Multiphase Flows: Analytical Solutions and Stability Analysis
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Lecture - 11
Normal and shear stresses on arbitrary surfaces: Force balance

Okay, so like to welcome you to the lecture today where we got to talk about how to find shear stresses and normal stresses on surfaces which are arbitrary, okay and the basic motivation for the problem is as follows.

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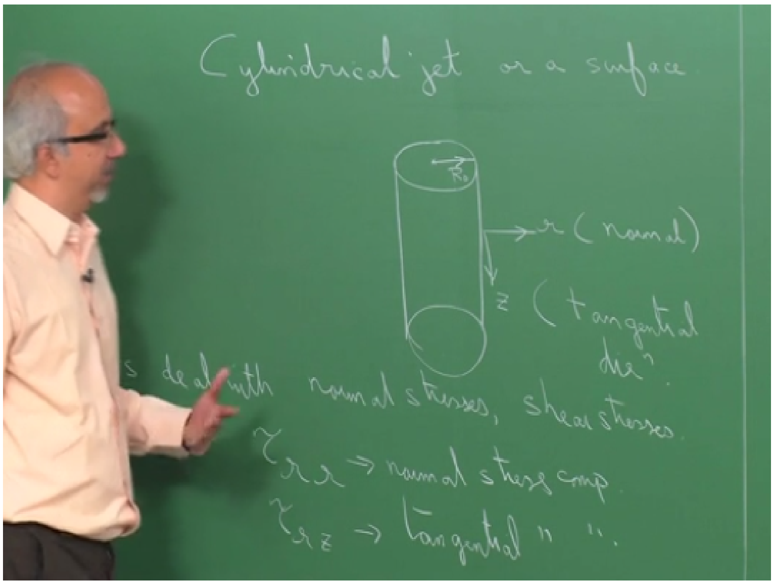

If we have a cylindrical jet or a cylindrical surface and that is a circle, okay of radius R0 and if I am looking at this curved surface the direction of the outward normal is going to be in the radial direction, okay and the direction of the tangent on the surface is going to be in the axial direction which is Z, okay. So this is the direction of the normal and this is a tangential direction.

Now normally what we have to do is we have to apply boundary conditions on surfaces of this kind and these boundary conditions usually have a physical significance. So the boundary conditions would invoke things like continuity of shear stresses that is tangential stresses or would involve normal stresses, okay. So when you write the boundary conditions which we will be doing later on the course you will be the boundary conditions deal with normal stresses and shear stresses.

So in this case the normal stress component is going to be given by tau rr this is the normal stress component, okay. And what about the tangential stress component or the shear stress component that is going to be given by tau rz for example. the tangential stress components are going to be given by tau rz.

So, when you are going to write the boundary conditions you are going to write the boundary conditions involving tau rr which you going to relate to velocity gradients depending upon the constitute of relationships. Suppose for the Newtonian fluid you will write it as a constant viscosity multiplied by the gradient of the velocity, okay. So that is what you would do. And since these are the standard directions r and z it is easier for you to actually know what these quantities are, easily compute them, okay.

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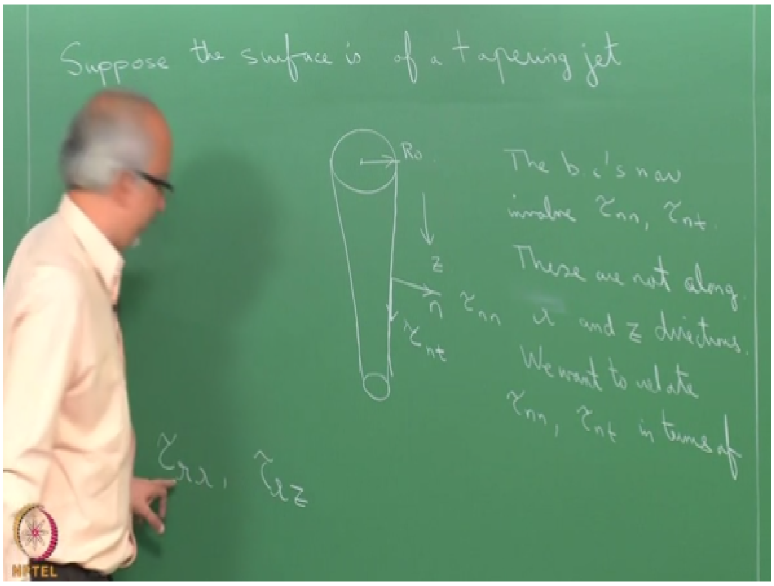

But suppose now, suppose the surface is of a tapering jet and so now what is happening is the liquid is accelerating and the jet is getting tapered as becoming small and smaller. The area here is R0 and R0 keeps decreasing as a function of z, okay. This is the z direction. So now if you were look at the surface and now when you are going to apply the boundary conditions these boundary conditions are also going to be in terms of the normal stress component and the tangential stress components.

But what about the direction of the normal component is going to be in this direction, this is you draw a tangent plane and then you draw the normal to that so this is the direction of the outward normal, okay and what is the stress component that you are interested in? If this is the vector n the stress component you are interested in is tau nn. That is our normal stress component at that surface, at that point.

What about here, the tangential stress component is going to be tau nt, okay. The point I am trying to make here is that tau nn has both the component in the radial direction as well as in the axial direction. The tau nt also is having a component in the radial direction as well as in the axial direction. So one of the things which you would be doing is you would be using tau nn and tau nt when you have arbitrary shapes of surfaces.

So you want to be able to calculate what tau nn is what tau nt is. And what we want to do is we want to relate tau nn and tau nt in terms of the classical shear stresses around fix coordinate system. Around the x direction or the r direction or the z direction or the y direction, okay. So the idea is the boundary conditions now involve tau nn and tau nt these are not along r and z directions. I like the earlier case, r and z directions, okay.

So, what we want to do is we want to relate tau nn and tau nt in terms of tau rr and tau rz. These are the classical directions that we have. So r theta z for the cylindrical coordinates is the, are the different axis and what we want to do is we want to be able to relate the normal stress and the tangential stress components, okay. I think on an arbitrary surface to this.

The reason is the differential equations they contain the shear stress components in the classical directions tau rr and tau rz and I also want to have my boundary conditions also in the same tau rr, tau rz. So, what we want to do that for is to relate tau nn and tau nt in terms of tau rr and tau rz, okay. So that is the idea that is the motivation as why we are doing this. So what we will do is we will do 2 approaches.

First we will use the physical approach and we will get this relationship but then you will see that the physical approach involves using a specific geometry and getting a specific relationship. What you want to do is you want to have a mathematical framework by which you can generalize this so that you can actually apply it across any arbitrary surface, okay. So that is the idea.

So, first we will do the physical approach try to get this relationship between tau nn, tau nt in terms of some classical known component tau xx, tau yy whatever it is tau xy and then we will generalize it, okay.

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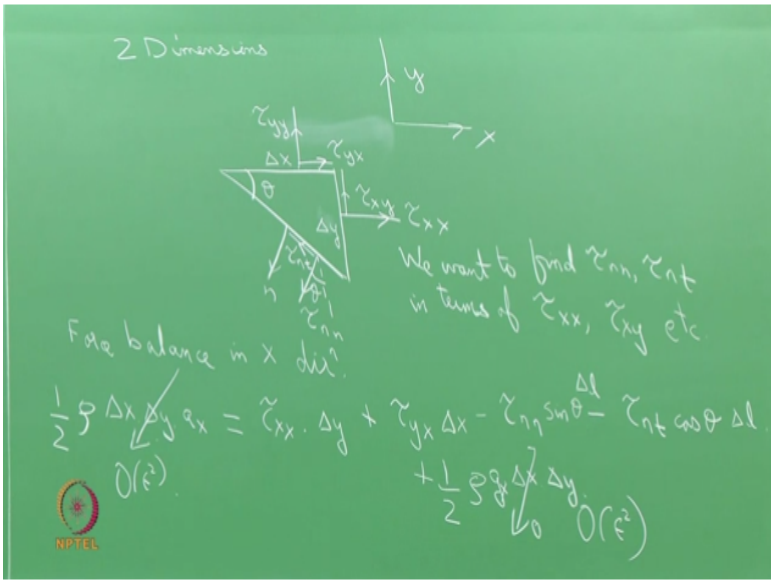

So, let us use, now we are going to simplify things and one simplification we going to make is we going to use a rectangular geometry, okay. And we are going to make a second simplification which is we going to look at things in 2 dimensions, okay. And look at 2 dimensions. So this makes it easier for me to draw pictures on the board. So it is going to be a better picture that what we had yesterday, okay.

So what we are going to do is x and let us say that is x direction and this is the y direction. And I am going to take a triangle okay this is definitely in the direction of the y and this is in the direction of x y. So this is my arbitrary surface so this side of my triangle represents my arbitrary surface the normal to this is going to be given by this direction, okay that is perpendicular I have drawn.

And let us say that this is theta. What is this height? This is delta y and what is this length here along x direction that is delta x, okay. What are the stresses which are acting on the vertical surface here? This stress is basically tau xx because tau normal in x direction and the direction is also x and this is tau xy which is in this direction. Normal and the y are both positive.

So while writing all these stresses they are all positive, okay. The y direction is positive upwards this is positive that way. What about here? I have stresses here which is tau yy which is in this direction and I have this surface here which is tau yx, okay. Now, what I want to do is basically find out what are the, what is the normal component of the stress here and the tangential component of the stress here and related to this.

So, the normal component of the stress here I am going to denote that by tau nn. So, this tell you it is on a surface whose normal is in the direction n and in the direction of n. And tau nt is let me just say it is upwards for tau nt is this way, okay. This is tau nt. So if I had a boundary of this kind and if I wanted to write my boundary conditions I would write the boundary conditions in terms of tau nn and tau nt.

But what I want to do is I want to be in a position where I can write this boundary condition in terms of tau xx, tau yy, tau yy, tau yx, okay that is what we are trying to do. So our job now is to relate this to these stresses in the classical direction, okay and the way we are going to do this is by simply a force balance.

A force balance which tells you that, in this triangular element which is infinitesimal, we look at all the forces we are acting on the system and say that that should basically give raise to an acceleration of the fluid element inside this, okay. So basically we look at the acceleration, okay so the force balance in the x direction is something here before I proceed. We want to find tau nn and tau nt in terms of tau xx, tau xy, etc. That is the job, okay.

So let us do the force balance in the x direction what we have, sorry, yes in the x direction that is right. Basically, the forces acting on the system must be equal to the change of momentum is your mass time acceleration, okay. So I am going to write the easy part first. Suppose the fluidic element is actually accelerating with let us say an acceleration in the direction of x given by ax.

Then the acceleration of the system so this because is the triangle the total volume element is half times delta x times delta y, okay. And acceleration of the system is ax. What about the force acting on the element here? You have tau xx which is acting on delta y and you have tau yx this is in the y direction. So we do not worry about this. We have tau xx acting on this we have tau yx which is acting over delta x.

And what I want to do is I want to resolve these forces arising because of this stress components in the x direction, okay and clearly the angle between 2 lines is = to the angle between the perpendiculars. So what does that mean? If I want to draw a vertical line here this angle is going to be theta, okay. This angle is theta so the component of the tau nn in the x direction this is the negative x direction is going to be – tau nn sin theta.

Cos theta is vertical and then I have tau nt has to go with the cos sin and that is also associated with the negative sign cos theta, okay. You also have the body force which is acting on the system which is going to be given by half times delta x times delta y. And what I have done is I have also forgotten something here which is the length element dl, okay so the length is delta l.

I need to mention that this is multiplied by delta l and this is multiplied by delta l, okay. Because this is stress and you just look at it as if you are looking at the unit depth into the board so that the other dimension is just unity, okay. So, this is the equivalent of the delta x. So, now you have everything consistent. This is also gx, yes. But I am not going to worry too much about it and you will see why.

The reason I am not going to worry about that this term and this term is in these 2 terms the one on the left and the last term on the right they are given by a product of 2 infinitesimal quantities delta x, delta y, okay. All the other terms are of only a single power. So since these are basically infinitesimal quantities these are all of order epsilon. So the first term and the last term of actually are of order epsilon square all other terms of order epsilon.

So, basically what this justifies and allows me to do is to drop these 2 terms and I am just going to write this here, this is of order epsilon square and that for this goes off with 0 and this goes off for 0 because this is of order epsilon square. And all these other terms are of order epsilon, okay. The y component balance a balance of the forces again in the y direction.

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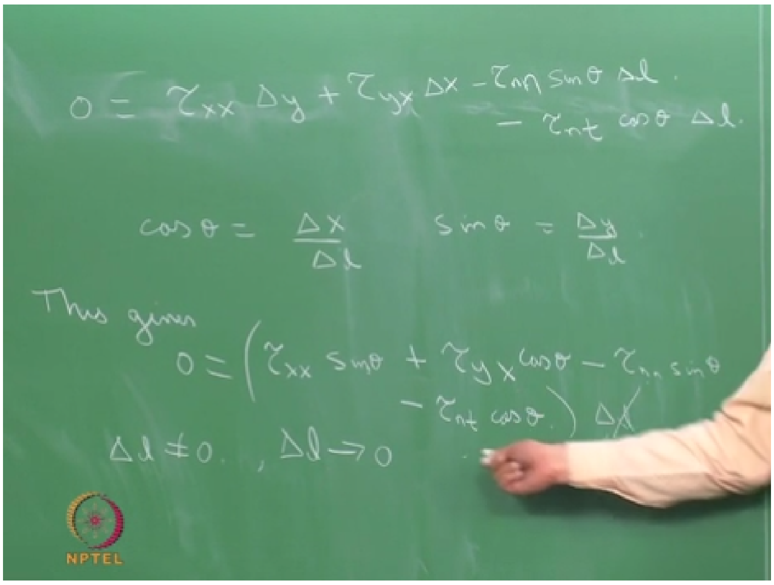

Okay, before I do the solving maybe I should simplify this and write the x direction balance. 0 as tau xy, sorry tau xx delta y, tau yx delta x and – tau nt with the sin theta delta l – tau nn. I got the other way, sorry cos theta delta l, okay. That is my balance. The other thing I can do is I can relate delta l to delta x and delta y, okay by using my trigonometry.

So, what is cos theta? Cos theta is delta x/delta l and therefore sin theta is going to be delta y/delta l. And what I will do is I am going to write delta y as sin theta multiplied by delta l and delta x as cos theta multiplied by delta l and this gives 0 = tau xx times sin theta + tau yx cos theta – tau nn sin theta – tau nt cos theta and the delta l I can take it out because delta l is not = 0. It is only tending to 0, okay.

So, this goes off and so that means this may force balance in the x direction, okay. So, delta l is not = 0, remember that. Delta l only tends to 0, good. So, now I need to basically do the same thing in the y direction because I have one equation remember I know tau xx tau xy and all these quantities here I know tau yx I know tau xx. I have 2 unknowns, tau nn and tau nt. I have only one equation.

I need one more equation and the other equation comes from the force balance in the y direction, okay. So the force balance in the y direction going to give me, again we are going to neglect the order terms. So, I am going just write the only the contributions arising from the forces on the surfaces, okay because they are of order epsilon.

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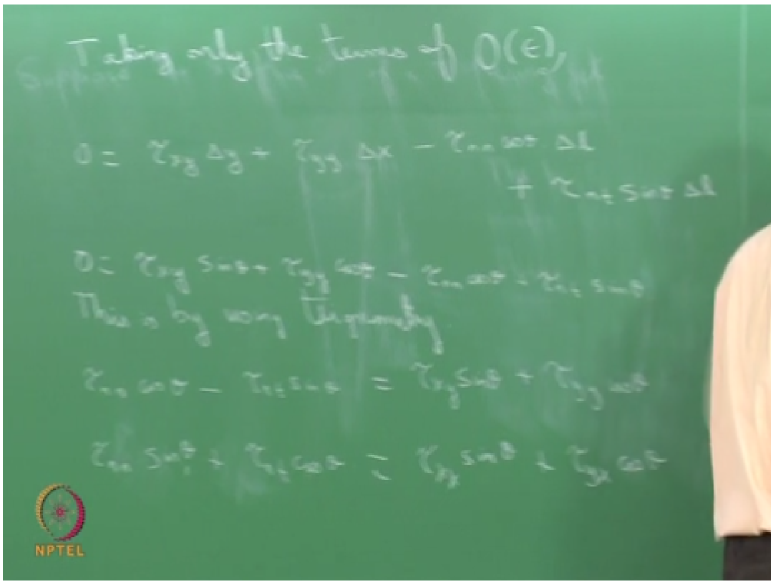

So, taking only the terms of order epsilon we have 0 = tau xy multiplied by delta y + tau yy multiplied by delta x, okay. And I need to do the resolution of the forces arising from this surface in the y direction and clearly these guys are the positive direction the way I have drawn this is outward normal. So this is going to be tau nn cos theta but it is pointing downwards.

So, it is going to be – tau nn cos theta and I will remember to write my delta l and I get – tau nt. The tau nt sin theta component is going to be in the upward direction so that is going to be associated with the + sign sin theta times delta l, okay. Again do the same thing, eliminate delta x and delta y by using our trigonometry get everything in terms of delta l and what we have 0 = tau xy times sin theta + tau yy times cos theta – tau nn cos theta + tau nt sin theta, okay.

So, this how would I get this? This is by using trigonometry, okay. So, now I have 2 equations and 2 unknowns and the 2 unknowns that are tau nn and tau nt. What I want to do is I can therefore solve for this right. The fact is you note tau xy tau yy from your constitute of relationship from, if it is a Newtonian fluid you know tau xy is there is derivative of the x component of velocity with y component of velocity with x.

So you know all these, okay. So I am going to move do a little bit of rearrangement of these equations and I am going to write this as tau nn cos theta – tau nt sin theta, I am moving this to the left hand side = tau xy sin theta + tau yy cos theta, okay. And this is my component balance let me write next component balance the same way move these guys to the left I have tau nn sin theta + tau nt cos theta = tau xx sin theta + tau yx cos theta, okay.

So, these are right if I not made any mistakes. So, what we are going to do now is you know I got 2 equations and 2 unknowns we can solve this using anything that you learned in mathematics earlier but I am just going to do one more step and leave it.

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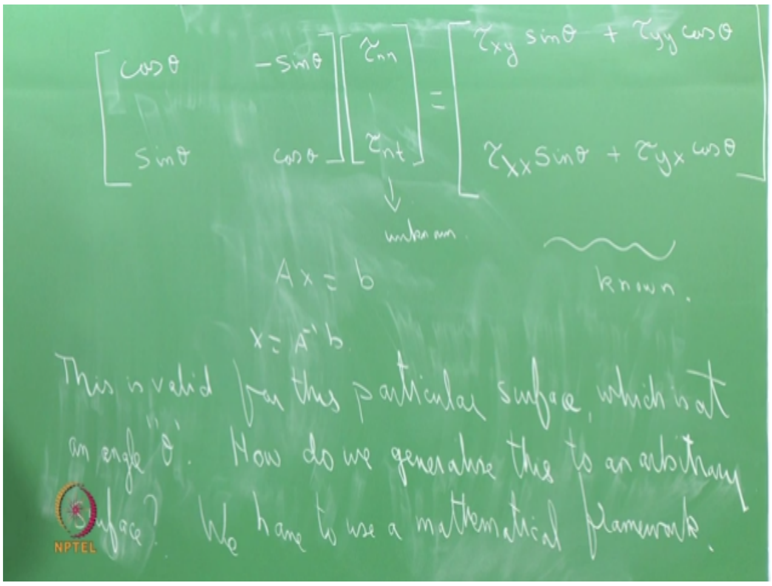

I like to write this in a vectorial form which is, I am going to write the left hand side as cos theta – sin theta and sin theta cos theta multiplying this vector tau nn tau nt, okay = tau xy sin theta + tau yy cos theta tau xx sin theta and + tau yx cos theta, okay. So, the idea is very simple, I mean what I have done is I am just written those 2 equations in a vectorial form. So this is a vector now, okay.

This is of the form Ax = b and what we can do is you can find x as A inverse b. So, I am not going to do that that is something for you people to do. So, the unknown is this. So, clearly the left hand side has my unknown tau nn and tau nt I want to find out tau nn and tau nt in terms of my known quantities tau xx, tau xy, tau yy, etc. That was what we started off with. So that is what we are doing here, okay. So, this is known and this is the unknown, okay.

And x is clearly A inverse b and so you should able to find out what this tau nn and tau nt are in terms of my classical directions x and y, okay. You can of course do the same thing in 3 dimensions also then you have a 3 by 3 matrix, okay. Now, what is the problem with this method? This is of course a method which works and you have of course found the solution for a surface which the arbitrary angle theta incline, okay.

We just plot have an angle theta to the x axis which what we started off with. But clearly what you want is I mean when you are solving a problem you cannot be actually do not know what the surfaces. Most of the time what is going to happen is you would not know what the surfaces, okay. So this is the reason why we want to go away from this physical approach. The physical approach gives me an expression which is valid for this particular specific case.

But what I like to do is, I like to generalize this, the form of some kind of mathematical equation or a mathematical formula which helps me to consider things for an arbitrary surface, okay and that is what we want to do, okay. So, can this is valid for this particular surface which is at an angle theta, okay. So, how do we generalize this to an arbitrary surface? That is the question and what I am saying we have to use a mathematical framework.

I just want to emphasis that this mathematical framework is basically to help you do the generalization, okay. So you know kind of have a formula so you can just do your calculation very quickly. I mean you cannot be in a position where you are actually sitting and drawing surfaces and doing the argument of what the forces are and I am trying to do this relationship, okay and that is what we are going to see now.

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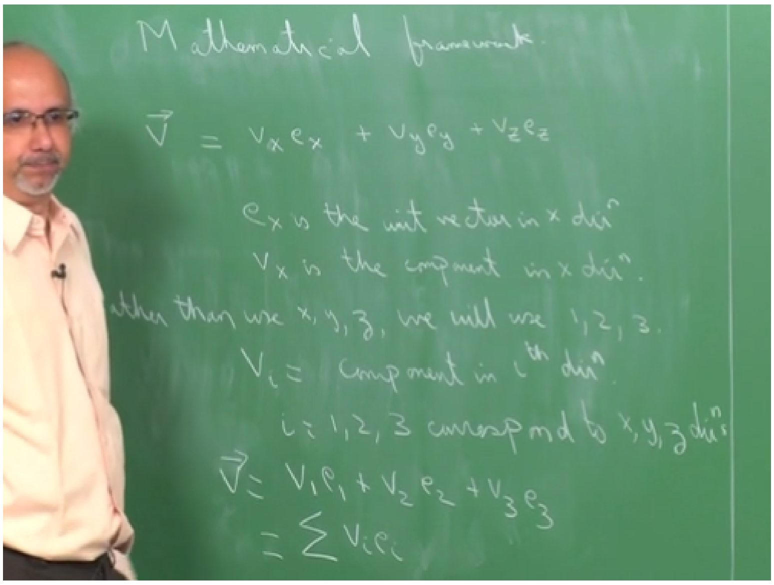

So before we do this generalization to this mathematical framework, so let me just write a few things. It will be a small recap of mathematical framework. When you have a vector, let us say a velocity vector what is the typical thing you are possibly used in a course in physics? You would write this as Vx times ex, ex is the unit vector in the x direction, Vx is the main component in that direction + Vy times ey + Vz times ez, okay.

Now, rather than and these are the ex is the unit vector in the x direction and so on and Vx is the component in the x direction, etc. So, rather than talk in terms of x, y and z what people like to do is like to talk in terms of indexes. Because that is kind of helps you make things more compact, okay. So because you have R theta z in the cylindrical coordinate system in a spherical coordinate system and there are several other coordinate systems that have different variables.

So, what we want to do is we are going to write rather than use x, y, z we will use 1, 2 and 3 the index, okay. We will use 1, 2 and 3, so I am going to say Vi I mean it is basically the component in the ith direction. So the component in the first direction and the first direction could be x for the Cartesian coordinates it could be R for the cylindrical coordinates and so on and so forth, okay.

So, what we can do is we can write the vectors suppose using x, y, z I want to write in terms of indexes 1, 2 and 3 okay. So, let me just say that, let me just write one more step here. So i = 1, 2, 3 correspond to x, y, z directions, okay. I would like to therefore this velocity of vector now becomes V1e1 + V2e2 + V3e3, okay. And what you would like to do is you would like to write this in a slightly more compact formats summation of Viei, okay.

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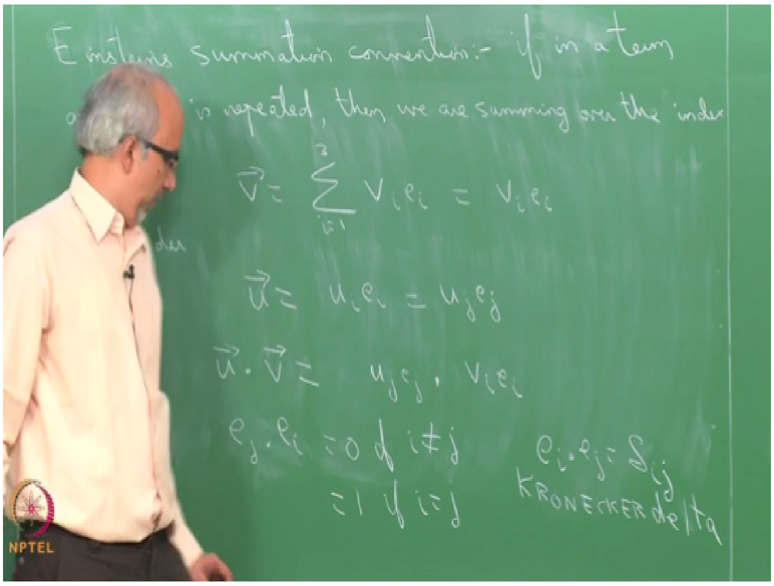

I just honor to introduce this concept of Einstein's summation convention which basically tells you that it just says that if in a particular term you have an index which is repeated that means you are actually summing over that term, okay. So if in a term an index is repeated then we are summing over that index, okay over the index. So rather than write the velocity vector as I written that sigma Viei, I can just write this as Viei term.

This is what you would have normally done, okay. So, here in this particular term the index I is repeated so it automatically implies that I am actually summing over all the i's. So I do not have to explicitly write the sigma sign. So, whenever you see a term with the repeated index that means you are summing, okay even if the summation sign is missing. Now, suppose you have another vector u which is therefore going to be given by ui ei, okay consider U.

Now, how do you find out the dot product or the product of these 2 vectors, okay. What I would do is since this is being summed over i rather than write it in terms of i that is now interested in writing this finding out this dot product. What I am going to do is, I am going to use a slightly different index uj ej because it does not matter. It was just an index, so this also going from 1, 2, 3 that is also going from 1, 2, 3 okay.

So, I am going to write this as uj ej dotted with Vi ei. Point I am trying to make here is that this dot product is going to contribute only when the 2 directions are identical, okay. If j is not = I the dot product of ej and ej dot ei is going to be 0. Only j = i you going to have a contribution to the dot product, okay. So ej because normally we are working with directions which are perpendicular. So ej dot ei is = 0 if i is not = j and is = 1 if i = j.

I can write this entire thing in a very compact way by saying that ei dot ej = delta ij, which is the Kronecker delta, okay. So the dot product ei dot ej gives me delta ej which is Kronecker delta and the Kronecker delta is defined as being = 0 if i is not = j = 1 if i = j.

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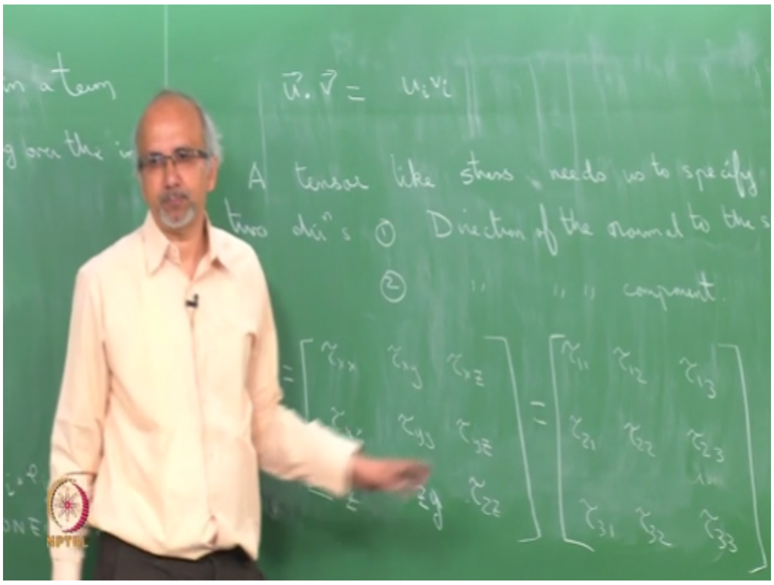

So this dot product now is going to contribute this dot product U.V and therefore we written as since it is going to be contribute only if i = j, I can write this as UiVi. So, what this means is you are summing over the corresponding components of the 2 vectors, okay. Now, you are all comfortable with the way the vector is represented in terms of the unit vector because only thing you need to specify the direction of the vector.

When it comes to tensors you need to specify not only the direction in which the component is acting but also the direction of the surface on which it is acting. So, basically what it means is you need to specify 2 quantities, okay. So, basically a tensor like the shear stress or like the stress sorry, the stress components the stress needs us to specify 2 directions, I mention this earlier.

The direction of the normal to the surface and 2 the direction of the component, okay. That is what, you have the 2 indexes, the first index told you the direction of the normal the second index told you the direction of the surface was normal or tangential or whatever it is. So, now when it comes to writing the stress tensor what you are used to is writing this as may be in the form of a matrix where you write this as tau xx, tau xy, tau xz.

This as tau yx, tau yy, tau yz this as tau zx tau zy. Z is different in different places know? It depends on how much I have to bend. And this is tau zz, okay. So, basically this is the way you possibly seen the stress tensor earlier written in the form of a matrix. So, rather than having a one dimensional vector component you have a 2 dimensional matrix, okay and this case it is 3 by 3.

What I want to do now is rather than talk about it in terms of x and y I am going to go back to using my notation of indexes, okay. So, indexes why this would mean tau 11, tau 12, tau 13, tau 21, tau 22, tau 23, tau 31, tau 32 and tau 33, okay. That I did for the vectors instead of drawing about x, y, z I am just doing 1, 2, 3. This 1, 2, 3 could be depending on your coordinate system whatever.

So, I now want to write this again because see when I am trying to do my mathematical framework I want to be in a position where I can actually represent the directions of different components. That is tau 11 for example represents the component which is acting on a surface perpendicular to 1 also in the direction of 1, okay.

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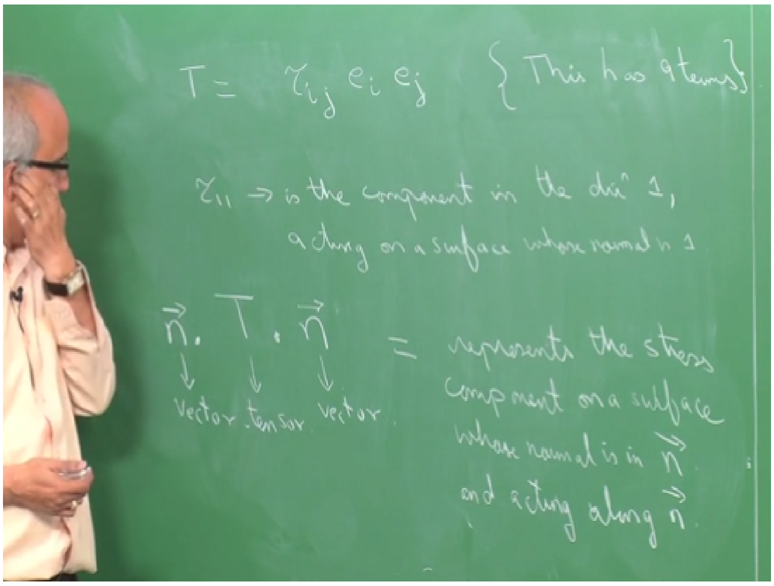

So, basically what I am going to do is I am going to write the stress tensor T as tau ij ei ej.
So, what does this mean? So remember this particular term has both the i component being repeated as well as the j index being repeated. The i index is repeated, the j index is repeated. That means you are going to summing over both i and j, okay. You first keep i constant some over j and then i and then some over j again.

So, basically you will have total of 9 terms, okay. So, this basically has 9 terms, okay. So, what I am doing is I am just trying to tell you that tau xx or tau 11 is the component in the direction 1 acting on a surface whose normal is also unity, okay. That is what it is. I will just do make one particular statement now illustrate it and then we will possibly stop. I like to define just like we have the dot product which was defined.

I like to define the particular, this particular quantity, will explain this. I just want to illustrate you what is in store for you on Monday, okay. So we have n.T.n and I want you all evaluate this. What does n represent? n represents the normal. So n is a vector, okay this is a vector and this is also a vector and this is a tensor.

What we want to show here is that n.T.n represents the stress component on a surface whose normal is in the direction n, okay and acting along n, okay. So what we will do is we will talk about how to evaluate this in the next class and you will then find out if this formula actually works for what we did during the physical argument. Using a physical argument, we got the expression for tau nn and tau nt earlier on, okay. That was just by doing a force balance.

So in order for you to verify that what I have written here is actually correct you would evaluate the term on the left and then verify by whatever we did earlier, okay. That would be one way. Now if you were to interested in getting the tangential component on a surface whose outward normal is n then what you would do is you would just do n.T.t.

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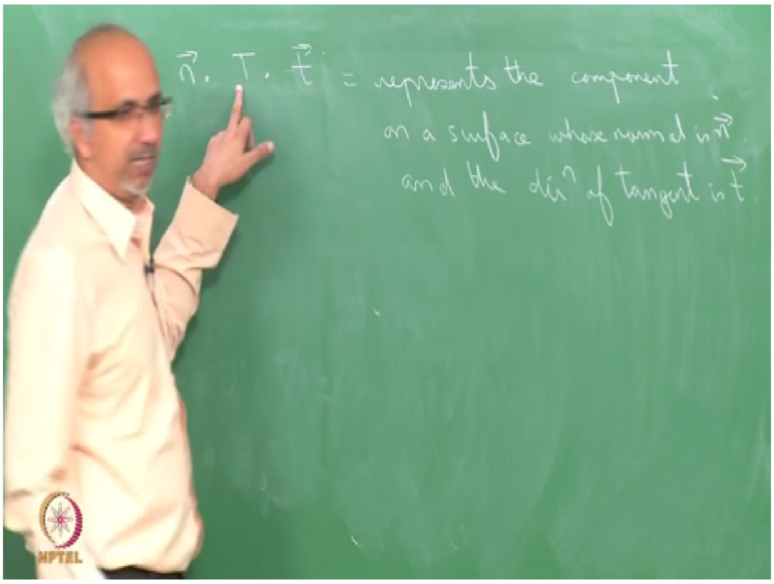

So this is the direction of the tangential vector represents the component on a surface whose normal is n and the direction of the tangent is t. So, what this allows me to do is if I have an arbitrary surface if the surface is given you can from calculus you can actually find the direction of the normal and you can find the direction of the tangent. So if you have a surface y = f of x or r = f of z in the cylindrical coordinate system you can actually find the direction of the normal and the direction of the tangent.

So on that surface if you want to find the normal stress component I would use n.T.n and if I want to find the tangential stress component I would use n.T.t. These would basically be because this is in terms of my classical directions x, y, z. So what I am doing is basically whatever ended earlier in the class today by using a physical argument but now I will just use this formula to get my components in the normal direction and in the tangential direction, okay. Thank you.