Multiphase Flows: Analytical Solutions and Stability Analysis

Prof. S. Pushpavanam
Department of Chemical Engineering
Indian Institute of Technology – Madras

Lecture - 14
Pulsatile flow: Analytical solution

So, we will start today's lecture, the 14th lecture of the course. And what I intend to do today is take a problem in fluid mechanics and explain the complete solution to that problem okay. And then explain how that same problem can be solved using the method of perturbations. So, this the same thing as what we did earlier when we have this quadratic equation for which we had the exact solution okay.

Because, it is a quadratic equation even when the parameter epsilon in it, we could find the solution in terms of the square root sin right the discriminant and all that. Then we said, we will do a perturbation series solution. So, I want to basically take you through the same process okay and in the course of this, we will measure a few things with I think our most of the times implicit in whatever is done in the classes.

So, I want to be explicit about certain things, which people do not explicitly mention okay.

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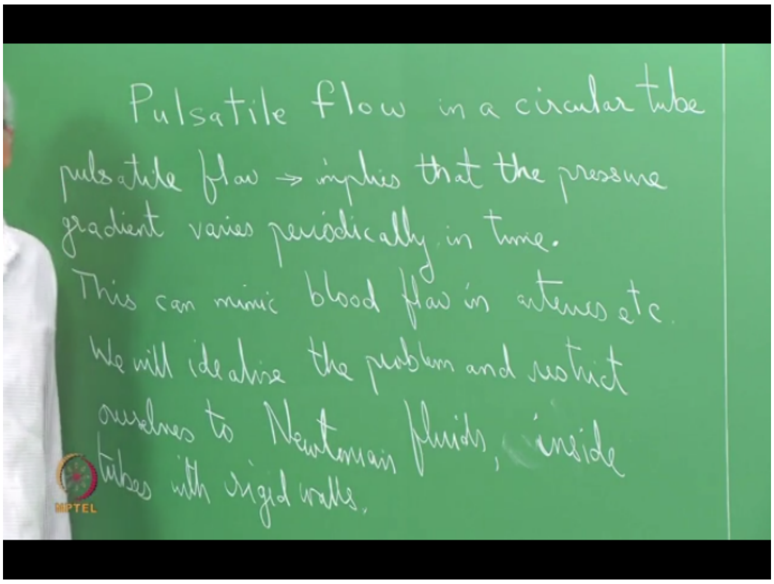

So, the problem here is a pulsatile flow in a circular tube. So, you have a circular tube as a liquid flowing through it and what do I mean by pulsatile flow? I mean that the pressure gradient that is imposed is going to be varying periodically okay. So, pulsatile flow implies that the pressure gradient varies periodically in time.

So, why would anybody be interested in this, I mean very classical example of a pulsatile flow is that of the blood flow through the capillaries okay, your heart is beating periodically. The pressure changes fluctuate okay in a periodic manner and so this kind of problem, just by way of motivation are so why anyone would really be interested in this is, this can mimic blood flow in the arteries for example etc.

The blood flow problem of course is more complicated in the sense that the zoology is not Newtonian. The valves are flexible okay. So, I mean those are additional complications. But, now we will keep life simple and we assume that the liquid is Newtonian, we will assume that the valve is rigid okay and if those of you have interest in making further studies on more complicated things can pursue okay.

So, what we will do is we will idealize the problem and restrict ourselves to Newtonian fluids inside tubes with rigid valves okay. So, that is just an idealization so that we can understand certain things about the flow. If you are not happy with that, then you go for relaxing some of this assumptions and then you proceed. So, clearly I mean this is an extension of a problem which you all have seen before. What is the problem you seen before? The Hagen–Poiseuille flow in a circular tube.

In Hagen–Poiseuille flow, what is the story? The pressure drop imposes constant okay and then you talk about the fully developed flow whether velocity does not change in the axial direction and you only have the velocity varying in the radial direction. But now the pressure drop is not a constant. Pressure drop is fluctuating with time. So, clearly what this means is the velocity that we are going to see in the channel is also going to be fluctuating with time okay.

So, earlier we were in a position to talk about steady state. We are now not going to be able to talk about steady state because the pressure drop is varying fluctuating with time. The velocity is going to fluctuate with time and very importantly I need to retain the term containing the time derivative in the Navier Stokes equation okay. But, we will still keep life simple in the sense assume that the axial velocity is a function only of the radial position and time okay.

Earlier for the steady state problem, you had velocity a function only of the radial position. Now, because of the periodic pressure drop, I am saying look sin is also going to come in. I am not going to complicate life because I just want to illustrate some ideas and then we will just say now that the velocity changes with R and T okay.

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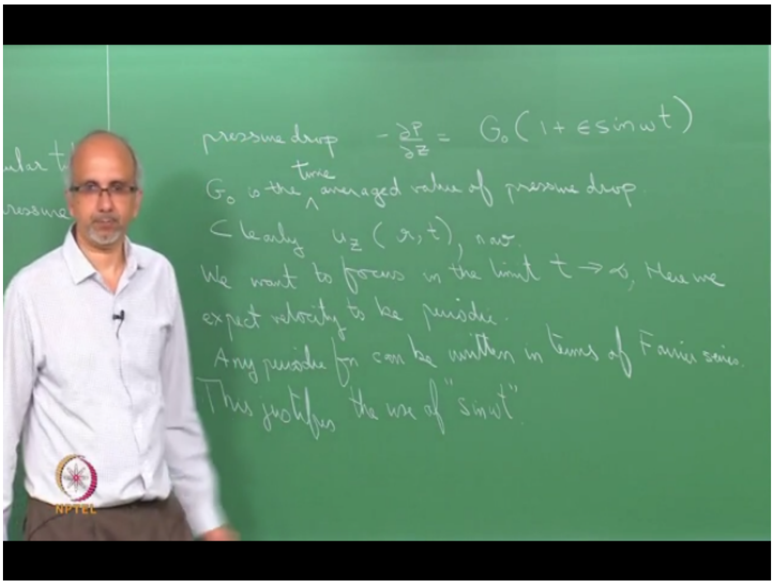

So, what I mean by this is when the pressure drop which is given by -dp/dz I am going to write this as some constant. Remember as epsilon sin omega t. what we are used to is if epsilon is 0 or omega is 0, that means there is a constant pressure drop. So, the way I am looking at it, the pressure drop is varying periodically G0 represents the mean value, the average value of the pressure drop okay.

So, G0 is the average value of the pressure drop okay. And the way I have written it I got to manage dp/dz here so that G0 is positive. And I should actually say this has got time averaged value of the pressure drop. So, clearly uz, which is axial velocity is going to be a function of r and t now okay. And what I am going to do is, see, I have a periodic pressure variation if I have this kind of a situation for a long time, the velocity profile is going to be periodic as well.

So, this is like the equivalent of my steady state. So, in order to reach my steady state, I have to wait for a sufficiently long time. Finally, I go to my steady state. I have an initial profile, supposing you have a fluid in a channel which is at rest, the liquid is not flowing okay. You have filled it with fluid. So, your initial state is rest. Now, you put a constant pressure drop. You will get your parabolic velocity profile.

But to attain the parabolic velocity profile, it takes some time and after you wait for a sufficiently log time, you have the parabolic velocity profile. So, that is in the limit of t tending to infinity, you have your steady state, your parabolic velocity profile. Same thing here, we today and tomorrow, we are going to concentrate only in the limit of t tending to infinity. I am not interested in how does the velocity change from the state of rest at t=0 to the final t value okay.

How does it go from the state of rest to the parabolic velocity profile? That I m not interested in. I am only interested in what happens in the limit of t tending to infinity, the final solution. The final solution, you expect it to be periodic. So, if you want to actually keep a proab in one of your arteries, you will see that the velocity is varying periodically with time at a fixed point. So, that is the thing we are interesting in finding out okay.

So, we want to focus in the limit t tending to infinity okay and here we expect velocity to be periodic, that is one thing. The other thing which I want to talk about is the fact that I have assumed sin omega t. So, of course the periodic function does not have to be sinusoidal, it can be some arbitrary periodic function. But, you know if any periodic function, we can resolve it in terms of the 4ier sine series or 4ier cosine series or 4ier series.

So, basically if you give me any periodic function, I will basically resolve it using my 4ier series and I have different components sin omega 1 t, sin omega 2 t, sin omega 3 t. So, that is the justification for using sin here okay. So, basically what I am saying is any periodic function can be written in terms of a 4ier series. This justifies the use of sin omega t. An epsilon basically represents the amplitude of the fluctuation that you have okay. So, this is the amplitude and this clearly is the frequency.

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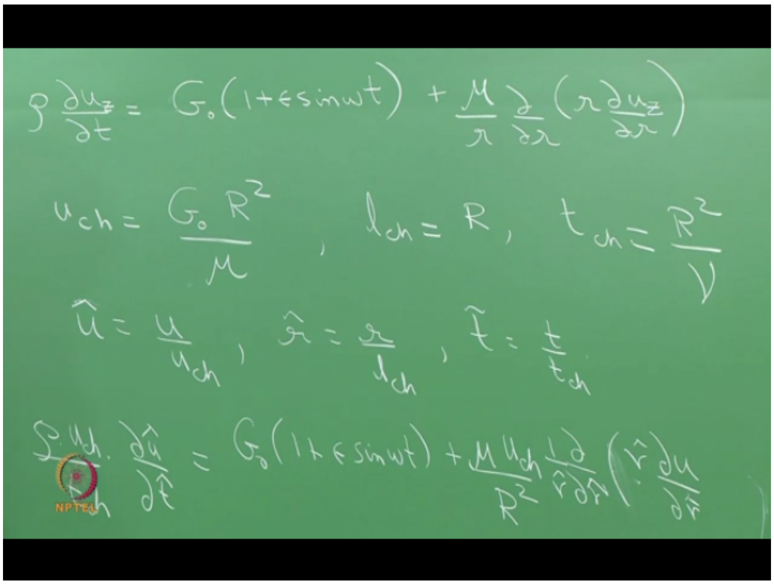

So, what I am going to do is just go directly to the problem that our objective is to find the solution du/dt, the axial velocity profile okay and this is going to be in terms of -dp/dz+my viscous term okay. The left hand side contains my initial terms I am just simplifying it and I am neglecting all the u.del u terms saying that they are very small.

But, I need to retain this because I need my time derivative okay and the dp/dz, I am going to write as G0 multiplied by 1+epsilon sin omega t okay and + my viscous term, which is going to be mu/r d/dr of r du/dr, that is my viscous term. So, that is my viscosity and that is my del square. Because this, I have only remembered the variation of uz with respect to r is captured here. The variation of uz at this time is captured here and this tells me how uz varies at r and t.

Now, I have not derived it from the Navier stokes equations but I made some assumptions of simplifications. Basically, I do not want to consider uz as the function of r theta z and all that. So, we are just assuming fully developed in some sense but it is unsteady steady state okay. So, like I mentioned earlier, we want to solve this problem in the limit of t tending to infinity okay and we also want to solve the problem exactly and you want to solve the problem using a perturbation series.

That is the idea. So, now the first thing that we have to do whenever you have a problem is to try and make it dimension less okay. Clearly, in this case I am imposing my pressure drop and the flow is going to be induced by my pressure drop. So, the characteristic velocity scale is going to be decided by the pressure drop which I am inducing, which is basically given by G0, the average value is G0.

So, I am saying that u characteristic is going to be measured in terms of the pressure drop and if I look at the terms here, I am going to choose G0R square divided by mu as my characteristic velocity scale okay. What have I done? I worked at these 2 terms, this term as units of G0 and I am looking at this term I have a length square in the bottom and have viscosity here so, I am just saying G0 and velocity is there.

So, G0 r square divided by mu has units of velocity and therefore I am just saying that is my characteristic velocity okay. What about the length scale? The characteristic length scale is going to be clearly the diameter of the tube, which is R or the radius of the tube which is R okay. And the time scale in this case is going to be given by I can look at these 2 terms okay and what I am going to get, I am looking at term on the left here, the second term on the right here and mu divided by rho is my kinematic viscosity and t characteristic is going to be given by R squared divided by mu.

So, in some sense, what we are talking about let us understand this physical meaning of this t characteristic. What does it represent? It represents the time required for momentum to diffuse in the radial direction okay. If you want R is the distance of the radial direction, mu is the kinematic viscosity, which basically is facilitating the transfer of momentum. Now, how much time does it take for the momentum to diffuse in the radial direction.

So, because once the momentum is diffused in the radial direction, then you will have your fully developed velocity profile. If you wait for that much time or longer than that. So, that is the idea. So, that is the choice of this scale. Now, one thing which I want to caution here is that the way you choose this case are not necessarily unique, you can always choose different ways of scaling a problem and proceed.

The idea is by choosing certain scales, you want to try and get insight about what is dominating, what is not dominating the problem and so that in some limiting cases, you can actually do some analysis and get some idea about how the behavior of the system is okay. So, please understand that I have chosen this, some of you for example, you could have chosen an alternative time scale which is alternative time scale? It is coming from the omega okay.

You can say look there is a frequency with which I am changing the thing and that is my timescale of my system. Sure we can proceed with that argument okay. But then what is important is not necessarily that the value of the omega what you will see is that there are 2 time scales in the system and the ratio of these 2 timescales is what is going to actually decide the behavior of the system okay. So, at the end of the day I think you have to be consistent.

Point is you can do the problem in different ways okay. So, let us make this thing dimension less now and what will that give me if I choose u tilde as u divided by u characteristic and r tilde as r divided by length characteristic and t tilde as t divided by t characteristic. Suppose, and you make the equations dimension less, what do you get? Rho multiplied by u characteristic divided by t characteristic du tilde I am forgetting; I am not to write the z dependency.

You all know that it is axial velocity dependency okay. Du tilde/d tilde=G0 times 1+epsilon sin omega t + mu times u characteristic/r squared times d/dr tilde and that is 1/r okay. So, that is basically your dimension less equation with a tilde here. Now, if you go to substitute, u characteristic as G0 r squared by mu and put t characteristic as r squared by nu substitute the u characteristic and the t characteristic. What do you get? U characteristic divided by t characteristic is just G0 =, you put tilde there okay.

Now the point I am trying to make here is this G0 cancels of. Anyway, this is a very small issue in the sense you can sit down and make the substitutions and we can find out for yourself what is happening okay. Now, some of you are thinking possibly that look I got this epsilon sitting here and this man is going to do a perturbation series about this epsilon. So, let me tell you I am not going to do a perturbation series about epsilon okay. Wait a second.

I got a problem right, I got a problem here in the sense when I am writing this in terms of t tilde, and I write t in terms of t tilde, I need to write this as t characteristic multiplied by that I need to write this as r squared/nu. Sorry about that. I need to include an r squared by nu omega here. Because, I have t here I am writing t in terms of the dimension less time t tilde. So, I need to get the characteristic time here so I get R square omega/nu okay.

Because that is important to me because that is the ratio of the time scale that I was talking about. So, remember I told u that you choose omega as your time scale, you can choose R square/nu as your time scale. But what is important is, this particular parameter here actually represents the ratio of these 2 time scales okay. And this is called a Strouhal number. So, whenever you have something like a periodically imposed pressure gradient, you will get this okay.

So, remember that the behavior of the system therefore is, I am going to write it correctly now. What I have done is basically rewritten those equations again and just want to make sure that the things are a bit more clear now. The left hand side contains the time derivative of the dimensionless variables, the dimension less velocity u hat with t hat as multiplied by rho times u characteristic divided by t characteristic and that =G0 times 1+epsilon sin R squared times omega times t tilde divided by nu.

So, that was the term which I made a mistake. Now, I am just going to emphasize that the argument of sin is R squared omega/nu t tilde + the viscous term which you have and what I have done in the next step is substituted the expressions for the characteristic velocity and the characteristic time and you see that the parameter G0 occurs in all the 3 terms and so you can actually nock it of okay. And the other change that I have done is call the term R square omega/nu R subscript omega.

This R subscript omega is something like my Strouhal number okay. And I am following (()) (23:00) for the rotation. The Strouhal number, R omega I have written it are the very last step. It is the ratio of 2 timescales. It is the ratio of the time scale for momentum to diffuse across a distance R, that is R squared divided by mu, the kinematic viscosity to the time scale of the oscillation. Omega, remember is the frequency of the oscillation.

So, the reciprocal of omega is the time period of the oscillation. And therefore, 1/omega is the time scale of the oscillation and the Strouhal number R subscript w represents the time scale for momentum to diffuse over a distance R to the timescale of the oscillation. This Strouhal number is the small parameter which we are going to be doing a perturbation series solution. That is, we are going to use R, the Strouhal number as a small parameter to seek a perturbation series solution.

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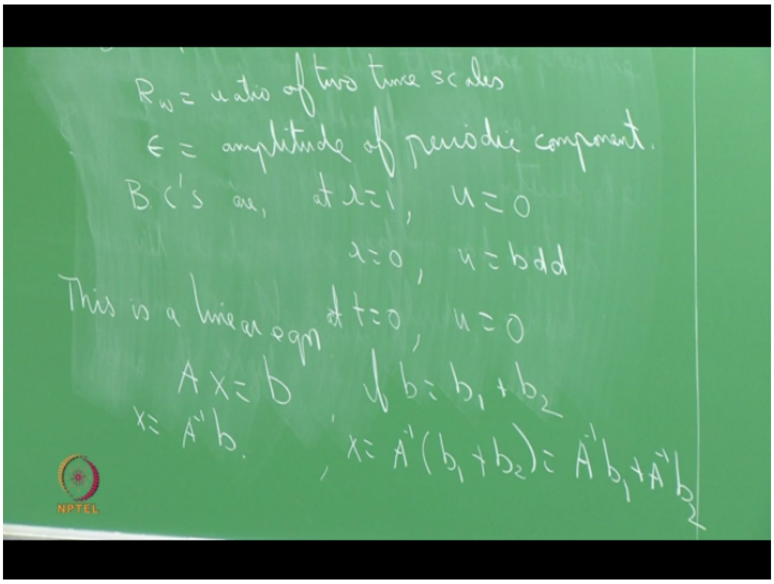

I have du tilde/dt=1+epsilon sin R omega t+ okay. I am dropping the tilde but now you know that this is dimension less okay. We drop the tilde but now the variables are dimension less. Why am I doing this? This is to reduce my chances of making mistake okay. So, now remember that this is a dimension less equation, all these are dimension less, this is a dimension less time, dimension less velocity, dimension less position okay.

So, these are putting the tilde everywhere, I am just writing it like this. What I am trying to tell you is that there are 2 parameters here, one is R omega, which is the ratio of the time scales and epsilon, which is the amplitude of the perturbation. Let us not call it perturbation. Periodic component. Now, this is R subscript omega, oh there is no omega here. You are right, there is no omega there. Thanks, so this is R subscript omega is the thing, yes that is right.

There is no omega there okay. Now, what I want to make you observe is in order to solve this equation, I need boundary conditions and initial conditions right. So, what are the boundary conditions? Boundary conditions are at R=1, we have u=0, that is my no slip boundary condition. At R=0, my u is bounded. And at t=0, u=0 that is I am just assuming that my fluid was initially addressed. Yes, everything is fine.

So, the fluid is at rest initially and now I am suddenly imposing this periodic pressure drop, I am trying to understand what is happening okay. The point I wanted you to notice that, this equation is a linear equation. And because this is a linear equation, I can be bold enough to look for an analytical solution okay. Now, this particular term, this is my source term okay, my pressure gradient.

That is my source term and I am going to look at the solution for u as being made up of 2 parts, one arising because of the constant component 1 and another arising because of the time dependent component the sin Rw of t. that is supposing you have a linear problem x=b1+b2 okay, you can solve that problem as x as a inverse of b1+b2 or you can solve or you can just look at x as a inverse b1+a inverse b2 okay.

So, that is basically what we are going to do. Let me just write the analog here see, this is a linear equation correct okay. Now, supposing I am just going to give you an analog of what we are trying to do so that you can relate to the solution strategy. If we have x=b and let us say, this is my equivalent of my b okay. If b=b1+b2, what I am going to do is, solve x is of course = a inverse b. But, x is also = a inverse b1+b2, which is a inverse b1+a inverse b2 okay.

Matrix multiplication just gets carried over. So, that is essentially what I am doing. What this means is, if I have 2 sources of non- homogeneity, I can find the solution, find the response of the system to one, find the response of the system to the other and then just add up. Because it is linear, I can do this super position okay. This is a principle of linearity and super position that you have are used to. So, that is exactly what I am going to do.

I am going to look at the solution to this problem as being composed of 2 parts. The first part being coming from this constant pressure gradient, the second part coming from a time dependent pressure gradient okay. The idea is that, you already know what the solution is for your constant pressure gradient. You get your parabolic velocity profile. For the time dependent thing is what we are going to do today okay. So, then once we do this, that gives you your complete solution.

Then we go back and look at the perturbation series approach for using in the limits of Rw, my Strouhal number being very low, Strouhal number being very large. So, the perturbation is going to be done about Rw okay. Epsilon is just a magnitude okay. Of course you could have done the other way also. If you wanted to, you could have done a power series about epsilon. But that is what we are doing today.

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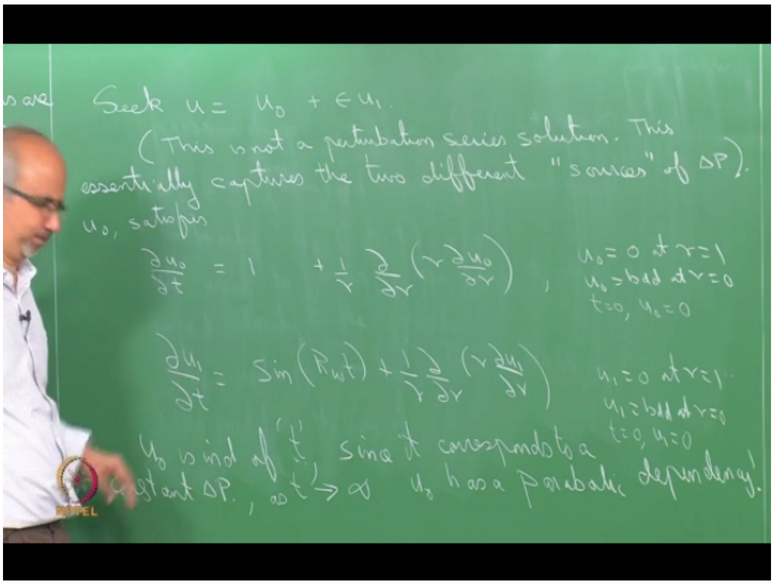

Seek u as being made up of u0+epsilon nu 1 okay. Now, what is this? This, I want to explain is not a perturbation series solution okay. This is just writing, finding the solution for 2 different sources of non homogeneities. But, I am putting epsilon here just so that that takes into account that epsilon which is there. So, u1 will be a solution of only sin omega t, that is all okay. This essentially captures the 2 different so I say source terms, sources of delta p okay.

The constant part and this. So, what I am going to do is, I am going to seek the solution for u0 and I am going to seek the solution for u1. So, u0 satisfies. So, u has to satisfy that equation. So, u0 is going to satisfy du0/dt=1+1/r d/dr of rdu0/dr okay. And du1/dt will satisfy sin of Rwt+1/r d/dr of rdu1/dr okay.

What I wanted you to do is, I want you to just add up these 2 guys, multiply this by epsilon, suppose you multiply this by epsilon, the second equation by epsilon and you add, you will get du/dt of u0+epsilon nu 1+1+epsilon that plus some second derivative operating on nu0+epsilon nu 1. So, that is my solution to my original equation okay. So, I mean, this is just to, i have just recomposed the solution just like I have explained here is like a inverse of 1 + this.

I am just looking at 1 separately sin omega separately okay. What are the boundary conditions to which this is going to be subject to? Same boundary conditions u0=0 at r=1, u0 is bounded at r=0, u1 is 0 at r=1, u1 is bounded at r=0, at t=0 u0 is 0, at t=0 u1 is 0. So, see the boundary conditions also have to be consistent with what the original problem was having okay. I mean you need to make sure that when you are doing this splitting up, the boundary conditions are consistent, the equation is also consistent okay.

So, now life is simple in some sense, in the sense that if you look at the first term and remember I am focusing on in the limit of t tending to infinity. In the limit of t tending to infinity, I am going to have something like for the first problem, a steady state because the first problem corresponds to the flow when you have a constant pressure gradient okay. So, u0 is independent of t since physically it corresponds to a constant pressure drop, del p okay.

That is the physical thing. Mathematically, you would have gone about solving it. If you are being a mathematician, you would have just oh let me solve this doing separation of variables or something like that. But now, you see I look this is constant pressure drop so the limit of t tending to infinity, I expect steady state. If you are not interested in t tending to infinity, then you have to worry about the change with time okay.

As t tends to infinity and that is kind of important. And then what you have is just your regular ordinary differential equation r and so u0 will be having a parabolic dependency. So, u0 has a parabolic dependency and that you know how to calculate. Just use a boundary conditions, you will get some 1-r square and you can calculate that. So, I am not going to do this. You will do that. I want o talk about the calculation of u1.

So, if I look at u1, what happens is, can the velocity or this term here represents how my pressure is changing with time. Is there a problem? It is fine. **“Professor - student conversation starts”** (()) (37:11) yes, the epsilon is not here because I have written my epsilon here. See, I am looking for the solution epsilon. If I have not put epsilon here, then I would have had to substitute u1 and then epsilon.

So, whatever the solution I am going to get u1, I am going to multiply that by epsilon and then I am going to add to get my original solution okay. That is the reason the epsilon got cancelled. Basically when I am substituting, epsilon nu 1 coming here, there is already an epsilon and there is already an epsilon here when I substitute this epsilon gets cancelled okay. **“Professor - student conversation ends”** I am just wanting to make sure I did not miss it over there okay.

Now, the pressure is varying periodically, sinusoidally with time. Will the velocity be in phase with the pressure, will it be out of phase with the pressure okay? And that is going to be decided by the only parameter which seems to be occurring in this equation, which is Rw okay. So, this particular physical thing we will discuss later. But I just want to point out that it is not the velocity in general will not be in phase with the pressure gradient.

For example, if assuming that the velocity is varying periodically with time and is in phase. What does it mean? Velocity will also be varying as sin Rwt. Only then, the pressure is varying as sin Rwt, velocity is also varying as sin Rwt, they are going hand in hand. Suppose now, the velocity okay the question.

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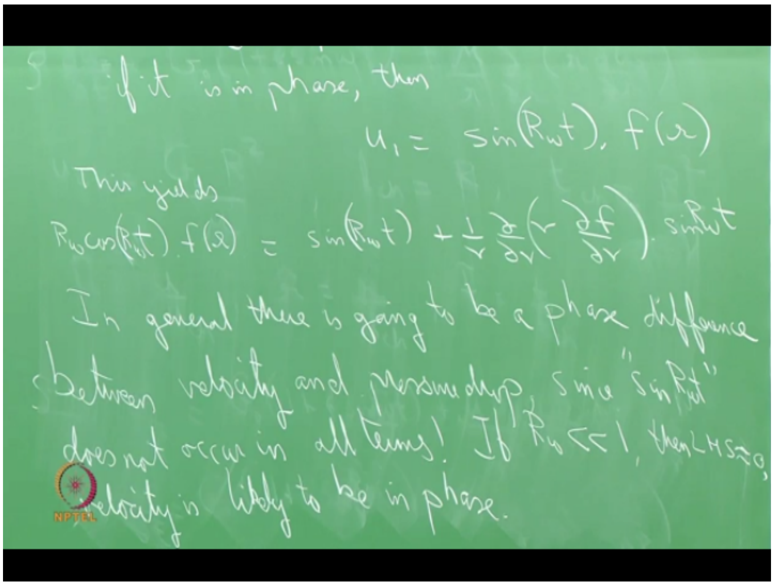

The question is, is the velocity in phase with the pressure drop? That is the question okay. Suppose it is in phase, if it is in phase then what do you expect? The time dependency is also going to be in the form of sin Rwt multiplied by something which is a function of R okay. Then u1 is going to be of the form sin Rwt multiplied by some function of r. I mean, otherwise would have been out of phase right.

So, now the point is, is this a likely solution to the partial differential equation. This is your typical separation of variables approach which you would do. You would assume that something is a function of time multiplied by a function of r okay. So here, supposing this is a solution, I am going to substitute this in my differential equation. When I substitute this, my differential equation, when I do the derivative with respect to time, I will get cosine.

This of course has the sin term and when I substitute it here since I am differentiating with respect to space, I have my sin term, I only have the derivative of f. So, what you would get is, this yields something like cosine Rwt multiplied by f of r may be multiplied by Rw because I am differentiating sin with respect to time I get that = sin Rwt+1/r d/dr of r times df/dr times sin omega t. Remember, Babitha will tell me that, this should be total derivative and not partial derivative okay.

So, since I am doing f and f is function only of r, this is a total derivative. So, now the point is, if it are be the in phase, then I would have a sin omega t in all my terms. And I could have actually cancelled out my sin omega t. but the fact that when I am putting this sin omega t here, I get a cosine omega t, tells you clearly that the velocity in general is not going to be in phase okay. In general, there is going to be a phase lag okay.

So, here in general there is going to be a phase difference, I should not use the word lag, I do not know if it is a lag or leak phase difference between velocity and the pressure drop since the sin R omega t does not occur in all the terms. If it has been sin omega t occur in all the terms, then that is the possible solution. That means the velocity is following the pressure gradient okay. The phase of this guy is not, so I got a problem.

So, what do you do? This also gives us some clue, it gives you some clue, in the sense for example, I have differentiated this thing with respect to time, I call it Rw here, if Rw is very, very low okay, these guys for all practical purpose, is periodic is bounded between 0 and 1, this is only telling you something about the frequency of the change. But the magnitude of the term, this is also bounded between 0 and 1, if Rw is very low, this guy is going to go to 0.

So, in the limit of very low Rw, Rw must smaller than 1, only these terms are going to be present. And then you can solve the equation okay. If Rw tends to 0, I expect my velocity to be in phase, but if Rw is much larger than 1, then this guy will be present, and my pressure guided and my viscous term will be present and then I am expecting it be out of phase.

So, that is the reason what we are going to do now is we are going to find the solution to this problem as it is and then we will try to find the solution the limit of low Rw using a bottle basin series and then do the comparison okay. So, I am just saying that, if Rw is very much lower than 1, then LHS is approximately 0 and velocity is likely to be un phase. What is Rw? It is omega multiplied by R squared divided by nu.

Rw is very low means, the omega is very, very small. My frequency is very slow. It is very low. So, my pressure gradient is changing slowly. If my pressure gradient is changing very, very slowly, that means my velocity is able to catch up with it. So, if I change it very rapidly, then my system cannot respond. But if my pressure gradient is changing very slowly because my omega is low, then the velocity can quickly respond to the change in the pressure and is going to just follow the pressure gradient and that is the reason it is in phase.

That is the physical way to look at it okay. I mean mathematically you can look at something. But because if I am giving you like you have slowly increase the pressure on you, you will be responding to the pressure slowly. But and you will be able to keep up with me. But if I am doing some very fast changes, I think you are going to go broncos right, so that is the what is going to happen. So, that is the idea.

As in the limit of Rw being much > 1, you have the other situation. And we will look at how to do and now that is the motivation for doing the perturbation series solution for Rw0, we know what is going to be in phase and when Rw is small, it is going to be slightly out of phase and we will be able to capture that okay. So, that is the reason we are going to be doing the perturbation series about Rw.

Let us go back to finding the actual solution. That is the idea. We are going to find the actual solution and then do the perturbation series. So the problem of finding a solution and since this is the linear equation, partial differential equation we can definitely solve by separation by variables right. And whenever you want to separation of variables you want that term to be repeated so that you can then convert the partial differential equation to an ordinary differential equation. That is the strategy.

What is preventing us from doing that? It is the time derivative, the first order time order time derivative. The first order time derivative when I differentiate sin, I get cosine. So, is there a function which on differentiating first time, gives back itself. The exponential function gives you derivative of the exponential of x, with respect to x, exponential of x right. And you also know from your complex variables, that e power I theta is cosine theta + i sin theta.

So, rather than solve this equation here, what I am going to do is, I am going to solve this equation.

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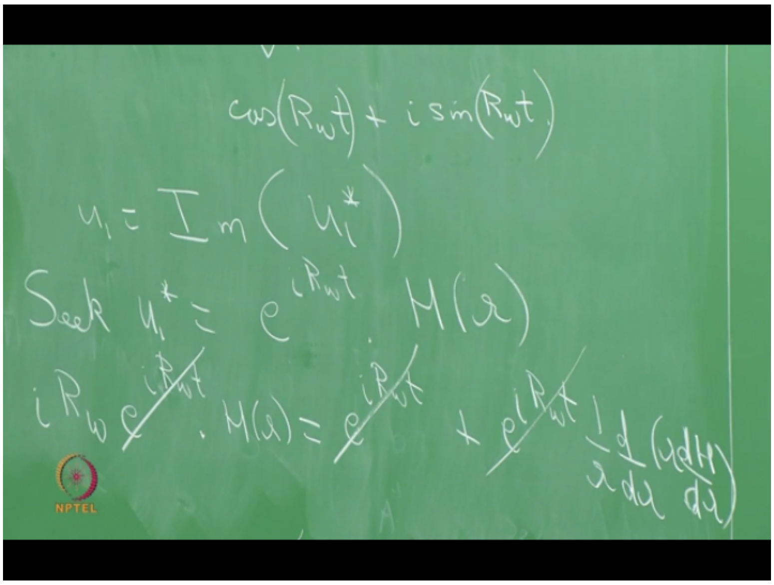

Du1 star/dt=exponential of iRwt+1/r d/dr of rdu1 star/dr. now, this is of course not the same as that problem. But, this is remembered e power i theta is cosine Rwt+i sin Rwt okay. Again from your knowledge of complex variables. So, if I actually found out u1 star and if I got the imaginary part. So, this is u1 star responds to this source okay. If I find u1 star and why do I want u1 star because when I now put the first derivative, I will get back my exponential.

So I am okay. I am happy with that because I can cancel of the time derivate and I have only my ODE and R, that means I can solve okay. So, after I found out the u1 star, I found out the imaginary part of it. Because the imaginary part is the one which corresponds to my source of sin omega t. So, basically what I am trying to tell you is, that u1 is nothing but the imaginary part of the solution of u1 star okay.

The u1 star is going to be complex, that is going to be real part, there is going to be a complex part okay. U1 star is in general complex, real part + imaginary part. The imaginary part corresponds to the sin term and that is what we are interested in okay. The real part corresponds to the cosine term but that is not of interest to me okay. So, clearly now as I am talking about the exponential term, I have a cross component and a side component.

So, this phase like business, phase difference business will be incorporated because I am taking both into account. Now, what I am going to do is, I am going to solve this problem, get the imaginary part, I get by what I wanted okay. How do I solve this problem? Seek u1 star as exponential of iRwt multiplied by some function H of r. I am seeking this as the usual separation of variables that you do. Now, I am going to substitute this back here.

When I substitute this back here, I get iRw I am differentiating with respect to time so I get that iRw times e power iRwt times H of r=e power iRwt+e power iRwt times 1/r d/dr of rdh/dr okay. Because H is of function only of r. Now, is this permissible choice? Is this a valid choice? It is a valid choice because now, the time dependent term e power iR omega t is present in all the terms and I can cancel of okay.

That was the reason what wanted me to go from sin to the exponential term okay. And now, I have my ordinary differential equation for H as a function of r. I should be able to solve this. If I am able to solve this, clearly I am going to able to solve this tomorrow. Because otherwise I would not have done this. I know H, I can substitute this back here. I get u1 star I get the imaginary part; I get my solution okay.

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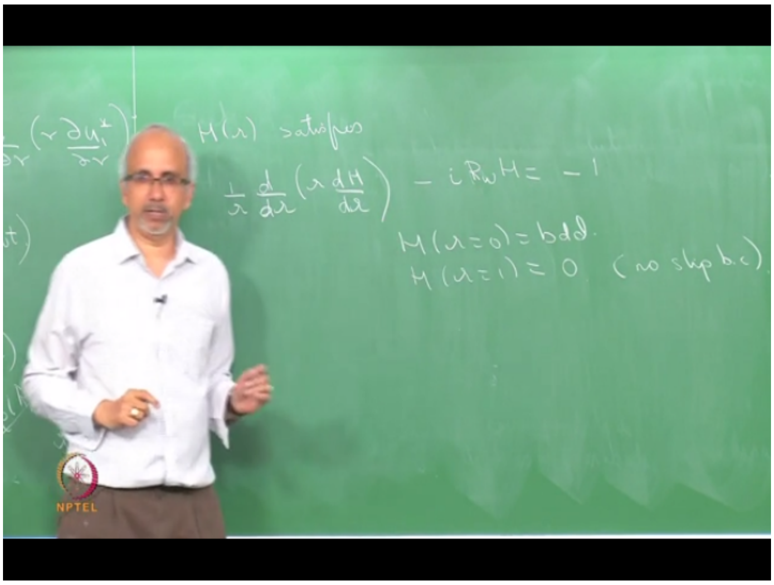

And remember so clearly H satisfies 1/r d/dr of r dH/dr I am going to bring that thing here -I omega H=I am going to take this -1 to the other side -1 okay. And now, I need to have boundary conditions on this. I will just write them down and we will stop. H at r=0 is bounded because only then my velocity is going to be bounded and my velocity has to be 0 at r=1 is 0. That carries from my no slip boundary condition okay.

Now, we need to be able to solve this. This is a linear equation, ordinary differential equation only thing is the complex number sitting here but we need to be able to take care of that. I just want to give a hint that this is an equation which you guys have seen before in your calculus scores when you are talking about Bessel's functions. So, the solution to this will be in the form of Bessel's functions okay.

And so that again is a nice close form solution what we get. That is my total solution. Then what we do is, this should be R omega okay. I think I need to stop because people are not correcting me. This is R omega. So, now in the limit of Rw being 0, I will know what the solution is. In the limit of Rw being small, I can do a perturbation series solution and then we will see. We will do that tomorrow okay. Thanks.