Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture – 13
Stresses on Deforming Surfaces: Introduction to Perturbation Theory

So, I like to welcome all of you to the 12th lecture of the course. What I will do is, I will just, we start working out some of the problems and the things like that in the second half of today's lecture. I want to just wrap up few things, but I think it is necessary, so that there is some clarity. The first thing I want to do is talk about the problem of the cylindrical surface and actually evaluate n dot T dot n.

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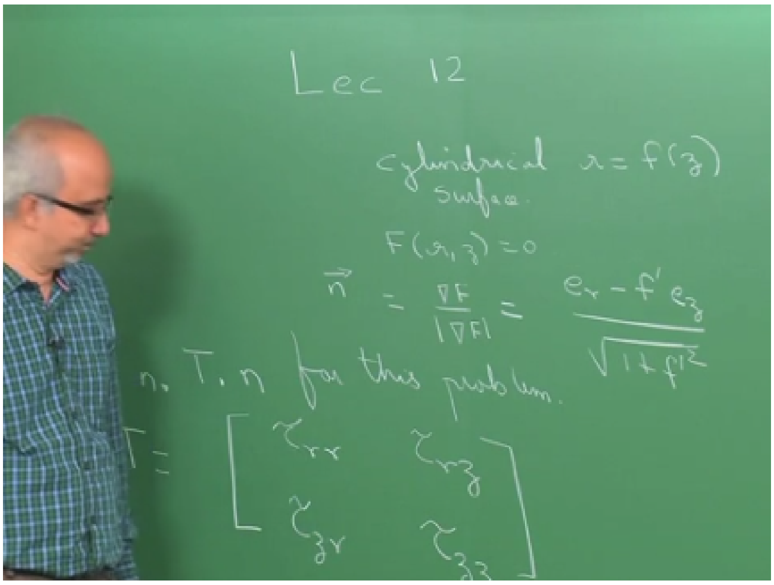

So, we want to have a feel for how it looks, okay. So, what we want to do is, we have a cylindrical surface like we had a last time. We just have given by r equals f of z, okay. This is the cylindrical surface and like you saw last time, what we do is we write this thing in an implicit form = 0 and the normal vectors happens to be the gradient of F divided by the mod of the gradient of F which in this case will be er – f dash ez divided by square root of 1 + f prime square, okay.

This we saw last time. What I want to do is, see when you actually working out a problem, you need to evaluate the quantity n dot T dot n to get the normal stress, okay. I just wrote it you that is the formula, but then you are working on a cylindrical coordinates or when you working in a problem in a rectangular Cartesian coordinates, you need to be able to evaluate it.

So, let us just do one case where we actually doing the evaluation and since we have the (()) (02:13) problem of cylindrical surface, we will just use that to illustrate, okay. So now, our job is to evaluate n dot T dot n, find n dot T dot n for this problem, okay. So, n is we already found and wrote the n and in order to, since we are talking about the very specific problem, I am not going to talk in terms of indices i and j.

I am going to talk in terms of r and z, okay and T remember will be just 2-dimensional matrix, I can write it as tau rr, tau rz, tau zr, tau zz. So, this would be the vectorial form by which you normally write, represent your stress tensor. We have assumed theta symmetry, so that all the dependency is gone and before I proceed further, I just want you to know that tau rr can be written as 2 mu times the derivative of the velocity component in r direction by r for a Newtonian fluid.

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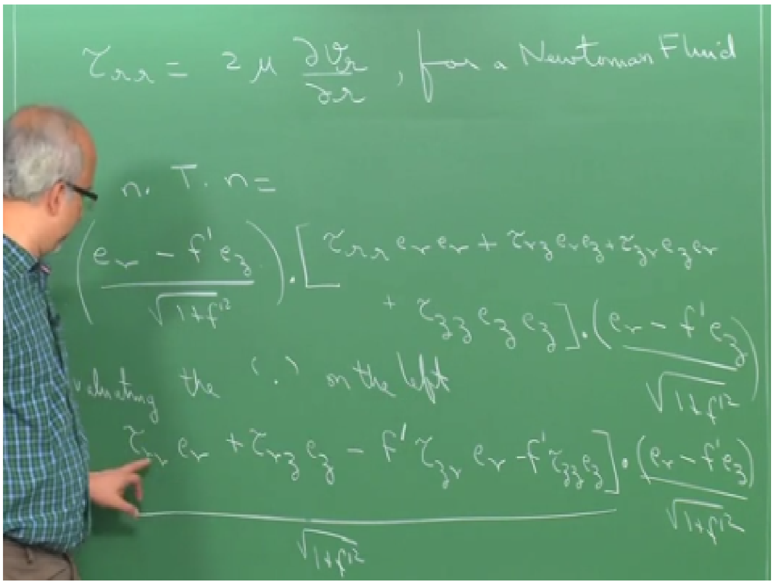

And similarly, you can express each of the stress components in terms of some velocity gradients or strains, okay and if you have a different rheological property, this particular relationship will change. Idea is that I have everything in terms of velocity now and my differential equation also has velocity, my boundary condition also is going to be in the form of velocity, so everything is consistent, okay.

Now, I want to evaluate this right. So, n is, how do I go about eval that, the 2 ways by which you can evaluate, both of them are equivalent, so n dot T dot n is going to be given by my unit vector which is er – f prime of ez divided by square root of 1 + f prime square dotted with my stress tensor. So, what I am going to do now is I am going to write my stress tensor, just expand it.

And each of this is going to be associated with a particular unit vector one representing the outward normal, one representing the direction. So tau rr for example, is going to be associated with er er, okay. It is a normal component. Tau rz is going to be associated with er ez, tau zr is associated with ez er and I do not like this tau zz is associated with ez ez. Again, it is dotted with er – f prime ez divided by square root of 1 + f prime square.

This is what I am saying when we are solving a problem, you have to be able to write n dot T dot n in terms of variables which are interested to you, which could be f prime and the velocity gradients, okay. How to evaluate this? I am going to take this dot product first, so this er dotted with this term, when I am doing this, I need to just look at the unit vectors adjacent to the dot.

So, I need to take this er with this er that is going to be unity, okay and I have tau rr. So, evaluating the dot on the left what do I have, tau rr er, okay. I have tau rz ez because er dotted with this er is going to be unity and I am left with this ez, tau rz stay as it. Tau rz ez, the dot of this er with the other 2 terms is going to be 0 because er dotted with ez which is 0, er dotted ez, so these 2 terms do not contribute, okay.

I am left with this term now. This term, - f prime ez dotted with this term is going to be 0, ez dotted with this term is 0, but ez dotted with this term is going to contribute and I have –f prime tau zr er, okay and again I have – f prime tau zz ez, okay that is just the dot on the left. The dot on the right is still to be evaluated, so I am just going to write that as it is, square and I must that is this denominator which is still there, okay.

Now a straight forward for you to evaluate this, this er dotted with this er is going to be give me unity, so I am left with tau rr, this simplifies to tau rr and then this er with this er is going to give me – f prime tau rz, okay. Clearly, this ez with this er is 0, this ez with this er is 0, okay. So, I got 2 terms here and now, I need to worry about tau rz ez with this –f prime, what did I do just now, tau rr er, I did these 2, right.

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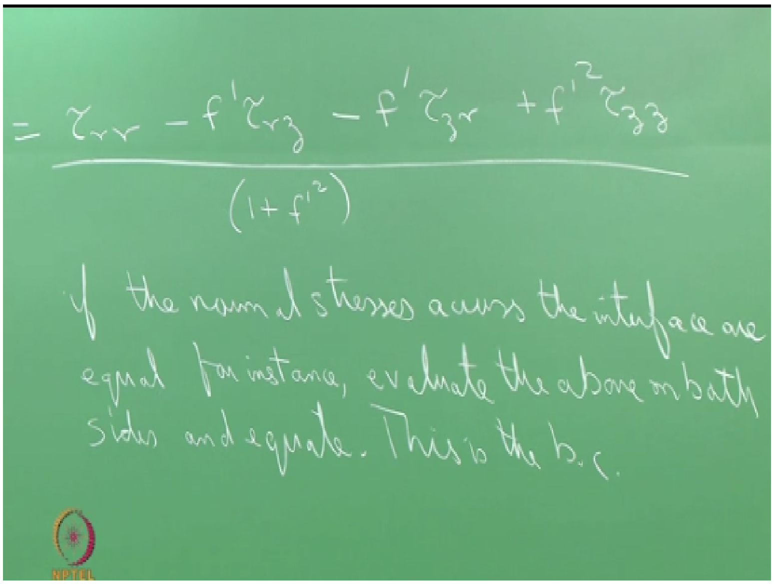

Tau rz with f prime, so I need to do tau zr with this –f prime tau zr and then I have +f prime square tau zz divided by 1 + f prime square. The square root and the square root together will combine and give me this. So, basically this is the normal stress balance or a normal stress component. So, when actually going to be solving an actual problem that n dot T dot n is actually useless.

You need to get this form. This form is what you are going to be using when you are solving a problem whether it is in cylindrical coordinates or in Cartesian coordinates or spherical coordinates, okay. So, well n dot T dot n is nice to put in a vectorial form when comes actual solving this is a guy was going to be used. So, this is one component. Supposing, you have a problem where you just say that the n dot T dot n from one phase = n dot T dot n from the other phase.

All you have to do is evaluate this in both the phases on the surface and equate them and that gives your boundary condition, okay. So, if the normal stresses across the interface are equal for instance, then evaluate the above on both sides and equate. This is the boundary condition. What is going to be different? On one side of the fluid you will use viscosity of the first fluid mu 1 on the other side you are going to use viscosity of the second fluid mu 2, okay.

And then you will go back to using this constitutive equation here, you will use mu 1, mu 2 and then use it, okay. So that is basically important when you are actually solving problems because this form of thing which you need and what you would do is, you would write the tau's in terms of the velocity gradient. Remember you Navier-Stokes equations is also in terms of velocities.

So, you have your unknown velocities, your boundary condition on velocities, everything is fine, okay and is important to do this because a boundary condition is usually an expression of a physical condition balancing of normal stresses, balancing of tangential stresses, so that is the reason why you need to find the normal component of the stress. There is another way, I will just mention what it is and then you can check for yourself that there indeed equivalent.

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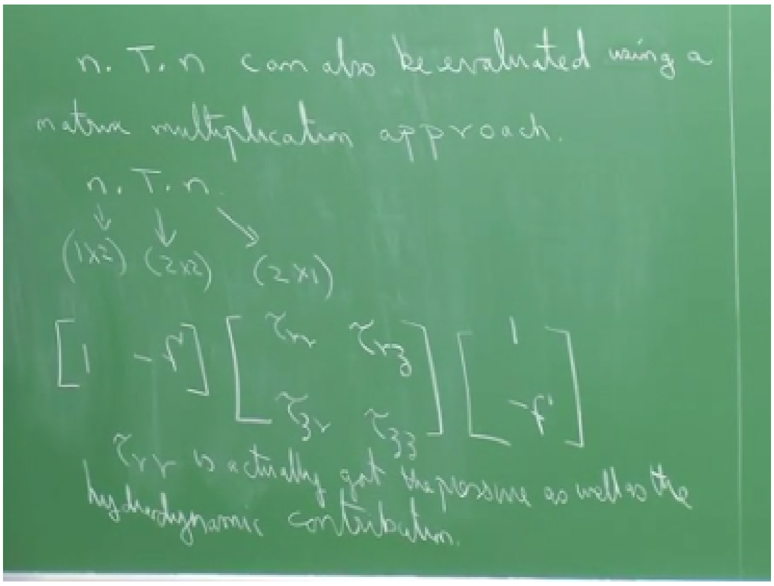

See n dot T dot n can also be evaluated using matrix multiplication approach. At the end of the day, I want to get the scalar, right. I want to get the component in the normal stress. So what is n? n dot T dot n, if I have this as a 1/2 vector and I am referring to my specific example which I solved just now. T is a 2 by 2 matrix which you saw already, okay and n can be written as a transpose and then if I write this n as a 2 by 1, okay.

You have already calculated n, it had an er and ez component. So, er is the first component and ez is the second component. So, if we write this as a vector, the 2 components and the matrix already know in terms of the tau rr, tau rz, n you already know again, but write it as a row, you write it as a column. So, get a 2 by 1, okay and you evaluate this, you get 1 by 1 scalar, okay.

Now, what I want you do is, which are the way you are comfortable with you are going to use, when it comes actually solving a problem, either a matrix multiplication approach or that approach, it does not matter, you will get the same result. So, you guys can do this and check if indeed getting the same result, I am not going to do this. So, what I am saying is, how do you write the n as er, the er component is 1, the ez component is –f prime, okay.

This is my 1 by 2 and the T is tau rr, tau rz, tau zr, tau zz and now, I am going to write the n as 2 by 1, which is basically 1 and –f prime, okay. I want you to understand that this tau rr is actually sigma because this is a normal component and remember what I did last time was, I actually when you did a force balance, you write this as sigma. So, I should have been careful right at the beginning where since I was it, I am just mentioning it now, okay.

So, remember the tau rr is actually got the pressure as well as the hydrodynamic contribution, okay. So, if you remember this is normal written in the form of –p + tau rr, -p + tau zz, okay and I want you do just separate out the p and you get a gradient of p term outside. **"Professor - student conversation starts"** Sir, yeah! For the component of n, there will be 1 divided by mod of square root of 1 + f prime square, yes, I have forgotten that.

You are right, so I want to need to do is I must remember, what he is saying is I have forgotten the normalization factor square root of 1 + f prime square and I must remember that there is a scalar which is 1 + f prime square here. Yeah, thank you, okay. "**Professor - student conversation ends".** This is coming from the normalization condition, okay. I kept telling myself I should use sigma that I did.

Just want to do the boundary condition and talk a little bit about the generalization of the boundary condition, okay and then, we will move on. So, you have already seen something like this earlier where we use the Young-Laplace law, okay and I believed was then for a cylindrical geometry. So, this has been done for a cylindrical geometry, the pressure here let us say P1 inside the thing as well as an outside it is P2, okay.

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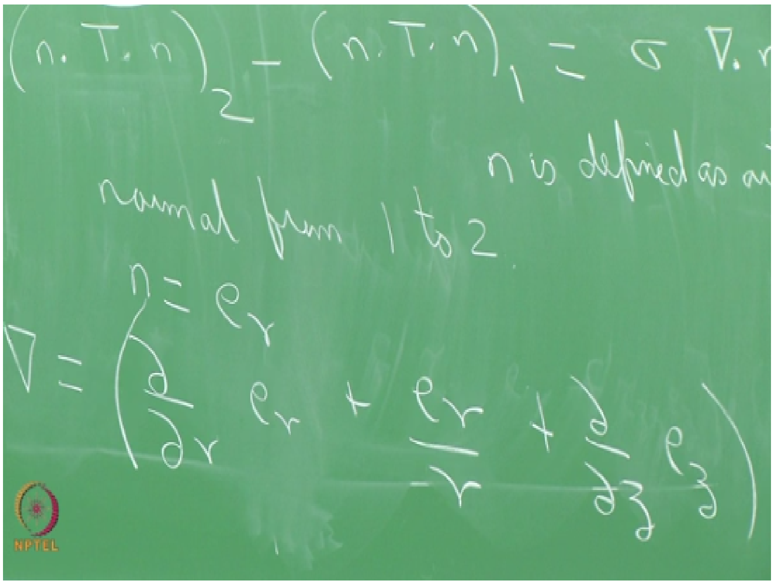

Now, what you have because of the curvature is, that is going to be a pressure difference, okay and you have P1 – P2 is given by sigma divided by r and r is the radius. In fact, what I should do is, I should write this as sigma divided by capital R, because the small r actually represents my radial coordinate, okay.

Now, I just wanted to, when I wrote down the boundary condition in the last class, what I did was, I just told you that n dot T dot n in the first fluid or in the second fluid, - n dot T dot n in the first fluid equals sigma times del dot n. This is the most general form of the boundary condition. I want to basically show that this particular condition that was derived last time using a work energy principle in terms of you know how much of energy stored on the interface when you change it by dr.

What is the work done then and equating that, that is how you got this relationship? So, rather than do a formal derivation of this which we will do later on in the course. I just want to show that this is basically a generalization of this and the way I am going to do this is evaluated each of these terms and show that it collapses to this and remember the way that the n is defined as outward normal from 1 to 2 that is the direction.

So, what is the direction of n, this is n, n happens to be equal to er because it is the radial direction, okay n is er and our job now is to calculate del dot n for this interface. So, what is the gradient operator in cylindrical coordinates is er + er/r + d/dz of ez. All I have to do is, get del dot n which means I must do the dot product of this with er, er dotted with er is unity. When I differentiate unity, I get 0.

So, this guy is not going to contribute. Er dotted er is unity. This r remains, forget that 1/r term here and I am going to be evaluating it on the boundary remember. So, therefore is going to be 1/capital R and ez dotted with er is going to be 0, okay. So, all I am trying is show here is that del dot n reduces to 1/R for this problem and we had a more complicated shape rather than you know worry about the interface, you would just do del dot n.

Possibly in your calculus courses, you always see the del dot n indeed represents the curvature. So, that is the idea here that del dot n is nothing but del dot er and that = 1/r and on the interface, we have 1/R because you are evaluating the boundary condition on the surface. So r = R, okay. What about the guys on the left, n dot T dot n. Now T, now I am going to be careful. How do I write T?

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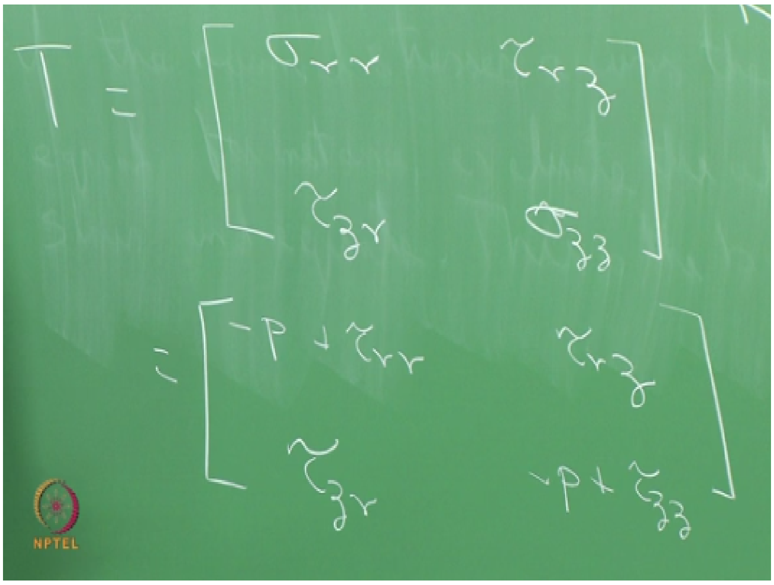

I am going to make sure, I do not make this mistake I made last time, sigma rr tau rz tau zr and sigma zz, okay. Just to tell you that in the normal direction, we are using sigma to represent the total component and now I am going to break this have been in terms of the pressure and that due to the flow, okay. So, now I am going to write this as – P + tau rr and this is what I sure, I have done last time – P + tau zz, okay.

So, since we are talking about scenario, our objective remembers you should derive the Young-Laplace law from this generalized boundary condition. This generalized boundary condition basically includes the effect on the flow as well. Now, we are in the case of the static situation, the only thing that is going to contribute is going to be the pressure terms in the diagonal element, okay.

These guys are going to be 0 because there is no velocity and if you want to now evaluate, n dot T dot n using whatever we did earlier, you will find that this is going to basically boil down just a pressure term, yeah. So, whenever if I want to substitute this particular form for the tensor T here and want to evaluate n dot T dot n using the method that I just discuss earlier what you will get is, -P2 coming from the first term and you will get – of – P1, +P1 coming.

And that is what which I want you to do, okay. I want to you to find out that this basically boils down to sigma divided by capital R, okay. On substituting all this, that is what we get. This is for the case when the fluids are at rest. So, what I want you to do, clearly is do the spherical analog. The spherical analog used the generalized formulation of this boundary condition and do it for a sphere and see what you get, okay.

And the relationship of course is classical, but I just want you to derive it this way. So, we get comfortable with this all business of evaluating the boundary conditions starting from this point, okay. **"Professor - student conversation starts"** we have defined n from 1 to 2, now if we have a complicated surface where n we defined it by capital F of r, z, then we should be careful with del F, right.

We can explicit in the other way also, the negative of that. That is the basis, I wanted to be explicit. If you want to define it from 2 to 1, then you flip the things on the left hand side. How do we make sure of that when we define a capital F of r, z. You have to make sure of that while looking at the way you defining the coordinate system. So, I think the question is how do you make sure that the outward normal is pointing from 1 to 2, okay.

The definition of the outward normal just depending upon the way we have define the surface in the sense, I am going to write it as R = f of z or z = f of x, so you are looking at when you write it as R = f of z, you are going in the radial direction. So, as you are going in the direction of increasing R, you are going from 1 to 2. So, I think if you are going in the other direction, it would be in the direction of –n, because n was equal in the way I define it.

N was in the R direction, as I was going in the direction n, I went from 1 to 2, this is going in the direction of the radius. If I want to go come in the negative R direction, then n would be automatically – of er, okay and then we are coming from 2 to 1. So, I think the best thing to do is to just make sure that you are going in one direction and calculate the normal and look at the sign, okay.

I think when we do a couple of problems, you will become clear, okay. "**Professor - student conversation ends"**. So, I think what we have done is we have actually established the framework in the sense we have got the boundary conditions, we have got the differential equations and since this course is basically on analytical methods. So, we will now talk about using some analytical methods and the basic concept is using a perturbation theory approach.

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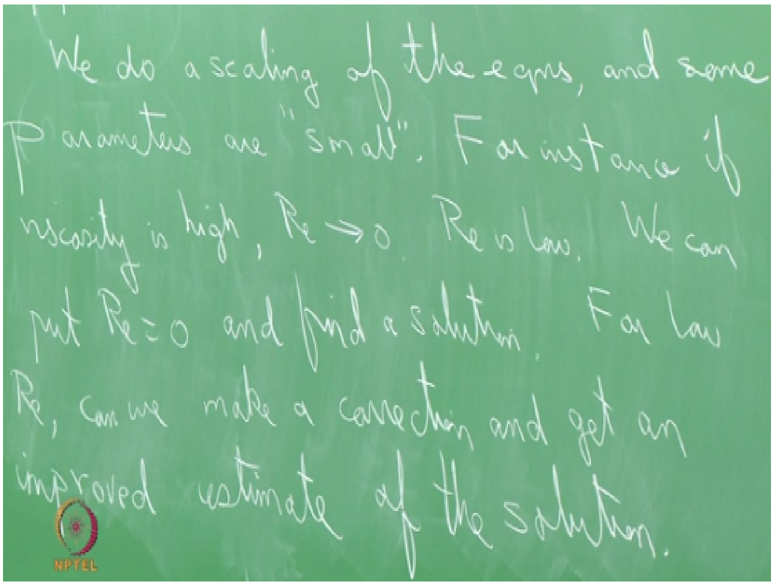

Now, here what is important is sometime back you had some couple of lectures on scaling, right. You transformed your equations in terms of dimensionless variables, you had Reynolds numbers, you had different kind of dimensionless groups which came up and under some conditions, these dimensionless numbers can have different orders of magnitude, so far example if your flow is highly viscous, okay then the Reynolds number is going to be very low.

And if the Reynolds number is very low, you can drop the inertial terms because the inertial forces are low. You can make a simplification. So, in the limit of Reynolds number = 0, you can possibly get a solution. Now, the question that arises is supposing the Reynolds number is not 0, but it does small but finite. You want to get a better approximation to the solution. How would do you go about doing that, okay.

So, what we want to do is, we want to be able to get some insight about a behavior of the system which in this case, is to be a flow problem. By looking at regimes, where some parameters could be small, so can I exploit the (()) (29:03) parameters as small to get some analytical solution, okay. I am going to illustrate this idea, this is a very simple problem now and then later on we will solve this thing with an actual flow problem

So, let us take, so idea is we do a scaling, okay of the equations and maybe some parameters are small. For instance, if the viscosity is high, then Reynolds number tends to 0, Reynolds number is low and can I actually use the (()) (30:14) Reynolds number is low to get a solution. So, one thing we can do is, we can put Reynolds number = 0 and find a solution. Why do I say that because when I put Reynolds number = 0, the left hand side basically contains the nonlinear terms, okay?

All I have on the right hand side are my pressure gradient and my viscous forces. So, that is linear system I should able to solve it, but the question is, is the solution valid for finite Re. Can I use the information that I have as a solution for Re = 0 and how can I use this information to make a correction and find the solution for a finite value of Reynolds number which is small? okay that is the idea.

So, this for low Re, can we make a correction, okay and get an improved estimate of the solution, okay because may be if I can give the improved estimate of the solution that I more confident rather than use the solution Re = 0, for Re = 10 or Re = 100, I would rather use this improved estimate to find out what the flow is when Re is 10 or 100, okay. So, how do you go about doing that?

So, we will take a very simple problem first to illustrate the idea and then, we will go back to solving fluid flow problems, okay. So, remember one of the important thing we are using in perturbation theory is you have to do the scaling, you have to make things dimensionless and you had a couple of lectures earlier on how to make thing dimensionless, so depending upon the values of those parameters, some parameter may be large, some parameter may be small.

If a parameter is large, you can treat the reciprocal of that as a small parameter. If a parameter is low, then you can use it as it is, okay. So, let us just look at very simple problem, just an algebraic equation. So, consider the algebraic equation x square – 1 equals 0, okay. So, everybody knows how to solve this, it might possibly do this in high school and you know the solutions are x equals + or – 1, okay.

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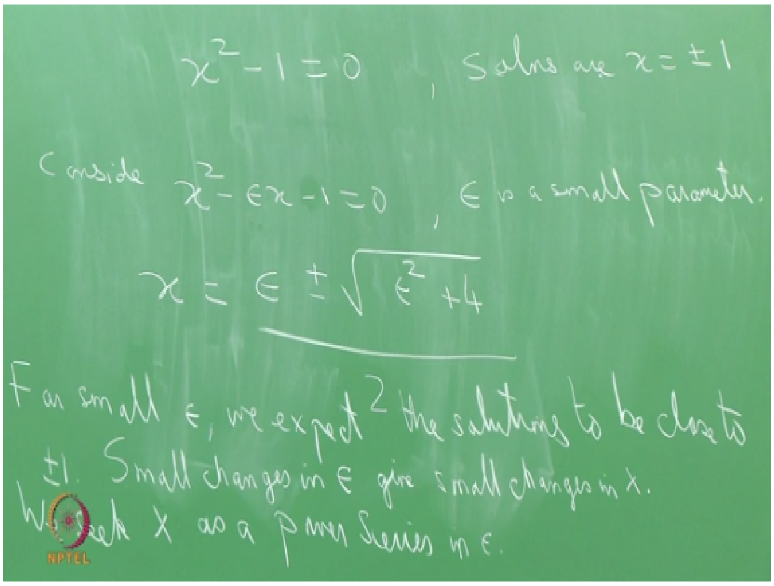

Consider now a modified problem, x square – epsilon x – 1 = 0. What I have done this is of course a fictitious mathematical problem, okay. I want to just illustrate ideas on the fictitious mathematical problem, then you can go back and do fluid flow problems where you can apply things. This epsilon is a small parameter, okay and here again, so epsilon is a small parameter and do you know the solution to this equation.

Of course, you know the solution to do this equation also. This x, the 2 roots the solutions to this equation is given by –B + or – square root of B square that is it. So in this case, of course, although you had small problem, is it okay, in this case, the problem is simple you have already know the solution, but supposing you do not know a formula, suppose this equation had been cubic or a 4th order equation, which contains this, may be you do not have an explicit relationship, okay.

In fact, if you have up to 4th order, 5th order, you do have explicit relationships, but we do not worry about this, but now the question that I want to ask here is, for the case where epsilon is 0, I know what my solutions are and is it possible for me to find out the solutions for this equation as because for small values of epsilon, I expect that my roots are going to be only slightly different from the case where epsilon is 0.

So, an epsilon = 0, have 2 roots + or – 1. When epsilon is small, I mean 10 per – 5, 10 to the negative 6, I do not expect that to be a very significant change, there would be a change, but may be not a very significant change. So, can I look at the variable x as a function of epsilon and look at the other, do a Taylor series expansion or do a power series expansion and seek the solution of x in terms of epsilon as a power series, okay, so that is the idea.

So, for small values of epsilon, small epsilon, we expect the solutions to be closed to + - 1. So, small changes in epsilon give me small changes in x, okay. Give small changes in x, the solution. So, can we do a Taylor series expansion or a power series expansion, so we seek x as a power series in epsilon. So, clearly you all understand that x depends upon the value of epsilon, okay.

We do not what the function is, but what we will do is, instead of varying it as f of x, we normally write it as a power series expansion, I am going to write the power series about a point which we have already know when epsilon is 0, x is + or – 1, okay. So, that is the idea when you are doing this perturbation series. So, we seek x as x0 + x1 epsilon + x2 epsilon square etc, okay.

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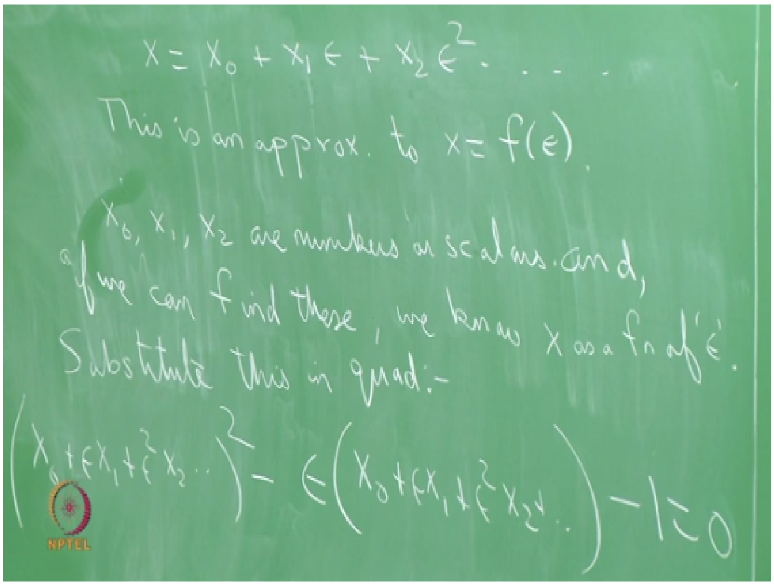

So, idea is x is the function of epsilon, okay. This is an approximation to x which is the function of epsilon. Clearly, defined what the epsilon give, defined values of x, so x depends on epsilon and the functional dependency is written in this form. What is x0, x1, x2, these are going to be numbers, these are going to be constants and if you actually can calculate what x0, x1, x2 are, you can actually calculate what x is the function of epsilon.

So, x0 in this particular problem it is very simple. So, x0, x1, x2 are numbers or scalars, okay and if we can find these, we know x as a function of epsilon, okay. So, whatever the arbitrary function, the function is given right there in terms of the square root sign, I am just going to write in terms of a Taylor series or a power series. So, how do you go about finding x0, x1, x2.

So, if x is going to be in this form clearly x must satisfy my algebraic equation, which I had. So, I am going to substitute this particular form of x in the original equation and I am going to invoke the fact that this particular thing has to be satisfied for every epsilon, okay for any arbitrarily epsilon. What that means is, I would get something like a power series and we would get equating terms of the same order of epsilon.

All the terms that of the order epsilon to the power 0, I group epsilon to the power 1, I group epsilon square a group, okay and that is the general approach. So, what we do is, we substitute this in that particular form there x square – epsilon x. So, substitute this in the quadratic and what you get whole square – epsilon multiplied by x2 – 1 equals 0, okay. I need x to satisfy the equation. So, this particular form has to satisfy the equation and substituting it here.

Now, you just need to expand all these and see what you get, so expand the quadratic term, the square term, I get x not square + 2 epsilon x not x1, okay + epsilon square which is going to arise from 2 x2 x0 + x1 square + higher order terms, I am going to neglect. What I have done is, I am just writing terms until order x1 square, okay. This is going to be coming from my x0 square + 2 this and some of you must tell me this is right or not, okay.

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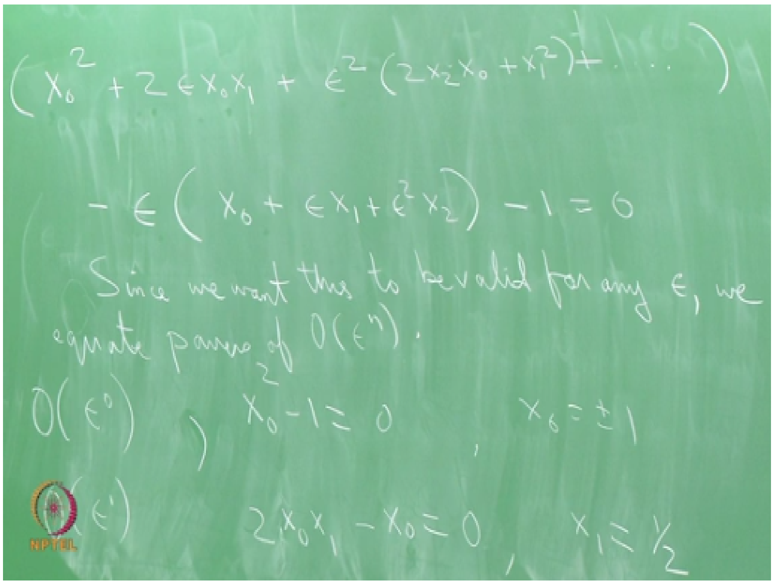

Then, I have the other term which is – epsilon times the same thing x2 – 1 = 0. I am going to now, I want this equation these terms to be valid for any epsilon, for any chose of epsilon, I want this to be valid, okay. So, what I am going to do is, I am going to equate terms since, we want this to be valid for any epsilon, we equate powers of the order of epsilon to the power n, okay.

So, I am going to group the terms that of order epsilon to the power 0. So, which are the terms that are independent of epsilon? It is just x not square and -1. This must be equal to 0 that is what it gives me and it gives me x not is + or – 1. What about order epsilon to the power 1, I have a term here 2 x0 x1, all these are higher order terms. I have 1 guy here – x not = 0, okay.

This gives me x0 is already found out, so this implies x1 is 1/2, okay and you can similarly calculate x2 by looking at the term which is of order epsilon square. So, now we look at the next term, the second order term and at order epsilon square, the equation which has to be satisfied is 2 times x2 x0 + x1 square – x1 = 0 and we already know x0 and x1 and so what I can do is, I can use this to find out x2.

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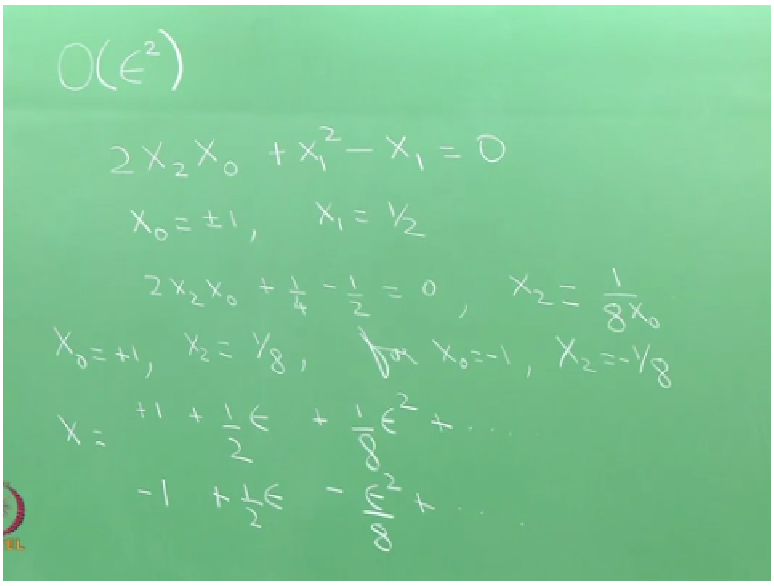

In fact, we know that x0 is + or – 1 and x1 is 1/2. So, what this gives me is when I substitute these values, I get 1/4th here, so I get 2 times x2 times x0 + 1/4th – 1/2 = 0, which implies that x2 = 1/2 – 1/4th, 1/4th divided by 2 1/8th, 1/8th of x0. Remember, x0 can take + or – 1 as 2 values. So, when x0 equals +1, I have x2 equals 1/8 and when x0 for x0 equals -1, we have x2 equals -1/2.

And now, since I know x0, x1 and x2 that I substitute the values in my power series and this gives me the 2 expressions, x as being + 1 + 1/2 of epsilon, where x1 is +1/2 and x2 is +1/8 of epsilon square + etc. and the other root gives me -1 + 1/2 of epsilon – epsilon square divided by 8 + etc. So, these are the power series expressions for x in terms of epsilon which we have obtained using the perturbation series method.

And depending on the level of accuracy that you want, you would take you know higher order terms should you be more interested in getting more accurate values. So, when epsilon is 0, I get back + or – 1 which are my roots. My first order correction is the same for both, but my second order correction has a different sign. So, x was given by epsilon + or – square root of epsilon square + 4 divided by 2.

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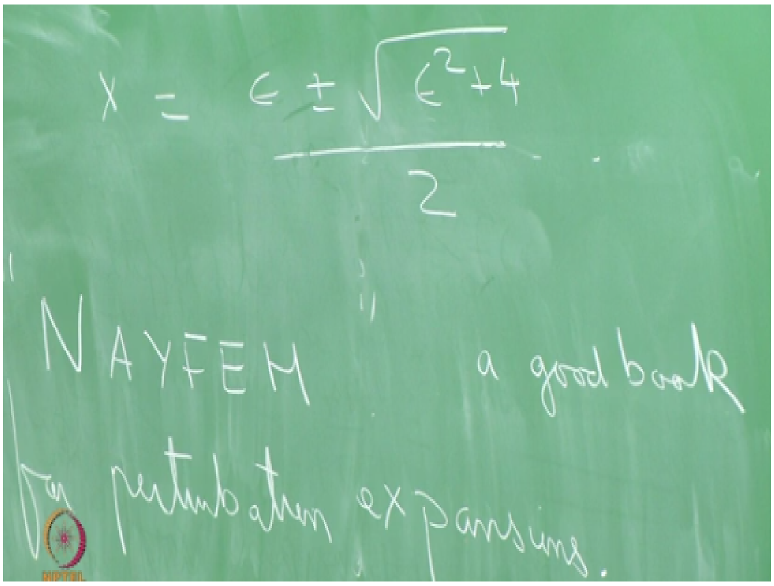

This can be expanded in the form of a binomial series, right and you know how it is, epsilon square + 4 to the power 1/2, you can factor things out and you can do this. I want you to do that and see if it boils down to this expression that we have and then you will get a better feel for what exactly is going on. So, one way to do an approximation to this particular exact solution is to do a binomial series expansion, take a few terms in epsilon.

The other way is to do what we did, which is just assume a power series expansion and you get the solution and you know when you see that both of them are equal which is how they should be, then your convenience the things are working fine. There are of course some limitations to this. There are times when this method is not going to work, okay and those are things we will see as we go along.

One particular book which deals extensively with perturbation series expansion is authored by Nayfem and I would you know recommend those of you are interested in understanding this a bit more deeply, this is a good book for perturbation expansions. So, what I have done today, just give you an idea about how this perturbation method works and what you will do when any problem where you have a small parameter, you can actually go about exploiting the existence of the small parameters and finding solutions.