Multiphase Flows: Analytical Solutions and Stability Analysis
Prof. S. Pushpavanam
Department of Chemical Engineering
Indian Institute of Technology – Madras

Lecture – 09
Derivation of Navier-Stokes equation

Yeah, welcome to the lecture, where we are going to be primarily applying Reynolds transport theorem and derive the conservation of momentum, the Navier Stokes equation which you are already familiar with okay and then we will talk about how you need boundary conditions to solve this and we will do it in a slightly different way from what you are already exposed to; in a different way in the sense that the boundary may not be a straight line.

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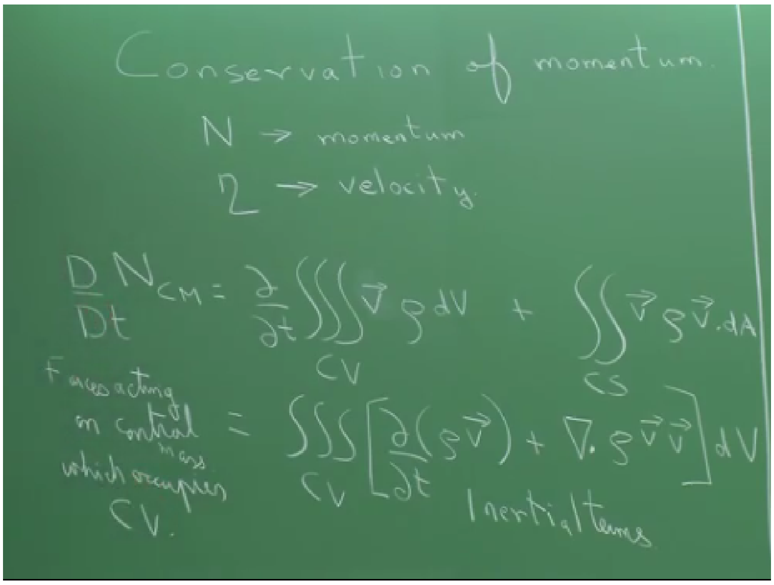

The boundary could be curved and then how do you generalize the formulation of the boundary condition, so that is the strategy okay. So, let us look at the conservation of momentum, so now M represents momentum and the corresponding intensive property like we saw yesterday, eta is going to be the property per unit mass, which is nothing but the velocity okay. Now, what I am going to do is; write down the Reynolds transport theorem for; and just substitute wherever we had eta velocity.

And we have the material derivative and remember now, N is momentum okay and this is associated with the control mass and this is equal to the rate of change of momentum in the control volume like I said, I am going to write this as my velocity vector v DV +; we have the efflux term across the control surface, which is eta rho V dot dA. All I have done is substituted wherever we had eta, the velocity vector okay.

We are going to convert this again to a volume integral and group these 2 terms okay, that is just one calculus and since the control volume is fixed in time, I move the time derivative inside like we did yesterday and so I have the control volume, I have d/dt of rho v +; I am converting this also to a volume integral and I am going to write this as the divergence of rho v, v okay and this entire thing I am going to integrate over the volume element.

What I want to do now is to tell you that these terms on the right hand side are basically the inertial terms, which you have familiar with in the Navier Stokes equations. Normally, when you write the Navier stokes equations, you would write this on the left hand side okay, so these are the inertial terms and the term on the left hand side represents the rate of change of momentum associated with the control mass.

What we going to do now is; use the fact that from Newton's second law, the rate of change of momentum associated with a control mass is going to be equal to the forces that are acting on the system okay. So, what we want to do is we actually want to study the rates of change of velocity at a fixed point, so I want to get this in terms of the forces okay. So, this is going to be written as the forces acting on the control mass.

The control mass, which is occupying the control volume okay, so that is basically what the Lagrangian approach tells me that the forces are equal to the rate of change of momentum and so now what we need to do is; identify what are the different forces which are acting on the control volume, the mass contained in the control volume and then we should have the Navier Stokes equation.

Since our objective is to derive the Navier stokes equation, what we going to do is and the Navier Stokes equation talks about how the velocity pressure etc vary with space, we going to consider an infinitesimal control volume and to find out what the forces are that are acting on the control volume okay.

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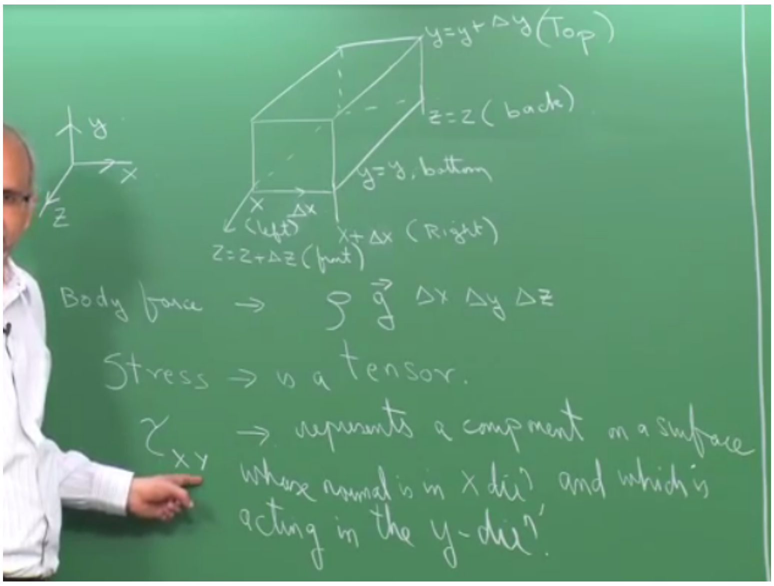

So, to keep things simple, we are going to look at a rectangular coordinate system and let us say this is my x axis, this is my y axis and this is my z axis okay and I am going to consider a cuboid, this is the x direction and the 2 surfaces are going to be given by; the surface on the left is at x, the thickness here is dx, so that the surface here is given by x + dx, okay, this distance is dx or delta x, maybe I should use delta x.

Because then, when I do the limit as delta x tends to 0, I can write this as dx, just to be on the safe side okay. The lower surface is given by y equals y and the upper surface is going to be given by y equals y + delta y, okay. Similarly, this surface here is going to have the z axis is coming outside the board, the way I have drawn it, so the surface of the back is given by z equals z.

And this surface in the front; the front face is going to be given by z equals z + delta z, okay. So, the back face is z = z, the front face is z = z + delta z. The 2 things on the sides; right and left are given by x coordinates and this. Now, the forces that are acting on the system, there are basically 2 kinds of forces, one which is the body force and then whatever is being exerted along the surface, okay.

So, let us do the easy one first, which is the body force and the body force is going to be primarily because of gravity okay that is the only thing we are considering right now and that clearly is going to be; what is the body force is acting on this control mass? It is going to be the density times the volume multiplied by the gravitational acceleration, right. So, rho times g times delta x times delta y times delta z.

Is something bothering you? Sure, everything is fine, okay, okay, so that is the body force which are acting on the system. Now, we are going to have surface forces acting on all the 6 surfaces and so let me just maybe write this thing, this is at the back and this is at the front and this x + delta x is on the right and this is on the left okay and why it was y is the bottom and this is on the top.

So, now as far as the viscous stresses or the stresses were acting on this, I will just say for right now, let the stress is a tensor and what does this mean? It means, I need to specify 2 things in order to completely specify my stress okay and normally the stress components are given by this symbol tau okay and let me just say that these are the 2 subscripts; x and y. What does this represent?

This represents the component of the tensor, which is acting on a surface whose outward normal is in the x direction and with this in the direction of +y okay, so this basically represents component on a surface, whose normal is in the x direction and which is acting in the y direction okay. So, we need to specify 2 things, I supposed to a vector but I only specify the direction of the vector, okay.

So, here I am specifying the normal to the surface as well as the direction, so it seems, there are 2 things which are specifying, okay, this becomes a tensor. We see more of this in some detail later on when we do it more mathematically but just for the sake of illustration physically because now we are going to get into some physical derivation for the Navier Stokes equation. The other important thing, which you should know is that there is a sign convention which you have to follow.

And the sign convention, which is followed is that if both the x and the y directions are positive, then the stress component is taken to be positive okay, this could be one of the sign conventions you can use or if both of them are negative, then the stress component is again positive.

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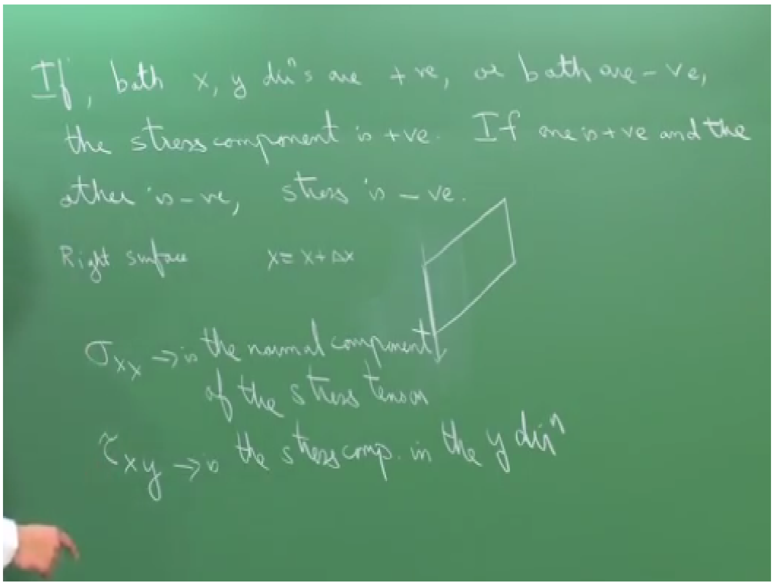

So, basically what I am saying is; if the tau; if both x and y directions are positive or both are negative, the stress component is positive, whereas one of them is positive and the other is negative. So, when I am going to illustrate this, it will become clear to you okay. If one of them; if one is positive and the other is negative, the stress is negative, okay. So, we will just stick to this sign convention.

And then we will find out what are the forces, which are acting on the system. So, now clearly if I consider the face on the right; on this right surface, I will have different forces, which are acting okay. I will have 3 components for the stress okay. So, let us look at the right surface, this is at x equals x + delta x, okay maybe, maybe; may be, it is better to draw it like the better way now, okay, I am happy with that, okay.

Now, this is the x direction right, sorry; this x direction, x is normal to this, so I have a component, which is going to be in the normal direction that is; that is going to be the normal direction to the surface is the x direction and I can have the force, which is acting on this resolved in 2 different directions x, y and z. So, the normal component, which is going to be in the x direction acting on the surface, whose normal is also in the x direction.

I am going to give it a slightly different notation, I am going to call it sigma okay and you will see, why that is? So, sigma xx is the normal component of the stress tensor, so what does this tell you; so whenever it is normal that means both the subscripts are the same because the direction of the stress is the same as the direction of the outward normal of the surface, so this is a normal component.

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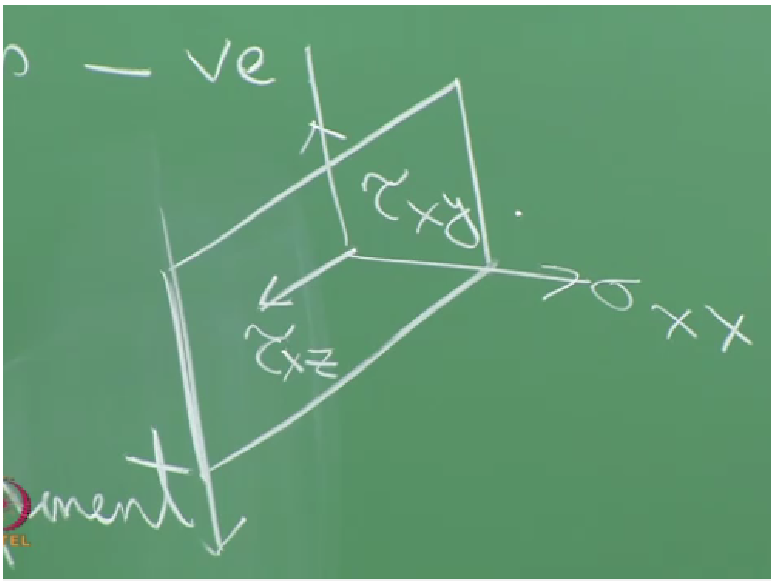

Now, what are the other components which are acting on this surface? Other components acting on this surface include tau x, y because my surface is fixed okay, it is a tau x, y and this is the stress component in the y direction but is acting on the surface x + delta x and tau x, z is the stress component in the z direction. So, the way I have drawn this tau x, z is going to be in this direction okay and tau x, y is going to be in this direction.

So, y is in a vertical direction, so I just stick to that and the way I have drawn these components, all of these are positive because the direction of the component is in the upward in the + y direction, the surface outward normal is also in the +x direction, so this is a positive component. This guy is also a positive component because a normal is in the +x direction, the direction of the stress component is also in the +z direction.

So, like I said if both of them are positive, then the stress is positive okay and the sigma xx is going to be a normal direction, so just imagine something coming out of the board or out of this thing, so that is sigma xx okay. Now, what we want to do is; do a force balance right, we are going to find all the forces, so rather than talk about all the forces acting on the entire control surface, what I am going to do is just talk about forces in the x direction okay.

So, I am just going to find what the force is on the x direction and then by similarity, we will get the forces in the y direction, in the z direction. So, let us just concentrate on what are the forces acting on the x direction, clearly in the x direction, the only component, which is going to play a role is the sigma xx because this guy is acting in the y direction, this component is acting in the z direction okay.

So, but what is going to happen is; the other surfaces will have components in the x direction, I need to take that into account, okay. So, now let us find out what are the forces acting on the other surfaces in the x direction and then we should be able to write; get all the forces acting in the x direction.

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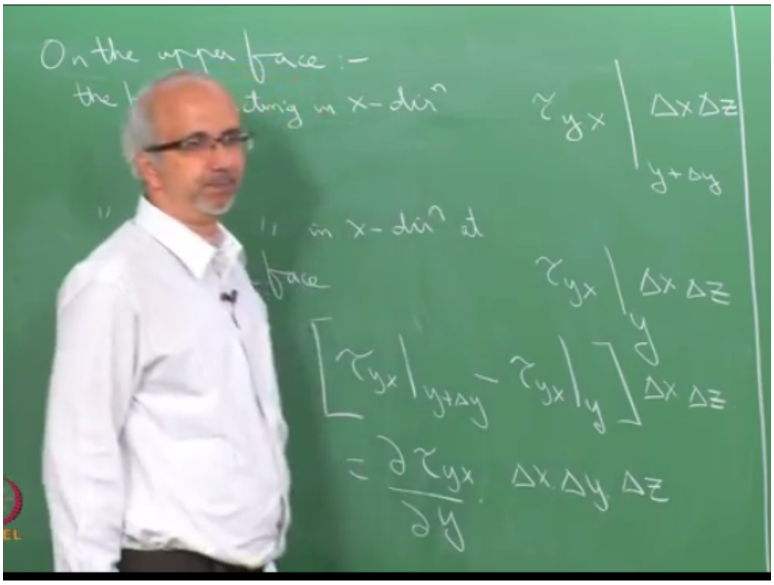

So, on the upper face; the okay, the force acting in the x direction is going to be tau, why? because this upper face is outward normal is y, the first subscript is y, the direction is going to be x okay, now yx and this is going to be evaluated at y + delta y, okay but this is acting on what area element, this is acting on an area element, which is given by delta x delta z that is the area on which this component is working.

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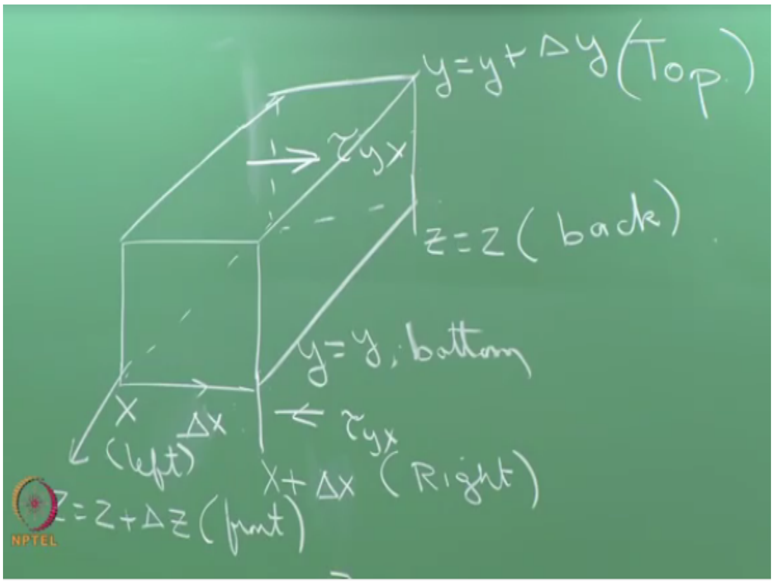

So, that total force is going to be delta x delta z, this is on the upper surface and what about the force acting in the x direction at the lower face? This is also going to be tau yx evaluated at y multiplied by delta x delta z, okay. I just, I am going to indicate that the x direction is positive in this direction, y is positive here. So, the way if you want a positive component; if you want a positive component for this; sorry, if you want a positive component for this, the positive component is going to be acting to the right; right.

Because this is the one where x and y are both positive but as here, the negative component; the component acting on the lower faces, outward normal is in the negative x direction, so in order for me to have a positive tau, the tau yx is positive is going to be acting towards the left, okay I am just going to indicate tau yx as being acting to the right here and tau yx as acting to the left here.

So, if you want to really look at the net force acting in the x direction from the upper and lower faces because the directions are opposite, you are going to have to basically subtract the 2 okay and so the net force is going to be given by tau yx at y + delta y - tau yx evaluated at y, the entire thing multiplied by delta x, delta z, okay and this you can write as; by just doing a Taylor series expansion, you can approximate this as for a very small delta y.

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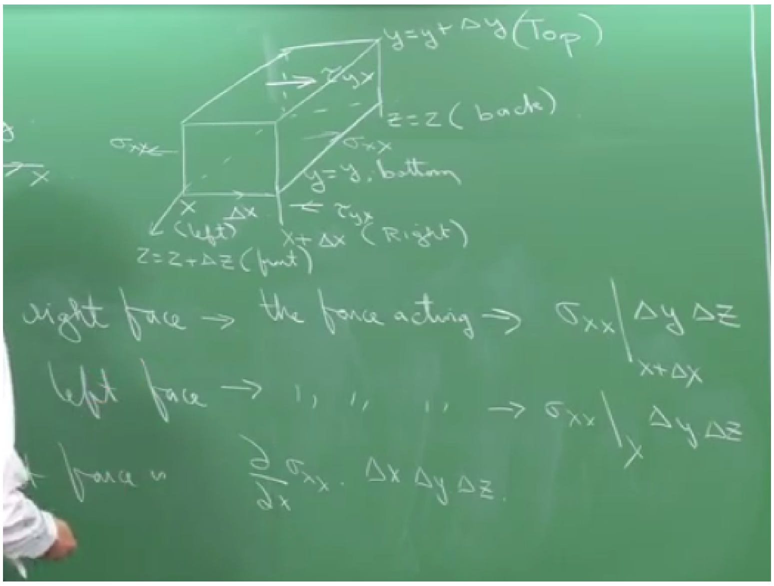

You can write this as tau yx/dy times delta x times delta y times delta z, okay. Now, let us look at the contributions due to the other surfaces. On the right face, the force acting and going to be given by x is in the outward normal direction, the direction is also normal, so this is the normal force that we are looking at, it is going to be given by sigma xx okay times the area element, which is delta y delta z.

And this is evaluated at x + delta x on the left face, the force acting is given by sigma xx evaluated at x delta y times delta z, remember here, the x component is; the x direction is the outward normal here, so your positive direction is going to be in the +x direction, here the outward normal is in the negative x direction, so in order for the stress component will be positive, you should be acting in the negative direction and that is what I am going to do now.

That sigma xx is acting this way, here sigma xx is acting that way okay and so the net force is going to be given by d/dx of sigma xx times delta x times delta y times delta z, so taking care of the right and left face, the top and bottom face, you are going to take care of the front and back okay.

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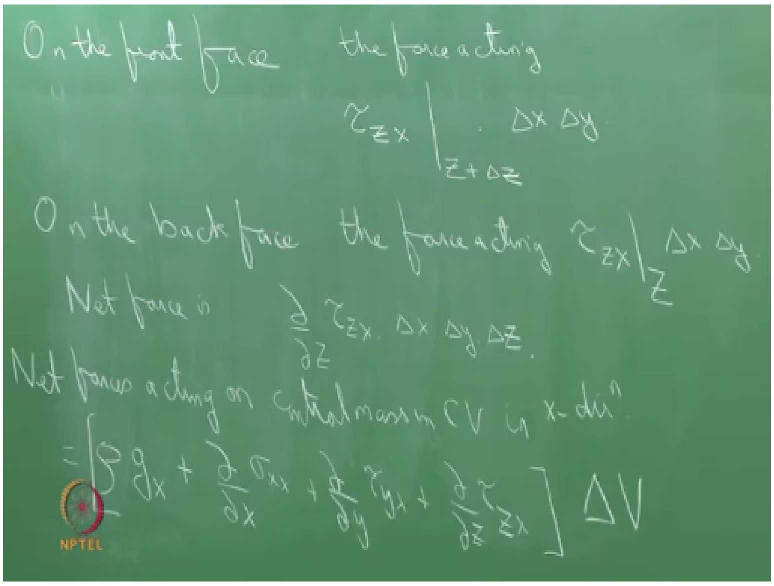

So, on the front face, we have the force acting is, it is going to be given by tau; the first component is that because the normal is in the z direction looking at x direction, x bar is what we are doing, so tau zx, okay but this is evaluated at z + delta z multiplied by the area on which it is acting, which is delta x delta y, okay. On the back face, the force acting is tau xz, evaluated at z multiplied by delta x delta y.

So, as far as the; I got it wrong here right, I got; I should write this as zx, as so the net force which is acting on the system because of these 2 faces; the front and back face, net force tis going to be given by just subtracting the 2 because of the fact that the directions are different with the same convention that we following and in fact, if you had followed the other opposite sign convention again, you will get the same thing, okay.

We are going to get d/dz of tau zx times delta x times delta y times delta z, okay so clearly the delta x, delta y and delta z, is nothing but the volume element; the volume of the infinitesimal cuboid that we just constructed. What I want to do now is substitute these expressions for the forces. Now, if you had an electric field and let us say your fluid had some kind of an electric dipole or you had a magnetic field, then you would possibly have included those forces okay.

Right now, we are just going to keep life simple, we neglect all of those, okay and it is likely that those forces are going to be more in the form of a body force. So, I just wanted to mention that but then we are going to keep life simple right now and then we just got to neglect all those forces and write out the sum of the forces, so this net forces acting on the control mass in the control volume is going to be given by the sum of the body forces that we already wrote down.

And these 3 surface forces that we have okay, so I will be going to be given by rho g multiplied by the volume element, I am going to keep that common and I am going to write this as d/ dx of sigma xx + d/dy of tau yx + and this is an x direction remember, d/dz of tau zx multiplied by the volume element delta V, okay. So, these are the forces, these are the surface forces that we have and this is the body force that we have.

And maybe what I need to do is be a bit more careful here, I need to tell you that we are looking as x component, so I need to change this to the gx, okay, net force is acting on the control mass in the control volume in the x direction therefore, okay. So, now I should be able to do the same thing in the other directions, in the y direction, in the z direction, so rather than go through the entire process, we will just you know look at this and we should be in a position to actually write down the components for the other directions okay.

But before doing that what I would like to do is; simplify things a little bit, so rather than talk about things with tensors, we like to keep things simple. So, let us go back to the Reynolds transport theorem and say that we are going to do conservation of momentum only in the x direction, what that has helped me do is instead of writing the velocity component in the vectorial form eta, I would write my eta as vx, okay.

So, rather than do a general conservation of momentum, I am going to do a conservation of momentum in the x direction. So, let us come back here and this that just to improve clarity okay and then we will definitely be doing things more tensorially but right now I do not want you guys to get tense looking at the tensors, so we will just continue with this.

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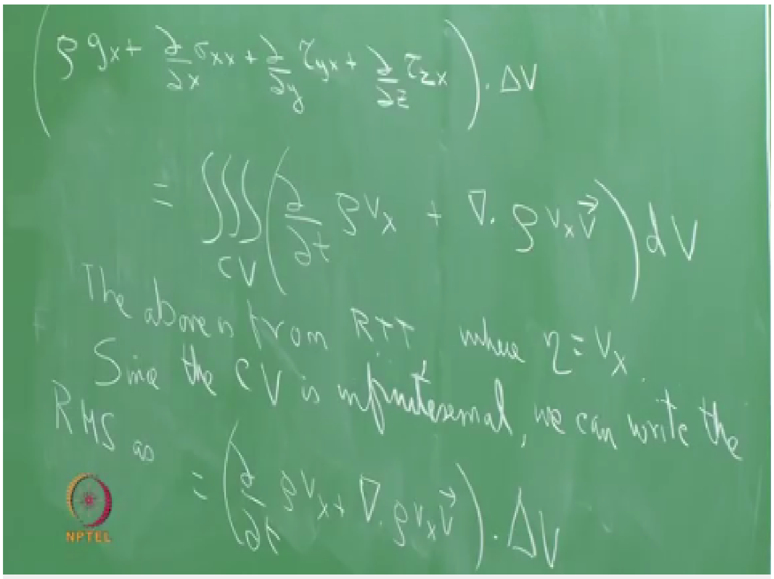

So, let us look at the momentum balance in the x direction okay, so what this means is, I already have the forces acting on the x direction, so I can just write this rho gx + d/dx of sigma xx + d/dy sigma; sorry, tau yx + d/dz of tau zx, okay multiplied by the volume element and this is the left hand side, which I had and the right hand side from a Reynolds transport theorem is going to be a triple integral over the control volume of d/dt of rho vx.

Because I am looking at the specific property, eta now is vx, when I started the lecture today, I just put eta as the velocity vector V but now I am doing only the momentum balance in x direction. So, I am going to restrict myself when we substitute in eta vx, okay + divergence of rho vx * v that is what we have, multiplied by dv, okay. So, this is the above is from the Reynolds transport theorem where eta equals vx, okay.

Since, we have an infinitesimal control volume, a very small control volume; we can basically say that this integral over the control volume can be written as this function multiplied by delta v, okay, since we are talking about a very, very small control volume. So, the right hand side can be simplified as since the control volume is infinitesimal, we can write the right hand side as d/dt of rho vx + divergence of rho vx v times delta v, okay.

But delta v is nothing but delta x, delta y. delta z, basically what I am trying to tell you is that this is some kind of an average thing inside the control volume okay and it is like; it is very small integral fx dx can be written as fx times the delta x using a trapezoidal rule or a rectangular rule for doing the integration okay. So, here we have this, so now what I can do; what as you can see here.

What is happening is I have gotten rid of all my integral signs and since this is going to be valid for any arbitrary delta v, so delta v cannot be 0, so I can actually remove the delta v and we get our equation that we want to do; want to get, which was the objective, which is the Navier Stokes equation in the x direction, okay.

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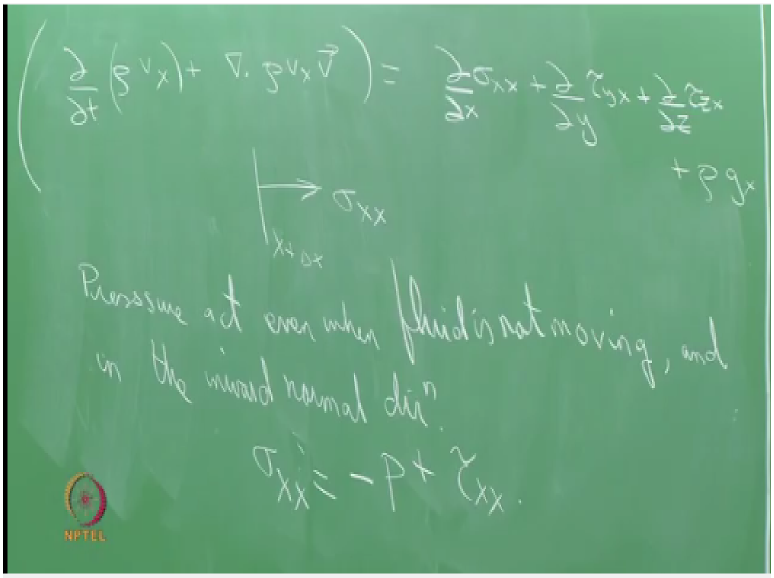

So, in the x direction, the momentum equation is; and I am going to now just flip the left hand side and the right hand side because then things would look very very familiar to you, rho vx + divergence of rho vx v okay, equals d/dx of sigma xx + d/dy of tau yx + d/dz of tau zx + rho g subscript x, okay. Now, I am going to focus on sigma xx; sigma xx is my normal stress component okay.

And the way I have drawn sigma xx is it is positive when the x axis is the surface outer outward normal is the positive x direction and the direction of the force is also in the positive x direction, so if this is x + delta x, this force sigma; this is sigma xx, so on the surface sigma xx is positive this way. Now, supposing you have a fluid, which is at rest okay, then what is the force that is going to be acting on the surface?

It is basically the hydrodynamic pressure okay, P and what is the direction of the hydrodynamic pressure? It is in the direction of the inward normal, so the hydrodynamic pressure acting on the surface is going to be basically because of molecules which come here and then they get reflected okay. So, as far as the force on the molecule is concerned, it is acting in the right direction towards the right.

But as far as the force of the surface is concerned is acting towards the inward normal, so the reason why I wrote this particular term as sigma xx is basically because I wanted to separate out the 2 effects; one coming from the hydrostatic and one coming from the fluid flow, so the effect coming from hydrostatic in the normal direction is basically the pressure term and remember pressure acts even when fluid is not moving okay and is in the inward normal direction, okay.

So, we are going to write sigma xx as - p + tau xx, this –p especially tells you that this pressure is the inward normal direction and sigma xx is in the outward normal direction, so basically what I am doing is; in the normal component, I am resolving it in 2 directions; one is in the; what you have when you have statics the pressure free and this is due to the flow okay. So, I am going to substitute sigma xx now in terms of this.

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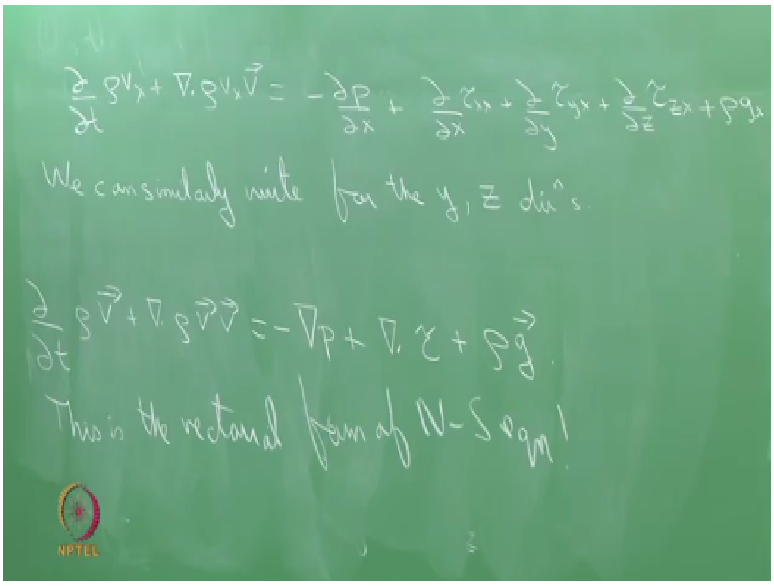

And then you get what you are actually familiar with; which is d/dt of rho vx + divergence of rho v xv equals – dp/dx + d/dx of tau xx + d/dy of tau yx + d/dz of tau zx, okay, you can write a similar equation for all the 3 directions and of course I have missed out the body force term, which is rho gx, okay. We can similarly write for the y and z directions okay and you can; you know write things in a very compact vectorial notation, which is d/dt of rho v + divergence of rho vv okay, it was - gradient of p + divergence of tau + rho g, okay.

So, this is the vectorial form of the Navier stokes equation. Now, I just wanted to derive it this way, so that I could illustrate to you that starting from the Reynolds transport theorem, we use Newton's second law of motion only to take into account the rate of change of momentum associated with the control mass, so that is where we bring in the forces okay and the other terms; the accumulation term and the convection term, where you have the flow across the control surface.

Those are you are familiar with as inertial terms from your undergraduate class okay, so basically these are your surface forces, the pressure is also a surface force but in the normal direction and tau represents all the components, which is going to arise because of velocity, whereas p will be present even when there is no flow, even when we have a hydrostatic situation okay.

Of course, I have jumped a little bit in the sense trying to put this thing together but we will take a look at how to evaluate divergence of rho vv, divergence of tau in the next couple of days. What I want to do is just talk about the fact that what we have derived the equation of continuity and the Navier stokes equation is basically a differential equation okay and in order to solve these differential equations, you need boundary conditions.

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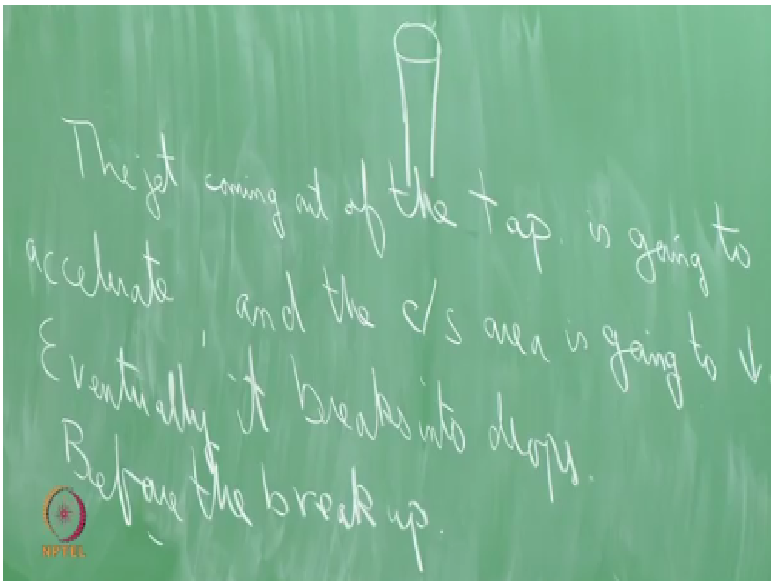

So, tomorrow what we are going to do is; we are going to see how these boundary conditions can be put in a slightly general framework. Let me tell you what the motivation for that is. So, I mean, consider thin liquid jet coming from a tap okay, so if you go the bathroom tomorrow or your sink and you just open the tap a little bit, what is going to happen is water is going to fall right.

So, and clearly the water is going to be surrounded by air. What do you expect? I mean this is the mouth of the faucet and this is the jet which is coming out clearly, as the liquid is going to come down because of gravity, the velocity is going to keep on increasing okay and if the velocity keeps on increasing, the cross sectional area through which is going to flow is going to keep on decreasing.

Eventually the cross sectional area is going to be so small that you are going to have this particular jet break up into drops okay, so that is a problem is we are going to look at in some detail later on in the course, how does a jet break up into drops but what I want to talk about today is talk about the portion, where the jet has not broken up into drops but the cross sectional area is changing okay.

So, the point is the jet coming out of the faucet of the tap okay, is going to accelerate and the cross sectional area is going to decrease, okay. Eventually, it breaks into drops, so why did your flow rate is sufficiently low, okay. So, tomorrow, when you go to the bathroom and you are trying to fill your bucket, you possibly do the experiment and see, does this indeed happen but I want you to focus on before the break up.

The surface is not going to be a uniform; is not going to be that of a uniform circular cylinder, it is going to be of a cylinder whose radius is actually, let us assume is going to be circular whose radius is actually going to change okay. So, what we want to do is we want to be able to apply boundary conditions to surfaces, which are not going to be given by you know, R equals constant, where R is actually in our function of z, okay.

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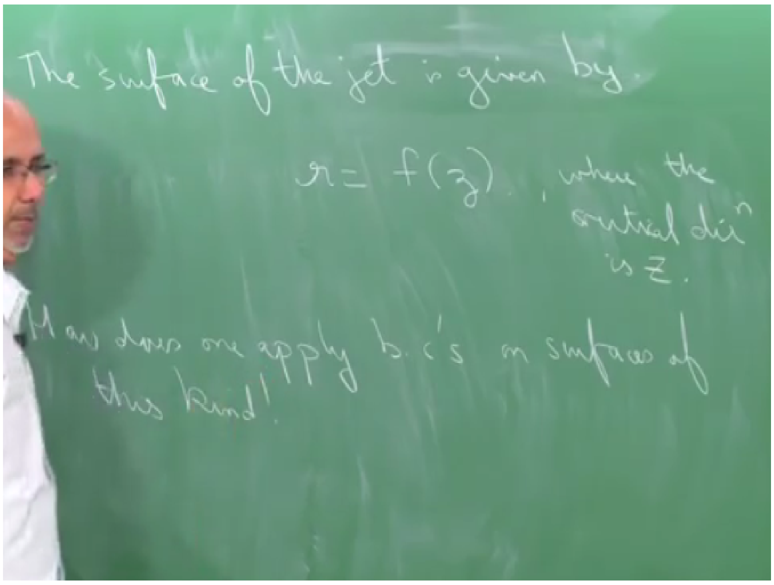

So and that is one of the things we are going to be doing in this course, where your surface is not going to be given by lines where the coordinate is equal to a constant okay. So, the surface of the jet is given by r equals some function of z, let us say where, z is the vertical direction; the vertical direction is z, so yeah okay, normally you are used to applying boundary conditions to r equals a constant, r0.

Because and the kind of boundary condition that we are normally used to is applying things like continuity of tangential stress, continuity of normal stress, if there is continuity of normal stress okay. So, we would now like to find out how to go about applying boundary conditions here, so how does one apply boundary conditions on surfaces of this kind, okay. So, before that the reason is your coordinate system is r and z okay and we have derived our different stress components in terms of tau rz, tau zz, tau rr etc.

But now the stress that is going to be acting on this surface is going to be in the normal component and in the tangential direction, when the normal and the tangential directions are not in the classical r theta z because the surface is actually changing, so we need to be able to relate the shear stress and shear components in an arbitrary direction acting on an arbitrary plane to the classical things acting on to classical directions r and z okay. So, that is something we go to see and then we will talk about the boundary conditions.