Multiphase Flows: Analytical Solutions and Stability Analysis
Prof. S. Pushpavanam
Department of Chemical Engineering
Indian Institute of Technology – Madras

Lecture - 10
Vector operations in general orthogonal coordinates: Grad., Div., Lapacian

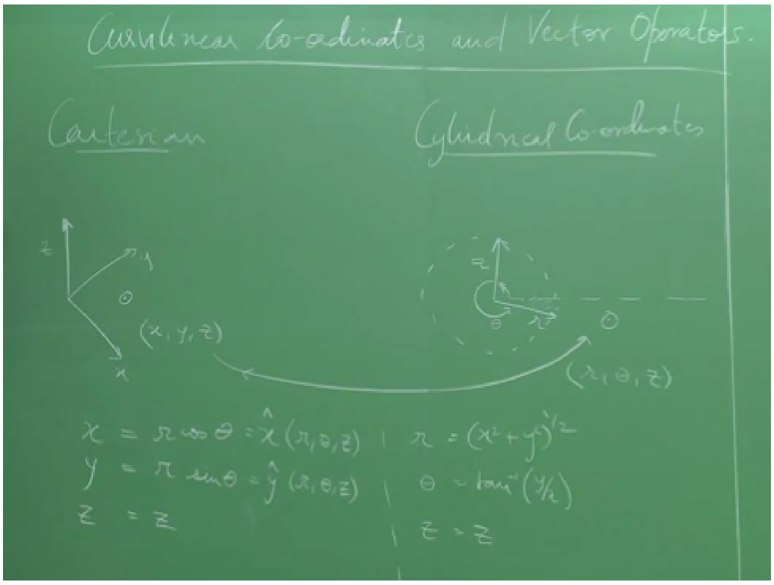
So, good morning, everyone and welcome to today's lecture. So in the previous few classes we looked at a few problems in different coordinates systems. So we derive the equations in the Cartesian coordinates but very soon we found ourselves having to deal with a spherical bubble or a cylindrical jet of fluid. In which case the idol coordinate system was no longer the Cartesian coordinate system and we moved into cylindrical coordinates and spherical.

And there when we had to use the gradient operator and the divergence we realize that those operators do not have the same form as they do in the Cartesian. So naturally the question will arise as to how one can derive these operators in a coordinate system of choice. More specifically, a question was raised by one of you all as to why the gradient operator for example in cylindrical coordinate has a certain form.

But the divergence which is the gradient dotted with a vector has it entirely different form. So what we will do in the first half of the class is actually look at the theory underlying these curvilinear coordinate systems and how one can derive systematically all the operators, all the vectorial operators. The second half of the class we will continue with the previous lecture where we looked at Perturbation theory.

And solve a simple problem which you all have, which you all have quite comfortable with but will solve it.

(Refer Slide Time: 01:48)



So to start off, to make it easier to understand I will work with this specific case of cylindrical coordinates and then we can generalize that later on. So in one hand we have the Cartesian set of coordinates which you are familiar with maintaining the right hand system of x cross y = z. And on the other hand for specific example we have the cylindrical coordinates. So, on the xy plane I now have a radius r.

This theta would actually be negative. So I measuring theta in the anti-clock wise direction. So, actually I should not put it that way. So the fundamental idea about having different coordinate systems is that there should be a one-to-one mapping between all the points in this space and all the points in that space. And that are totally reasonable because the space is the same we are just using a different reference frame.

So, that means if I have a point here whose coordinates are x, y and z they should get mapped to unique point which will in fact be the same point in space but having a different representation as r theta z. And this can be mapped back and forth. So naturally the starting point is to understand what this mapping is or in other words if I have the x, y, z coordinates how do I get the r theta z and then things will follow from that.

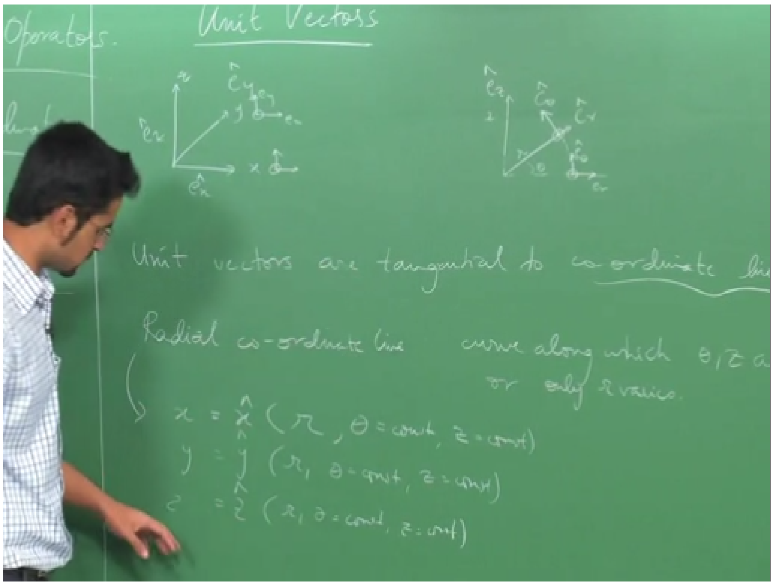
So, in the case of cylindrical coordinates that is quite straight forward. So x is simply r cos theta, y is r sin theta and z of course remains as z. So when you are working with a new coordinates system the first step is to always get the mapping 0. So if you have a cylindrical z and you envision this coordinates system the first thing to do is get this mapping.

There are some cases where you will want instead of having a z you may want to have a torus which means that you have a cylinder which also curves. So in that coordinate system you can also proceed in the similar fashion but you will have to start with a different mapping. So the even z will get mapped. Now this is the mapping where I am able to right x as a function of r theta z and y as a function of r theta z and so on.

Z of course remains like that. So, we just focus on x and y r theta because z is just identity map. Now, as I asserted if you can go forward we have to go backward. So, there will be the inverse mapping where I can write r in terms of x and y. So usually we require that as well in the calculations. So that we can write down and all can work it out with me. So here r should be just x square + y square, square root and theta is of course tan inverse of y/x.

So this you would be familiar with and there is no mystery there. Now what this means in terms of representing vectors and the Navier–Stokes equations is what we look at.

(Refer Slide Time: 06:46)



So, here we know how to represent any point. So any scalar function defined here will easily get mapped to this reference ring because we know how to go from xyz coordinates to r theta z. Now the next step is when you dealing with vectors is apart from the magnitude of course you have a direction. So the question is if I know the direction of a vector here, how do I represent that in the cylindrical coordinate system?

Or in other words if I have a vector pointing in the x direction for example then here the vector is simply that. But if I look at it here then that vector will be in some r and some theta direction. Or it will be a combination of the unit vectors and r and t. So that is why we need to understand how the unit vectors follow from when we make the coordinate system transition.

So of course in Cartesian, I just draw it again, the unit vectors are straight forward you have ex cap who is with magnitude 1 of course in the x direction and y you have ey and z you have ez. When we come to the cylindrical coordinate system, right, you have 1-unit vector in this z direction same as before. The r unit vector is pointing in the radial direction and theta is actually so if I draw a kind of circle here theta will actually be tangential to that circle.

So at any point the unit vectors at this point are radially outward tangential to the circle which is e theta and then z, ez vertically up. Now the point here and the crucial reason why the operators change and have different forms is that as I move around in the Cartesian space the unit vectors do not change. But the unit vectors in this cylindrical coordinates system actually keep changing as I move around.

So, for example if I am located at this point then my er vector is this guy here and e theta is this guy pointing perpendicular to er. But when I move from here to here you see that my direction of the unit vector er has now changed. So has the e theta vector. But the same thing if I did here in Cartesian's space I would get ex ey and it will still remain ex ey, right, there is absolutely no change.

Of course, it will displace the origin but that is not we are looking at. We are looking at the direction of the vectors and they remain the same. So the question is to understand how this unit vectors change with position and that is the first step and then we can look at deriving the gradient operator. So to understand that we need to, we able right down the vector say er in terms of ex ey and z.

That is the main step and we would like to do that because we know that these vectors do not change. So if you can represent er in terms of these and we can understand how it is changing. So for that we need to have the basic idea that unit vectors are tangential to coordinate lines and lines here in general could be a curve. Just the same. So, what this means is that it tells us what exactly the er vector is for example.

So, the er vector is in fact some unit vector that is tangential to the radial coordinate line. So what I mean by a coordinate line? A coordinate line is a curve in space along which any 2 of the coordinates are constant. So the radial coordinate line is the curve along which theta and z are constant or only r varies. So that is quite clear because along this line which I am asserting to be the radial coordinate line.

You see that only r will change whereas, theta as a constant value and of course z has a constant value. So, similarly you can find that the theta coordinate line is a line along which only theta changes which are simply the concentric circles in the plane of the board and z of course are different planes. So this kind of a picture you would have had intuitively in your mind. So now we are just formulizing that as coordinate lines.

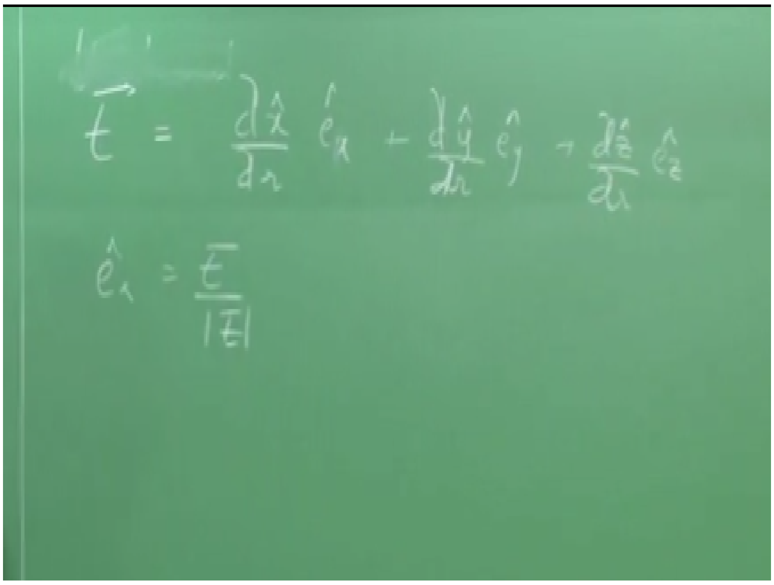
I am sorry, the z coordinate line is not planes they are just the z direction that comes out. So in x, y, z the coordinate lines are simply the Cartesian access themselves. But in the cylindrical frame is slightly different, so you have the radial line then of course this tangent e theta along which only theta changes, basically it is the circle and then you have just z side vector pointing out to it.

So, mathematically we can use this form that we have to represent any coordinate lines. So looking at the radial coordinate line it can be represented as x is of course I will leave it in the general forms some function of r because we know r is changing. However, theta as a constant right and z is the constant. Similarly, the y coordinate along the radial coordinate line will be given by this where again you have to put theta constant and zr constant and the same is to for 13:57 z.

So what this tells me is that if I want to find one of my radial coordinate lines all have to do is come to this mapping. Give some constant value for theta. May be 45 or in terms of radiance pi/2 or pi/4 and then zu gives some value may be 3. And now you will just vary r and it will trace out the coordinate curve and this case it is the trivial radial line but in a more complicated system if you cannot figure it out visually you can always do this and understand what exactly the coordinate lines.

So you can figure the same thing out for theta and z. So having established that the next question is we coming back to is what is the unit vector?

(Refer Slide Time: 14:54)



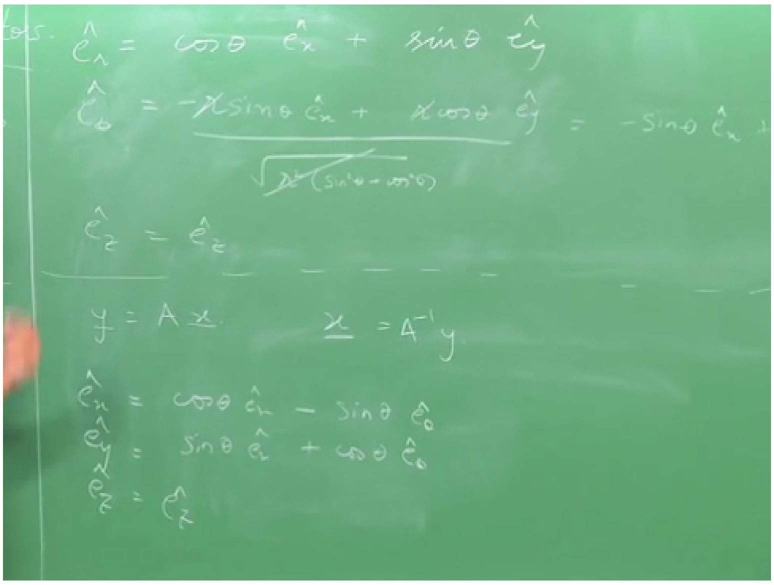
So, a little inspection will tell us that the unit vectors as I have said here, I already written that down. The unit vectors are tangential to the coordinate lines. So that means that all we have to do find out the unit vector er is to find a tangent, a unit tangent to the radial coordinate line. Now, the reason why I have written it in this form is that this gives us a particularly easy way to find the tangent.

So, if you look a little closely at this figure, I mean at these 3 equations what you will realize is that it is nothing but an explicit representation of a curve. Which means that I have given you the x the y and the z coordinates in terms of one parameter. In which case this is r and we know from basic theory of calculus that I can find the tangent which I will call t to the surface which is given by this and by simply taking the derivatives.

So what we are doing here is the x, y and z coordinates our changing only with the variable r. And if you want to find the tangent all you need to do is take the small infinitesimal distance that gets propagated as I change r by a small value and that will naturally lead to the different components of my tangent vector. So that is quite straight forward to do because we remember we know what x hat is what the y mapping is what the z mapping is.

So all we need to do is just take this derivative and we will get t. And now er is simply t by its modulus or say absolute values so I can get unit vector. So proceeding in the same way we can find out e theta and ez and that will be quite easy to do. So what we will do is apply this formula to calculate er and e theta and then take the next step.

(Refer Slide Time: 17:44)



Alright, so let us work out er. So the first is just the derivative of x with respect to r. Apologize, I should have put partial derivatives here. It is in general they are functions of any of r theta and z. Alright, so er dx/dr. So, that is simply cos theta. So this is cos theta ex plus sin theta ey and of course because derivative of z with r is 0. So, that means that my er unit vector lies only in the xy plane which is consistent of course.

Now e theta and this is possibly the unit vector which was the least obvious when you first encounter the polar plane. So now we can calculate that even if we could not intubate it. So now we have the derivative with theta. So you will get, am I done here? No, so I need to divide by the magnitude. So I will do that and but that simply r. So essentially what happens is that the r's cancelled out.

So I will just get – sin theta ex + cos theta ey and ez is of course ez, right. So, here we have our 3 unit vectors er e theta ez in terms of the unit vectors in the Cartesian reference frame. So that is the crucial point of proceeding with any coordinate transformation. So, now we can go to the next step which is to try and relate it the other way round, which means can I get ex zy easy in terms of er e theta.

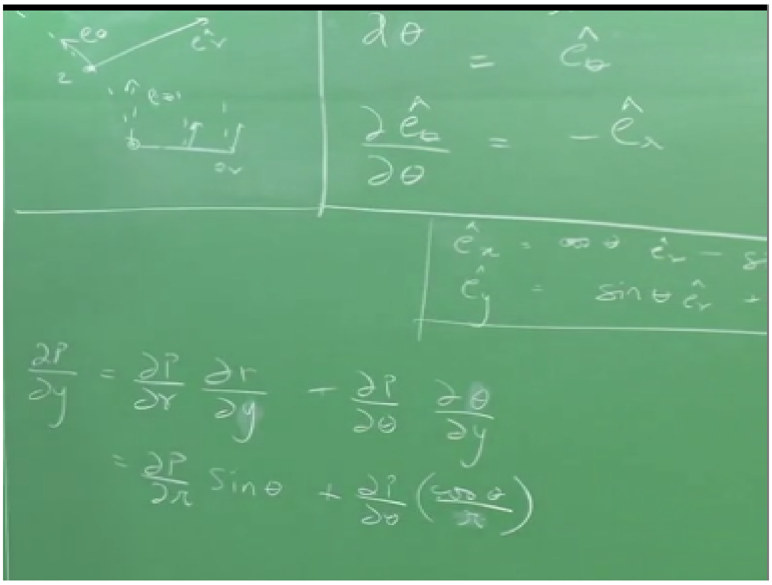
So that is not too difficult because what we have here is a system of 3 linear equations right with 3 unknowns. So all have to do is invert it and I will get ex zy and ez. In other words, what I have done here is saying something like y = Ax. Where y is the 3 cylindrical unit vectors and x vector is simply the Cartesian vectors. So, I just have to say that x is A inverse y. Where this A will have this cos theta sin theta and the other terms.

So you can work that out but since this is relatively simple we can right down a solution by inspection which is what I will do now. So maybe I will give you all a minute to tell me what ex is in terms of er and e theta. So you are saying sin theta er + cos theta. So you can of course check here. I do not think this is going to work out. Just double check that. So according to me, right.

So of course I will do the easy part first which is to say ez is ez. So of course let us look at it again, what I know for sure is that the left hand side, right hand side should be er so naturally ex should have a cos theta terms so I can retain the er guy. So if I put cos theta here, I think that is work out. Right, so if I substitute ex here I will get cos square theta er and yes ey will give sin square theta er. So er remains and you will that the other terms cancelled out, fine.

So now we have represented back and forth just as we had the forward and backward mapping for the points we now have that mapping for the unit vectors as well and that is puts us in a good position because we can now calculate how the vectors er e theta ez change as we move around space in the cylindrical coordinate system.

(Refer Slide Time: 23:46)



So, let us look at the er vector. So I will draw it here again for reference. So we have er now here if I look at a point I have e theta here in the cylindrical coordinate system. Z point 1, point 2, so at point 1 my vector er is here. So as schematic shows us that as I move along r for example my vector is do not change. So as move along r er remains the same right and e theta also sort of remains the same because I will just have, right.

So, e theta and er and ez do not change with r. However, theta is a troublesome guy because as I move in theta you will find that my er vector has changed instead of pointing horizontally it is now pointing at some inclined angle. So definitely the er has changed with theta. e theta of course also has change now instead of going vertically it is now going at some oblique angle. So I know that schematically my unit vectors going to change with theta.

So let us see if we can calculate that from what we have just derived. Let us look at the derivative with respect to r for starters of er. Or let us just make it more interest in directly jump to theta. So we are looking at the derivative of the unit vector er with theta and now it is extremely useful to use what we have derived here because I know that ex and ey do not change.

So I can substitute er as using my relationship here into that derivative calculate the derivative and come back. So let us put er this representation into the derivative and tell me what the derivative is in ex and ey. But that we can see from here is nothing but e theta. So now we have the neat result that the derivative of the er vector as I move in theta is simply e theta. So you can do the same thing for e theta with respect to theta and tell me what you get – er, exactly.

Of course the unit vector in the z direction and this case does not change. I just write this over there because I may have needed in the future. Okay and now we are in a position to begin our investigations into the vector operators. So let us start with the gradient of a scalar. So since we are going to be calculating gradients of pressure in this course I will just take P.

So, we know that the gradient pressure in the Cartesian coordinate system is simply dou P/dou x ex. And the question is how do I relate this to its representation in the cylindrical coordinate system. So the first idea that we need to have is a vector is a vector as a vector. So that means is that just like you have a point in physical space that point is not going to change regardless of how you are going to view the system whether cylindrical or Cartesian.

In the same way when you have a vector entity that is entity that does not depend on how you look at it. Which means that whether I represent it in Cartesian or whether I represent it in cylindrical those forms have to mean the same thing in terms of magnitude and in terms of direction.

So what I am looking to do when I am translating the cylindrical is really find out how to write this guy in terms of r derivatives of r theta z, in terms of the unit vectors er e theta ez while maintaining its original structure and what it really is. So that in this case will involve 2 things first of all I need to transform ex, ey, ez to the representation of er e theta and ez and that I already have over there.

So, we have covered that but then the problem is, have derivatives with the x and I will want to represent P as a function of r and theta. So I need to rewrite the derivatives in x as derivatives in r and derivatives in theta. So once we do those 2 things and put them together we should get our gradient form in cylindrical coordinates. So we will do that. I will be demonstrated for x now.

So dou P/dou x can be written of course as dou P/dou r * dou r/dou x + and for the sake of completeness. So I just applied the chain rule of multivariable calculus and now we know r is the function of x, right here. So we can substitute that derivative and proceed. Some of you all might be tempted to calculate dou x/dou r from this relationship and then inverted and put that in. So you are welcome to try.

I have done it in the past and of course not met with the success because you cannot actually do that. What you need to do is actually write r out as a function of x and y which means that at this step I really need the inverse mapping and that can be a bit of a pain because generally u have x related to r and theta in most practical situations that is the easy mapping to get. To invert it may not always be possible.

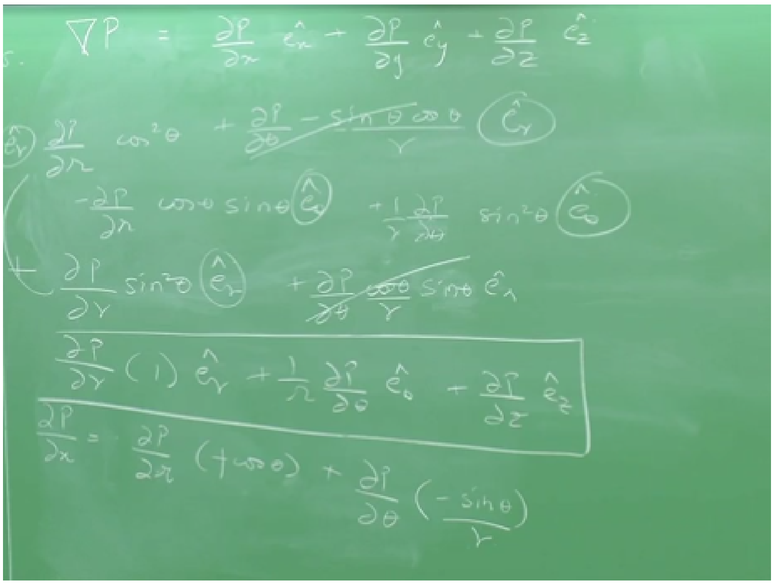
So this is something I will come to later. But right now we are lucky enough to have the inversions so we can proceed. So let us calculate dou r/dou x. So I will do this one for you. So then the 2’s go off and this is nothing but r so I will just get – x/r. In the same way I can get it is the derivative of time inverse 1 upon 1 + m square, right. If you use derivative of time inverse 1 + m square which I looked up yesterday.

And then of course if I look at y/x I will get y/x square the negative sign. So this is just – y/r square. So you can plug that in and we will find out that is a derivative with respect to r, alright now I do not really want to keep r hanging around. So what I will do is use, sorry I mean, I would not want x to be hanging around because I want to convert everything into r and theta. So I will use x is r cos theta here, so I will simply get – cos theta, right.

And here I will put y is r sin theta, so I will get. So now we can put that back in. Right so I differentiated that. Oh right sorry yes, correct. Yes, so it is just + value. Sorry, yes. Similarly, we can write out dou P/dou y and this case we just focus on this I mean dou P dou theta here. So dou P/dou y we can write out.

“Professor - student conversation starts” Can someone tell me dou r/dou y? dou P/dou r * sin theta, dou P dou theta * cos theta/r. **“Professor - student conversation ends”**

(Refer Slide Time: 37:19)



Fine? So, now that we have found the derivatives in terms of r and theta. What is left us for us to do is to now plug it back into this formula. So we have dou P/dou x so that I have here. So let us put this entire term there and then use my ex formulations that I have over here and substitute everything together and let us see what we get further I mean the first dou P/dou x. **“Professor - student conversation starts”** Are you sure with the expressions of this? Yes, that is fine. **“Professor - student conversation ends”**.

So dou P/dou theta and we just need to substitute that in there and I will get, maybe I will just do this systematically. Has someone done this? Essentially we need to just substitute, just multiply dou P/dou x with the x that comes out there. So that will give me is dou P/dou r, so cos theta with ex gives me cos square theta and then the second term will be dou P/dou r cos theta with – sin theta e theta, right.

And then dou P dou t I have – sin theta/r I will get cos theta er and then dou P/dou theta times – sin theta/r that is sin square theta/r e theta, that is fine. So you can see here that I have these 2 terms, sorry those 2 terms. Yes, we are fine here actually. There is no problem, so this is what we have for dou P by for this first guy. Now we will do the same thing over here right and proceed.

So if this + this is er, alright. Now we do the same thing here. We just calculate dou P/dou y ey by substituting from there. So you will see I will get dou P/dou r sin theta * sin theta er, correct. S I will get dou P/dou r sin square theta er right that comes from the first term. Sorry that comes from here right with the ey. Then the second term gives me dou P/dou theta cos theta sin theta/r.

Right that come from the, sorry that comes from the last term. have dou P/dou theta cos theta sin theta/r and then you can carry it out. So since we are running short of time I will just show you the calculus works out. So if I look at the er terms that I have so far you will see these 2 guys which basically amount to right because cos square theta + sin square theta becomes 1.

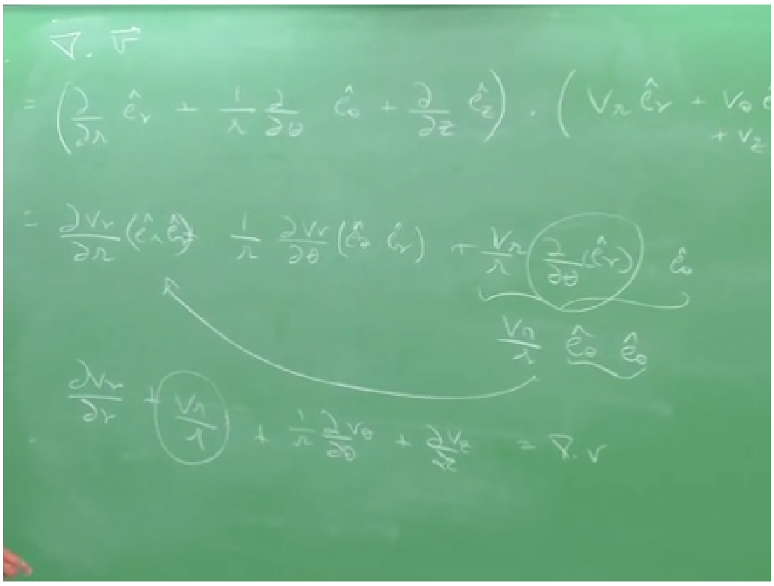
Then you will find that the troublesome term which was this guy because I mean this is not what there you know in the final formula er. So that is just cancels of with the guy here and if you continue for the next 2 terms you will find another cos square theta term that comes for this. That is in fact here, see. If I multiply this with this, I will get dou P/dou r cos square theta/r.

So that should be combine with the sin square theta/r, so I will get +1/r dou P/dou theta, alright? So, what we have done here is started with the gradient representation in the Cartesian coordinate system which was the one which you are familiar with. Then rewrote the derivatives in x,y and z in terms of derivatives in r and theta using the, I mean using calculus.

And then because we had the representations of ex zy ez in terms of er and theta we simply substitute that back in. And ultimately what comes out is the form of the gradient in the cylindrical coordinate system. So you can do the same thing in spherical coordinates or any other coordinate system and you will be able to derive what the gradient acting on scalars. So this is how we got the form for the gradient.

So, now that we have this let us look at the divergence of a vector and see what is it exactly that is different over there?

(Refer Slide Time: 44:00)



So now we move on to del dot v where v is a vector in general but we will be calculating it with the velocity or del dot n where n is the normal vector for the value. So now that we have the gradient operator and the gradient operator by itself is in fact the same as gradient of P. So this is just dou/dou r er alright and this has to be dotted with Vr er + V theta e theta + v sin e sin.

Also if you look back at the form for er and e theta which you will have in your notes you will find that er dot er = 1 whereas, er dot e theta = 0 and so on. So even the r theta z coordinate system is also orthonormal, just like x, y, z. So that means that when we apply the dot product we only need consider the er dot er terms e theta dot e theta terms. So er dot e theta will go to 0.

So if you just look at it that might tell you that all I will get is dou / dou r Vr + 1 by r dou/dou theta v theta dou/dou z Vz. Which looks exactly like gradient P but we know that is wrong.

So the difference that comes in here is that my er e theta and ez vectors are not constant. They are also functions of r theta and z which means when I operate on them I do not only operate on the component I must also operate on the unit vector itself.

So I just take the term which makes a difference, so we know that nothing changes with r basically. So when r acts on any of the unit vectors it causes no change. So there, there is no mystery I will just get its straight forward. So this will come by operating taking the dot product of this with everything else. Because dou/dou r e theta remains e theta and e theta dotted with er will give me 0.

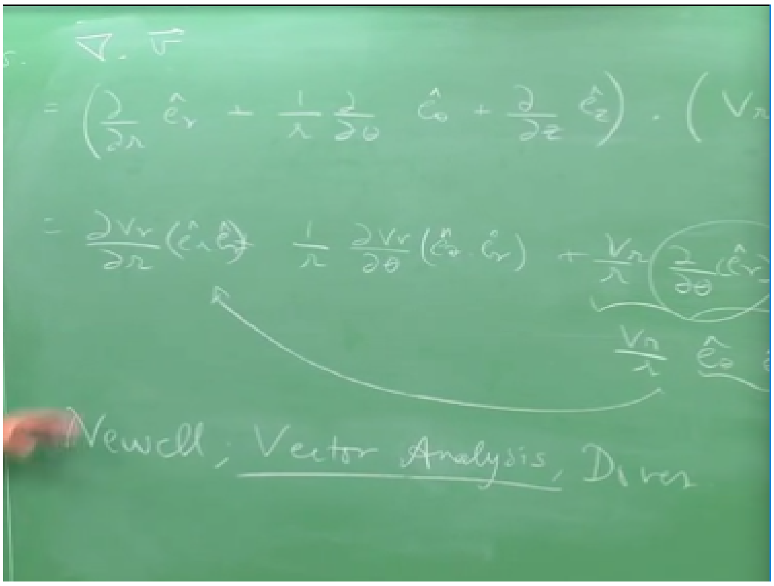
Now let us come to the second term. So, here I will get 1/r, let us look at this guy, dou Vr/dou theta e theta dot er, right? I am operating this on this. So first I am going to operate the derivative. So here I will use the product root. So I first differentiate Vr and er remain out here. The second guy of course will be where I differentiate er now and Vr comes out, right 1/r dou/dou theta and this whole thing is dotted with e theta.

Now here is the catch that the derivative of er with e theta is not 0. It is in fact e theta right? So this term becomes Vr/r e theta dot e theta and this is non-zero. So I have another term here. This guy gets added in and that is how you get dou Vr/dou r, sorry this is er dot er. Dou Vr/dou r + Vr or and this is that additional term that comes up from the divergence.

And now if I move on you will find that I will have to operate the derivatives on the remaining coordinates and then I will just get the rest of it is the same. Of course nothing much happens in this z level. So this is how we calculate the divergence and this is why you get some additional terms that crop up here purely because of the change of the unit vector er in this case with theta.

Now, the nice thing is that if you proceed you will find the same thing happening when you go to the del square V.

(Refer Slide Time: 48:33)



So another challenge is to calculate the Laplacian of the velocity vector which is a vector and this is what comes in all the viscous terms. So you will find in cylindrical coordinates of the viscous terms also have a slightly different form. So what we need to do here is realize the del square is simply the dot product of 2 gradient operators del dot del and then at this del square once we calculate it on V.

But here there is again the same complication that when a this del square will involve some derivatives, basically dou square/dou r square and so on. So you will have to operate not only on Vr but also on er. So when I have the 1/r square dou square/dou theta square term right this has to act on Vr er + V theta e theta. So I do not only get the second derivative of Vr I should also take the second derivative of er and see what I get.

So this term will give me some components along e theta and so on which is by you get these additional terms in even the viscous part of the equation. So you can possibly look at doing that yourself and comparing it with the cylindrical coordinate equations in any book and that should give you confidence about working through these calculations. So, what I have shown you here today is a sort of, I mean mechanistic approach based on the rudimentary of calculus and what the vectors really mean.

So that gives a nice understanding of the problem but there is a much more elegant treatment in the book by Newell, think it is Vector Analysis, Dover publications. So you can get it quite easily and what he does is he derives a series of formulas which can very easily be implemented on a computer for example. So with Mathematica I can actually get all the equations in any desired coordinate system by using those formulas and the basis of the formulas also quite nice and very powerful application of the Einstein notation.

So that allows you to easily generalize what we have done here for cylindrical coordinates. So, I suggest that you work out the Laplacian and for this derivative of calculations you can try to use Mathematica that will make things easier and if you are feeling particularly adventures you can do the same thing for spherical coordinates.