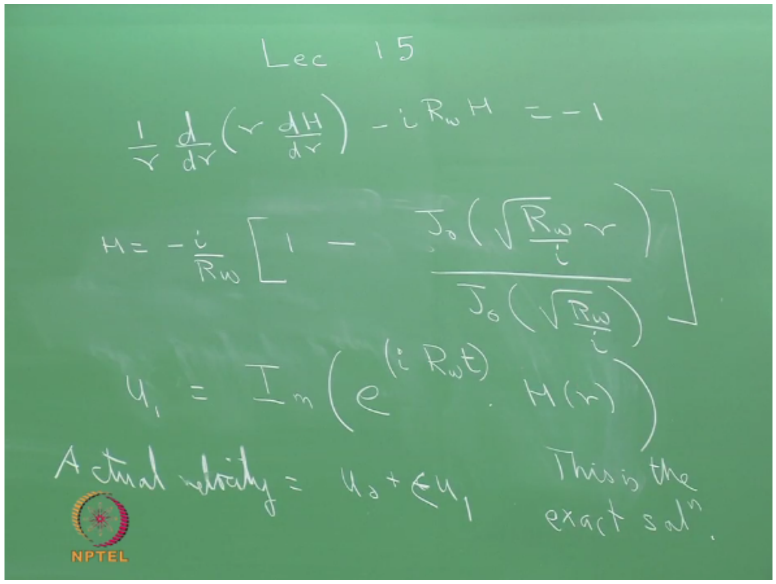

So the H is –i/Rw time 1-J0 of square root of R omega/I r/J0 of square root of R omega/i). Okay, remember this only tells you this is my analytical solution which tells you how the radial dependency is. Now how I-- but what I am interested in is actually my velocity, right that is what we want to find out. This H is only giving me a part of the thing.

How is my actual velocity defined? The u,1 I think that is what I use I am not sure, u1 remember was the imaginary part of the exponential of i times Rwt times H of r. Is not this correct? You need to check your notes and tell me if this is consistent with what I wrote earlier, okay. And so this is basically what you need to do in order to get the velocity is you need, you found the radial dependency.

You have found the exponential dependency on time which we had assumed to begin with. And now I have got H I am going to multiply that by to the i R omega t and I am going to calculate the imaginary part of it and that gives me my velocity. And that gives me only one component of the velocity remembers, because there is the other component with the constant pressure gradient. Basically it gives you the second (()) (14:05).

So I got to add this component of velocity to that and then find my actual velocity. Okay, this is u1 and the actual velocity is I think u0 + epsilon u1, is it, something like this. I am not sure what I used yesterday so I just want to make sure subscripts, okay we will just use this as a subscript just to be consistent, great. Okay, so yesterday – so this is just I am trying to be consistent with the thing.

So the actual velocity that you are going to get there is going to be composed of two parts. We found this as a result of the constant pressure gradient get your second (()) (14:52) parabolic velocity profile. u1 is the imaginary part of this and then you multiply that by ε and you get your solution. So this the analytical form of the solution. Now what you can do is you can possibly go to one of the software packages like MatLab or Mathematica and plot the velocity.

Remember it is a function of Rwt and r, okay. So what you need to do is you need to decide what you are interested in. If you are interested in a specific value of Rw you just fix it. If you are interested in a specific position fix R and plot the velocity as a function of time. And you can do this for different Rs, you can do it for different Rws that gives you your actual velocity.

Now since this is a complicated expression one may want to say like can I get a simplified expression, okay. And I would not call it a simplified expression, an approximate expression. And one way to do this approximation is to invoke the fact that Rw can be either very large or very small, okay. So now let us look at like we did yesterday look at the limit where Rw is very small, okay.

And now so this is the total solution, this is the exact solution if you get u0 from yesterday and U1 from today and put it together. But so yeah but what I have done is – no this is r=1. It is a function of r. **“Professor-student conversation starts”** Yeah. J0 a function or r star. Yeah, but what I have done is. (()) (16:44) No, this is the r=1. So it is a function? It is a function or r yeah. This is a function of r but this is a constant.

The denominator is independent of r, the numerator has r. And this is r not r star, r star is a way we have defined it, okay yeah. **“Professor-student conversation ends”.** So this quantity here is a function of definitely r and the velocity will be a function of r, t and Rw, okay. Now okay.

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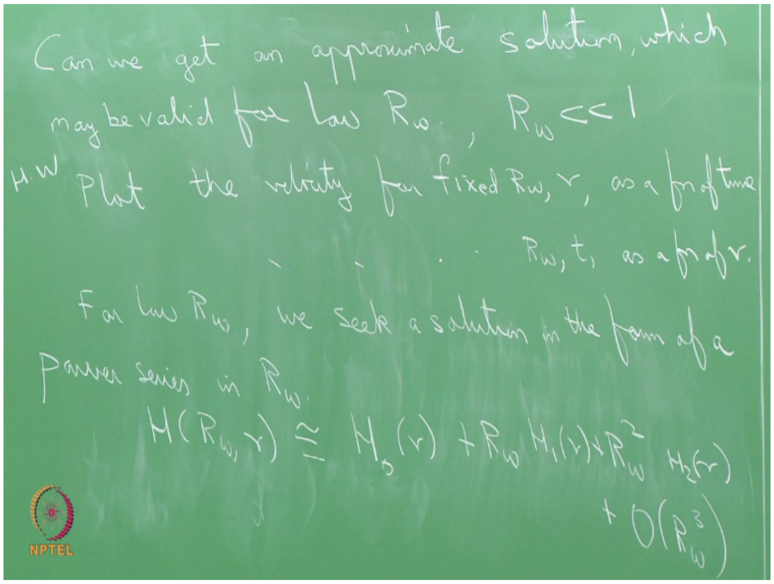

Can we get an approximate solution which let us say maybe valid for low Rw? And when I say low Rw I am talking about Rw being very much < 1, okay. Why would we be interested in? Okay, one of the things I want you to do is sketch or plot. Plot, okay this is a homework problem for you plot the velocity for fixed Rw and r as a function of time. So you have the expression. Now you just have to go and code it.

You need to go to the computer okay and then you need to go and solve and get the graph, okay like just getting a function. You can do the same thing. You can sketch the velocity for a fixed Rw and t as a function of r. So this I think there are many packages. If you have a fancy about a specific package you can use your package, it does not matter, okay. But I need the result at the end of the day.

So, I am comfortable with MatLab, some people are more comfortable with Mathematica, some people say Maple anything is fine. But the reason you need to do this exercise is we are going to approximate this result. And then we want to check how good our approximation is, okay. Clearly when I say low Rw how low is low. Is it 10 power -5, is it 10 power -10 so that is the thing that you want to know, right.

There are two ways of doing the approximation. One is you can take this expression and you can possibly expand the Bessel's function in terms of a power series and then do the approximation, that is one option. Or rather than work with the solution you start with a differential equation itself. And depending on your choice you can do this, okay. So what we are going to do is we are going to go back to the differential equation and explain the method of finding the solution using a perturbation series, okay.

Because I think that is easier in some sense rather than because you need not to know the exact form of the expressions of the Bessel's function and then you have to apply it. So for low Rw we seek a solution in the form of a power series in Rw. The idea is I am implicitly assuming the small changes in Rw give me small changes in the velocity profile, okay. So what do I do? I have H, right H clearly is the function of Rw and r, okay. H with the solution to this equation depends both on Rw and r those are the two parameters.

What we are going to do here is we are going to seek this as a power series expansion in Rw because Rw is my small parameter. And now the coefficients will all be functions of +. So this is like your Taylor series expansion or a power series expansion depending upon the level of accuracy that you are interested in you are going to keep the terms, okay. So now I am doing a power series expansion in terms of Rw.

So the coefficients will be depending on the radial position r, do you understand? So basically H remembers the function of Rw and r. What I am doing is the Rw is in the form of this power series. The small R dependency on the independent variable is captured in these coefficients. So our job is very simple. If I am seeking a solution of this kind I expect that this series should satisfy my differential equation, okay.

So I am going to have to substitute this series in my differential equation and like we did earlier we equate terms of the same order which means we equate terms of which are independent of Rw, Rw to the power 0, which have Rw to the power 1, which have Rw to the power 2, okay. And then I am going to get a sequence of differential equations and I use these differential equations I solve and I am going to get H0, H1 and H2, okay.

And once I get that I have a form and then I can go back and substitute it here. Your job is to verify how would is this approximation. Clearly, if you take one term you will have an okay approximation. If you take two terms you will have a better approximation to the actual value. So what you are going to do is you are going to find the actual solution and you are going to find out how good these approximation is to the thing.

And that would clearly depend upon the number of terms you take, okay. So for example, maybe Rw<10 to the power -3-- the approximation is good for first order. But suppose you want to push it for a larger Rw value then you may have to take a second order term as well, okay. So just like you take more terms you get better accuracy. So let us just do this extra side of substituting this in the differential equation.

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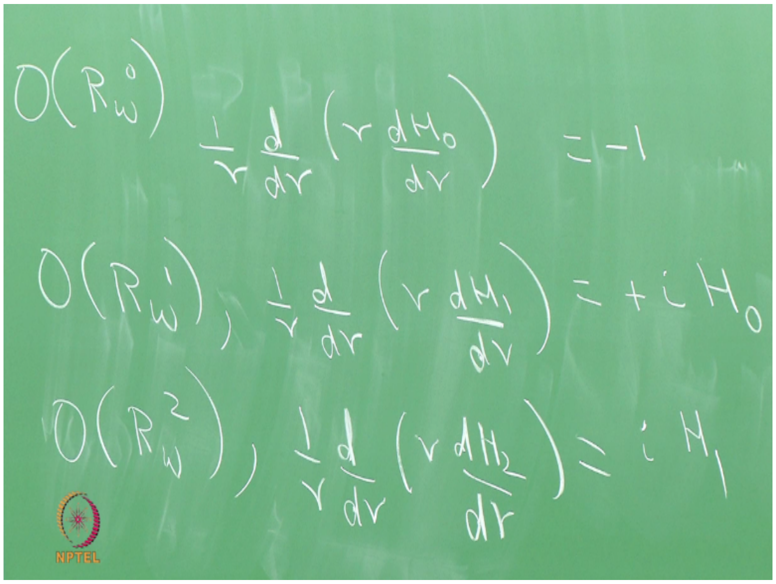

And I am going to retain this as it is d/dr multiplied by H0+Rw H1+Rw squared H2, okay. Remember H0, H1, H2 are all functions of r, okay –iRw times H0+Rw H1+Rw squared H2, higher order terms which I am neglecting = -1, okay. So our strategy is now to find out H0, H1, H2. If I find out H0, H1, H2 I can substitute it here and I can find H. How do you go about finding H0, H1, H2? By equating terms of the same order, okay.

So if I look at terms of the order of R0 that is terms which are actually independent of Rw. What do I get from this one? I get 1/r d/dr of r dH0/dr, okay. And this will contain Rw so I do not use this but this is independent of Rw so I get = -1. Okay that is the differential equation which I get. What about this guy? To the power 1. Clearly this is going to be contributed by this?

So I am only interested in the terms without the Rw. And what I mean is I am not going to write the RW explicitly this gives me 1/r d/dr of r dH1/dr = -i H0 + this guy, is this +, yeah, when I am taking it to the other side of course it becomes +. Yeah, you are right. I am taking it to the other side. Yeah, so it becomes +i H0. Now, I am happy when you guys correct me because then I do not have to redo the lecture.

And the second one gives me this, okay. And that was going to be +I H2, no +i H1 and so on and so forth. So what I want you to observe here is that we are solving a sequence of problems. I solve for H0. H0 is known okay then I am going to now the right hand side becomes the non-homogeneity. I solve for H1. We have a bunch of linear equations. I solve for H1 then H1 is known. I substitute it back here and I find H2.

So this way I am able to proceed sequentially in my calculation. And so once I know H0, H1, H2 I can substitute it back here and I have my solution H which is an approximation, okay. Just want to make sure I have not made any mistake. Yeah, I think everything is fine. So now in order to solve this we need to have boundary conditions because they are all differential equations anyway.

So what is a boundary condition on H. H the boundary condition is where this should be 0 at r=1 and so I am going to write it here.

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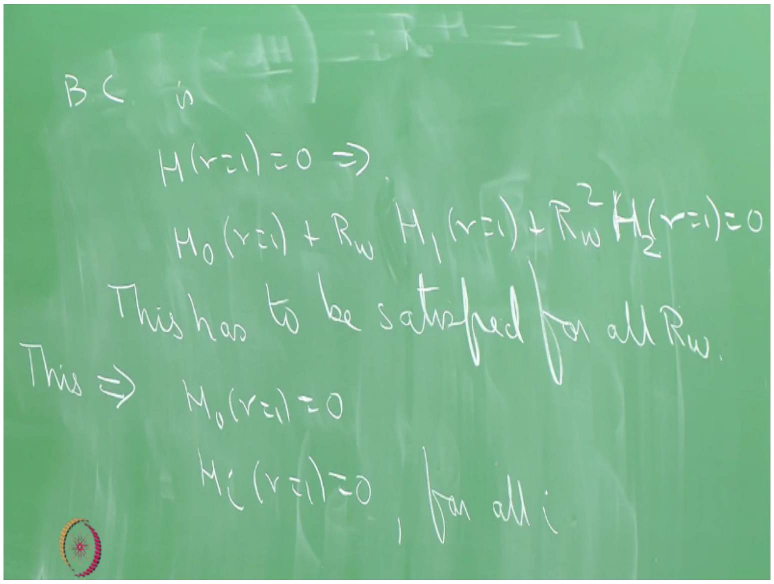

The boundary condition is H at r=1=0. This implies H0 and r=1+Rw of H1 and r=1+Rw squared times H2 of r=1=0 + higher order terms, which I do not write, okay. Remember I want this equation to be satisfied for any Rw, okay that is the idea. When I am doing my power series expansion this has to be satisfied for all Rw or any Rw. And this can happen only if H0 at r=1=0. H, I should be more smart, I should just say HI at r=1 is 0 for all I, okay. And this implies that.

So now I am all set because that is a very straightforward differential equation which we can solve. And now since this is straightforward I will be bold enough to make an attempt, okay and get the solution. What we will do is we will try to get H0 and H1. I suggest you guys work it out on your own and then we can compare so or you can just follow me, whichever way you are comfortable. I will leave you to calculate H2.

So this is you can just directly integrate and get the solution

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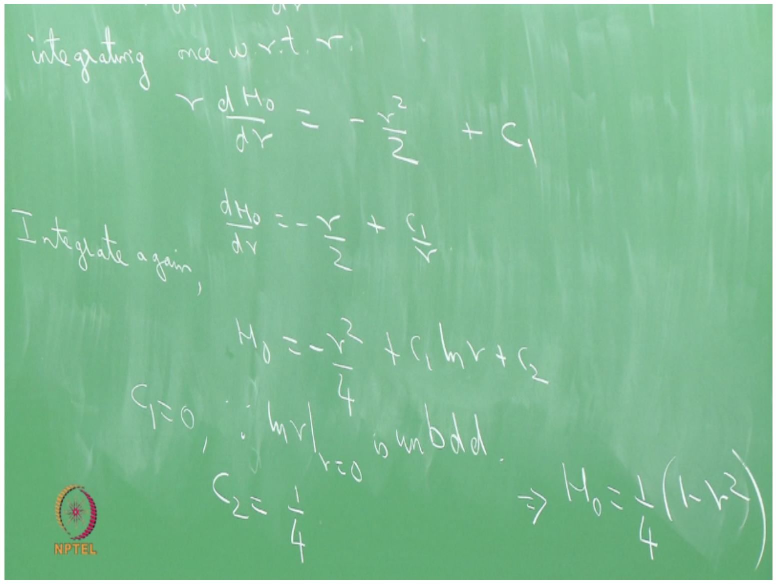

H0, okay. 1/r d/dr of r dH0/dr =-1. I can take the thing over there and integrate this out I get r dH0/dr = – of R squared/2. I have r dr I take the r here, integrate this with respect to this thing. I integrate this once I get a factor constant C1 okay, integrating once, once with respect to r. So I take the r there and do this. Integrating one more time but then before that I want to bring my R below, okay and I get d H0/dr = -r/2+C1/r. Integrate again, what do we get? H0 = –R squared/4+C1log r+C2, okay.

So these are the constants which I have to evaluate. I know that at r=0 my log term becomes unbounded so I use the same argument as last time. I say, C1 is 0 since log of r at r=0 is unbounded and I only have to evaluate C2 and that comes from a condition at 1, okay. And C2 is therefore = at r=1 I have H0=0 and I have C2 is therefore 1/4th. And this implies that H0 is 1/4the of 1-r squared.

So in some sense you can H0 corresponds to what. The solution meant Rw is 0, okay. H0 corresponds to the solution when Rw is 0 when there is no omega. Remember Rw is omega something multiplied by r squared/Mu. So that means there is no periodic part we can think of. And so the solution is that only due to the constant part and again your parabolic velocity profile. And basically this is what we always try to do.

Whenever we are doing a Perturbation series solution when I am trying to expand it in terms of a particular parameter I want to make sure that when that parameter is 0 I have a solution, okay. And then I am trying to improve on the solution for non-zero values of the parameter, okay. So when Rw is 0 I am getting a solution. When Rw is not 0 is some small finite value I am going to tell the solution is going to be different.

And that difference the correction is incorporated in that Rw H1 term, okay. And H1 is what we have to calculate now. So depending on your level of accuracy you take more terms. So now that I know H0 which is here I am going to substitute it in this equation and I am going to solve for the H1, okay.

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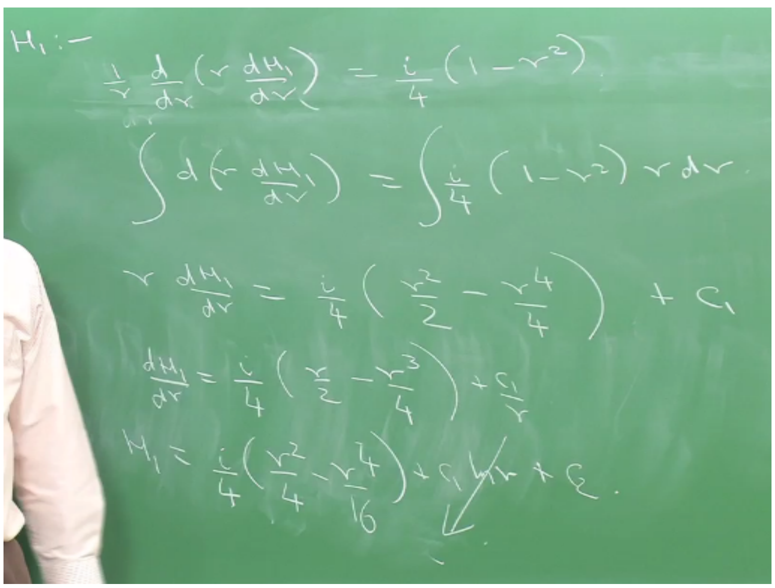

I have 1/r d/dr of r dH1/dr = i*1/4th of 1–r squared. This is the governing equation for H1, okay. I am just going to put that. I think I am fine. Okay, so now I can make space. Clearly this is a function of r and you know how to integrate this and we can get H1 now, okay. So I am just going to integrate this with respect to r again and what do I get. d of r dH1/dr I am integrating that = i/4 just doing it a little bit of a stepwise manner to reduce my chances of making a mistake.

And so this gives me r times d H1/dr = 1/4th of i. I get r squared/2 and I get -r to the power 4/4, okay + a constant of integration C1, okay. All I have done is just taken this r dr to the other side and I am just integrating it. I going to do exactly what I did last time, divide throughout by r. r/2-

r cube/4+C1/r. Integrate this one more time to get H1=i/4 times r/2–R cube/4+C1 something is wrong. I get r square/4 and r to the power 4/16+C2.

Same argument as last time I knock off C1, okay. And I need to calculate C2 and I am going to use the boundary condition that H=0 at r=1 and get C2, okay. And let me just do this.

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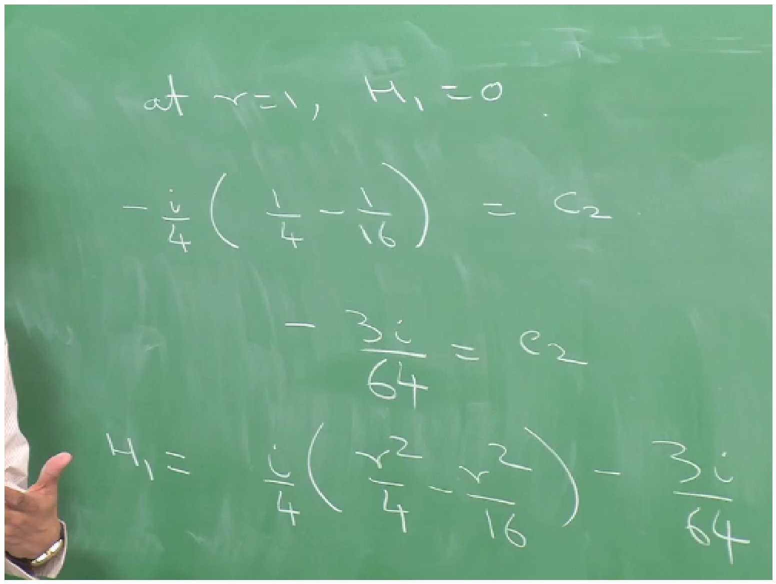

And r=1, H1 is 0, okay. This implies –i/4 times 1/4 – 1/16. Let me just do this because = C2, right. And I have what 3/4, yeah, (()) (37:22)/16 is that right, there is a 4 here, yeah, so 4/48 right = C2. 16*4=64, yeah, right, that is it. So that is my H1. As i/4 now again write it whichever way you want. Some people like to write it in a different way 16–3i/64. We can possibly write it in a slightly better way. This is slightly clumsy I got I occurring a couple of times. You can take things common and you can simply things, okay.

So now if you wanted to get a more accurate solution go and get H2. But we can see that this is a very simplified way of getting the coefficients of Rw. What are you going to do now? You are going to actually go back and H is now a composite H0+Rw H1. I want you to see that H0 is independent of i the imaginary number. H1 has I in it. So what is this going to do? If you go back to get the velocity you had to do the imaginary part of the exponential E Rwt * H.

So this guys are going to be exponent that is the real term the exponential of that is going to give you your cos theta + i sin theta. So the imaginary part is going to give you only the sin theta which means that particular component is actually in phase. So when Rw is 0 your velocity is actually in phase with the pressure gradient whereas when Rw is not 0 a small amount comes in, the I comes.

And now when you actually take the imaginary part instead of like sine term you will also get a cosine term because now it is exponential of i theta multiplied by i something so the imaginary part will have the cosine term now, okay. So this is going to give you that out of phase component of the velocity. So I mean, that is one way for you to actually figure out why when you have a finite value of Rw you have an out of phase component.

So I think this becomes very clear here. So this is something which you should let me write this down.

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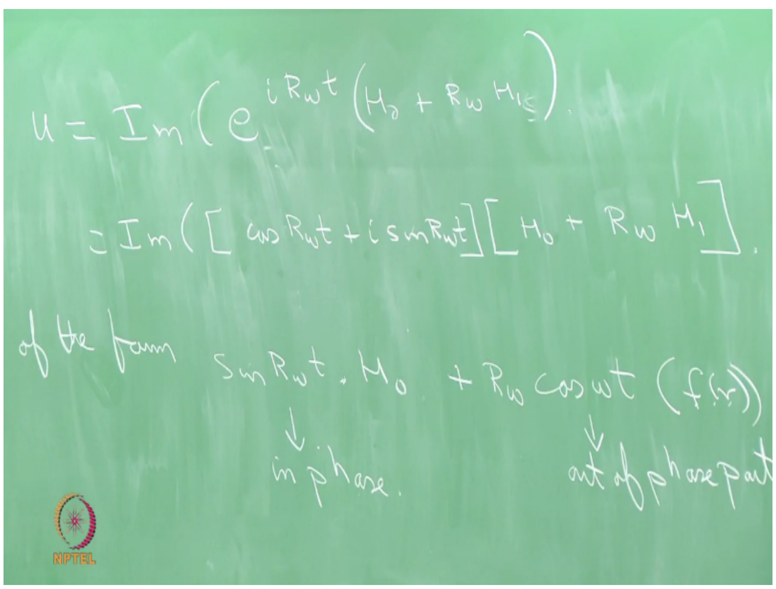

So clearly u is the imaginary part of e power iRwt times H0+Rw H1, okay. I mean, I am neglecting all the higher order terms. So this is imaginary part of what? Cosine Rwt+i sinRwt star H0 + Rw, remember Rw actually has no, sorry Rw does not have it H1. One I am trying to make here is H0 is real. So this multiplied by this is going to give me a real part. I am not interested in that. This multiplied by this gives me the imaginary part.

So this is going to be of the form this is of the form sin Rwt star H0, okay. So this H0 is contributing to my in phase part. So this is in phase, this is in phase. What about this guy? This remember has an I in it. So i star, this is a real part so I am not interested in that. i star this gives me the imaginary part which is the cosine part. So this is basically going to have a cosine component times something, okay some function of r. So this gives you the out of phase.

So what I am trying o tell you is and remember this is multiplied by Rw. So this is the out of phase part. So and Rw is very, very low, the out of phase will go off. The more the Rw, the more the out of phase component, okay. So that is basically for you to understand. So what you people will be doing is actually finding out the solution using this approximation, finding out the solution using the exact solution and making comparison just like what we did for the quadratic equation.

In the quadratic equation we have the exact solution in terms of the discriminant. Then you have the binomial series expansion all the power series that we actually did and then you can compare for how accurate is this thing for different epsilons. See only when you do that you will get an idea because I am saying epsilon is low. I am saying Rw is low but how low is low that is going to depend upon the problem of all the specific problem.

For some problems Rw maybe low is 10 for some problems Rw low could be 10 to the power -5. So how do you figure that out? Only by doing this comparison. So how good is your part of the Bessel's series solution and because it if is very, very low then possibly this approach is not very good. But supposing retaining the two terms I am able to push to Rw = 5, 10, 50 the more the better then I am happy with it, okay.

But then this can only be done when you actually sit down and do an actual calculation. So that is what we people are going to do, do an actual calculation and then verify how low is low. I think one last thing I want to do and then we will stop. Somewhere in the beginning I chose my time scale as R square/ kinematic viscosity, okay.

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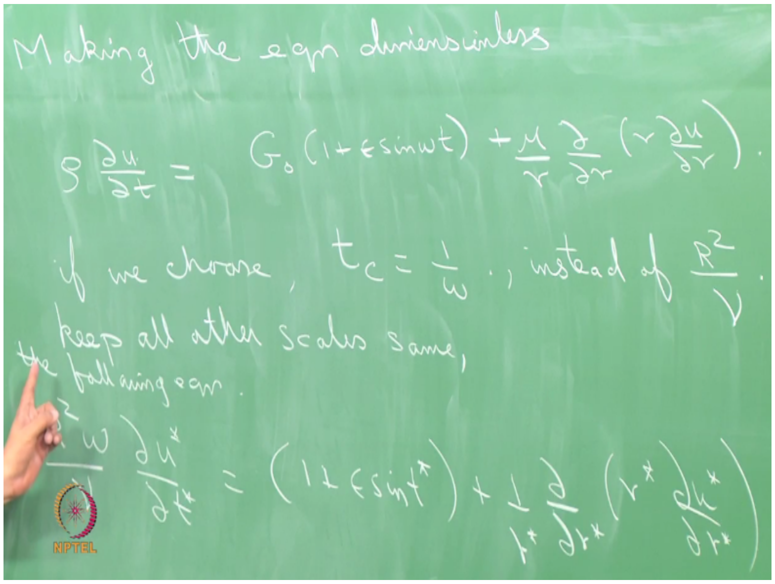

I want to go back to making the equation dimensionless, okay. And I have told you that there were two possible choices of making the equation dimensionless because you have two choices for the time scale, okay. What was the equation we had? So T = G0 1+ epsilon sin omega T+ Mu/r d/dr of r du/dr that is the equation we began with, okay.

I mean, assuming that they have only the actual component of velocity only varying in r and t. I am going to keep the velocity scale and the length scale the same, okay because length scale clearly r there is nothing else happening. And velocity is going to be decided by the pressure, okay. But if we choose the time scale the characteristic time as 1/omega instead of r squared/Mu, this is what we did earlier r square/Mu is what we did earlier.

What you are going to see and I am just going to write this here, you will have and you can do this, okay. Keep all other scales the same. You would get in a dimensionless form T the following equation equals okay. So by choosing a different scale I just wanted you to understand that I told you earlier that the scales can be chosen in different ways. I am choosing it this way now. The advantage of so anyway that is one parameter which occurs.

It is not that this Rw has disappeared. One parameter will occur. And this again the ratio of the same time scales, okay. This guy now is multiplying my inertial term. Now it is possibly easy for you to see that when Rw is 0 when Rw is very, very low, the inertial component is not going to be significant. In my earlier formulation I could not see that. Now I can see that when Rw is 0 this guy is going to get knocked off, okay.

So when Rw is 0 this guy goes off and remember this was the guy who was creating a problem with out of phase component. If this goes off then I can actually solve all the velocity and I can get my velocity directly, okay. So basically what I am saying is here in this formulation here Rw=0 knocks off the inertial term and so the velocity is in phase with the Delta P.

So possibly by doing this scaling instead of solving it and then trying to understand whether it is in phase or out of phase even by looking at the differential equation you can actually make this conclusion. So whenever you are solving any problem you need to be able to actually you need to actually work out by choosing different time scale and see what kind of information you are getting, okay because a lot of information by choosing the right scales or different scales and that gives you some insight into the problem, okay. Yeah, let us stop thanks.