Computational Techniques

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Module No. # 04

Lecture No. # 01

Nonlinear Algebraic Equations

And welcome to the module 4 of computational techniques. And in module 4, we are going to look at non-linear algebraic equations, and computational methods to solve the non-linear algebraic equations.

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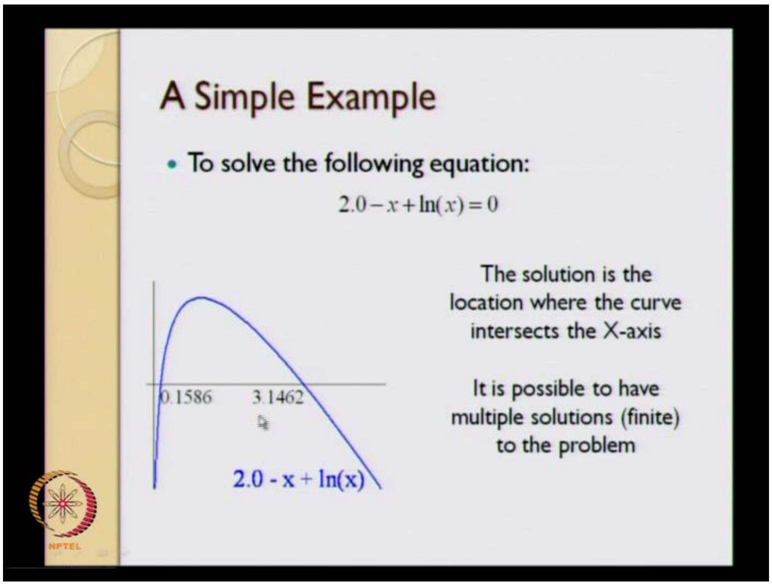

So, let us look at the simple motivating example for this particular problem; this is just a straight forward example of non-linear equation 2 minus x plus log of x equal to 0, and essentially what we are intended intending to do is find the points of intersection of this particular curve with the X axis. So,this particular curve intersects the X axis, at 2 different locations. So, this one location, and this second location are essentially the solutions to the equation 2 minus x plus ln of x equal to 0.

What we intend to do is figure out ways of numerically solving this particular equation; what I have shown over here in the graph is essentially a graphical method, we can employ graphical graphical methods for relatively simple problems perhaps - problems in 1 or 2 variables, but when the problems go beyond say couple of variables, we are not going to be able to use graphical methods, and for in general for more complicated examples, we will not be able to use analytical techniques in order to get the solutions.

So, we resort to essentially using the power of computer and numerical techniques in order to solve this problem. So, as I said the solution is the location, where the curve intersects the X axis, unlike the linear case that we saw in module 3; it is possible to have more than 1 solution, but a finite number of solutions to the equation.

For example, in linear algebra when we looked at the systems what we found is when the matrix a was non singular; that means, rank of matrix a was equal to n, we could get exactly one unique solution. And when the rank was not equal to n, we could possibly get infinite number of solutions, but we never got multiple finite number of solutions to the problem. However, in disagreement with that observation, what we see in numerical in sorry in non-linear systems is that they can have multiple solutions, and the example is right in front of you, is there are 2 solutions - one at this particular location, and the other at this particular location both of which are the solutions to the equation 2 minus x plus l n x. The first solution is 0.1586, and the second solution is 3.1462. So, these are the two solutions to the equation.

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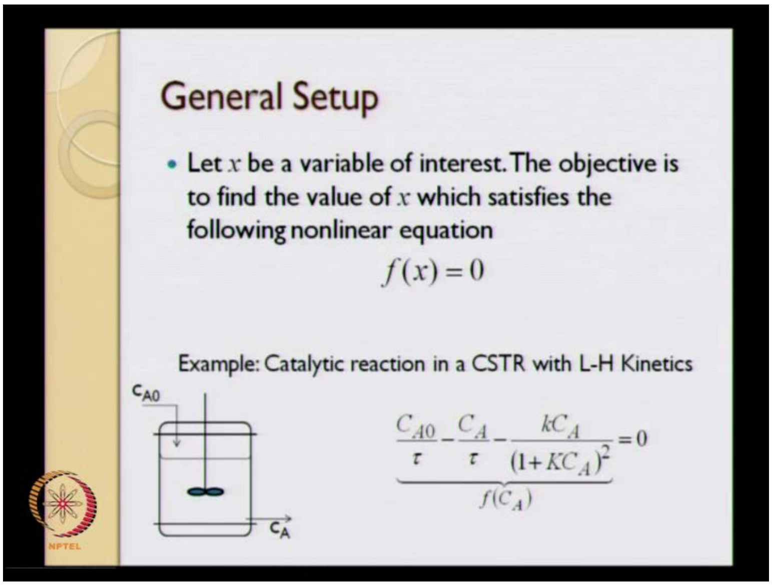

The general setup that we are interested in here is given a particular function f of x, we are interested in finding out the x that solves the non-linear equation f of x equal to 0.

This is a general setup in a single variable; in general this f of f of x could be a closed form closed form expression as in this particular expression 2 minus x plus ln x, what what what we mean by closed form expression is that, we can express x - we can express f of x explicitly in terms of x.

There are lot of problems in which, we would may not have a closed form equation or we may not have an algebraic representation of x, but x can be obtained through certain other numerical means. And we will look at those problems perhaps when we come to module 8, when we discuss about boundary value problems in ordinary differential equations, and when we talk about partial differential equations. But as of now, what what happens is all we need is given a particular x, we need a method to give us the f where the function f of x, and as long as we have that we do not care, whether it comes from an expression of the of the type 2 minus x plus ln x or whether it comes through any other means.

So, that is the general setup, and we are interested in finding out the point of intersection of the the curve f against the X axis. An example and non-linear systems are extremely common in chemical engineering for that matter in any engineering systems, and just simple example that we have taken over, here is catalytic reaction taking place in continuous stirred tank reactor, and let say the catalyst follows a Langmuir Hinshelwood type of kinetics. Then if we try to get the steady state relationship for for this CSTR, where tau - if tau is the residence time, we get inlet concentration divided by tau minus the outlet concentration divided by tau minus the rate of reaction essentially should be equal to 0 at steady state.

And if it follows Langmuir Hinshelwood kinetics of a particular kind, we have the rate expression expressed in terms of KC A divided by 1 plus capital KC A the whole squared, and this is a non-linear expression, and because of the presence of this non-linearity; the overall function f of the concentration C A becomes non-linear. So, in general now when you have to solve an equation of this sort in order to get the concentration of C A or in order to get the tau value that will give you a particular C A, what we will have is essentially a non-linear equation; a single equation - in a single unknown. And we can solve that particular equation through the techniques that we will learn in this module will be able to solve those equations, and we will get the desired solution for that particular problem.

So that is, for example in this case that particular solution is going to be the concentration C A that we are going to get out this particular reactor; if tau is going to be the residence time in this particular solution.

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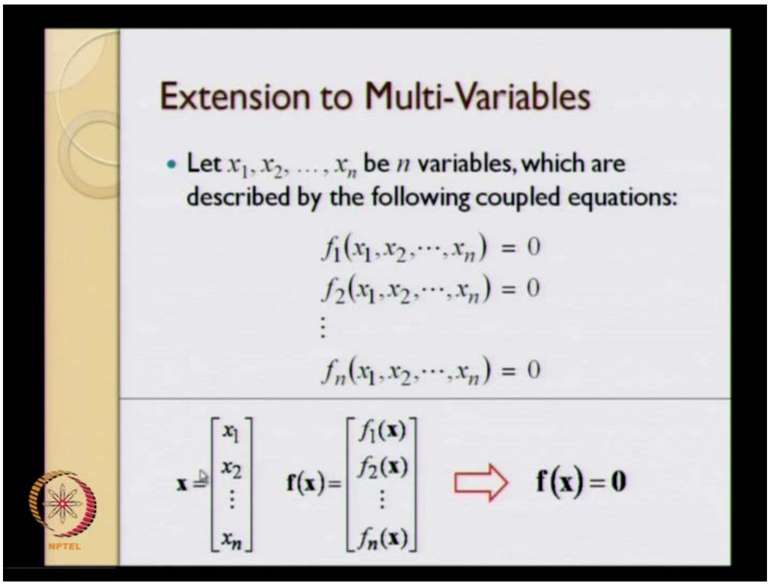

Now, that is when we talk about a single variable case; let us look at the extension to multi-variable case - in multivariable case we do not have just 1 variable, but we have in general n variables. So, we have x 1, x 2, up to x n; those are n variables, and for them to how for the entire system to be a well posed systems, we system we actually need n functions - f 1, f 2, up to f n equal to 0's, which and this particular condition needs to be satisfied. If if this condition is satisfied for each of those functions f, that time we say that the values of x 1, x 2, up to x n are the solutions to these equations.

And we will use a vectorial notation for multi-dimensional system, very similar to what we did in in module 3, our our x bar is going to be our in this case I have use the bold phase x our x is going to be x 1, x 2, up to x n. It is going to be a column vector, column vector, where the n variables from the n rows of that particular vector, and f is also going to be a vector, where the first first element of f is going to be just f 1 computed at x 1, x 2, up to x n.

The second is going to be f 2 computed at x 1, x 2, up to x n, and so on, up to f n. And this these particular n equations in n unknowns, we can write in shorthand notation in the form f of x equal to 0.

So, that is the shorthand notation for this particular system. So, there is our specific reason, why we are using this particular short hand notations. And essentially, it is because we are interested for single variable case in obtaining a numerical technique that will solve the problem, and give us the solution x. And in multi-variable case, we want a technique which is scalable to any general n, it is we are not interested in developing a technique that is suitable only for 2 or 3 or 4 variables, but we generally in general we are interested in figuring out a technique that can be applied to a large number of variables.

So, that is really the overall picture or overall introductory picture behind this solution solution to non-linear equations.

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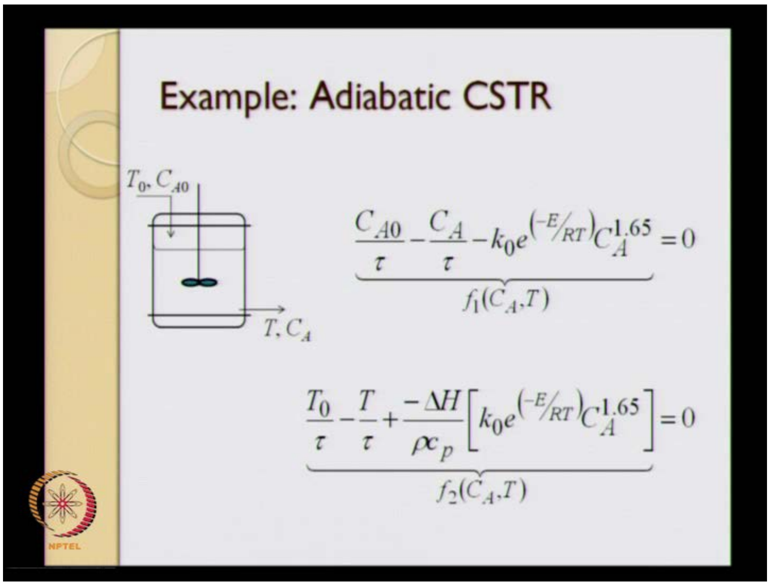

We will take up another example, when we have 2 equations, and 2 unknowns; this is an example actually of propane combustion in case of oxygen oxygen is going to be present in excess. So, if you have propane combustion taking place in a adiabatic continuous reactor, and and C A 0 is the concentration of propane at the inlet, and C A is the concentration of propane at the outlet. And tau is the residence time, then this is the overall equation that we get for obtaining the concentration at the exit. This is a non-linear equation, the 2 types of non-linearity - one non-linearity comes in because of the erroneous type of rate law, because of the exponent of minus E by RT; the other non-linearity comes in because this is not a first order reaction rate, but it is the reaction kinetics is as an order of 1.65.

Likewise, so this one is going to be f 1 of C A and T, likewise we will need an energy balance, also the energy balance will give you the second function f 2; and f 2 of C A and T is written over here. And basically this is the heat of reaction term and, this is the rate of reaction term so, this particular product is the total heat that is generated, because of the reaction taking place. This is the sensible heat entering into the system at the inlet, and this is a sensible heat exiting from the system at the outlet. And this becomes our overall problem. So our x bar in this case, if we refer to the previous slide f bar in this in case becomes f 1, and f 2 our x bar is just going to be C A and T, and n in this particular example is nothing but 2.

So now, we are not in general not just restricted to 2 equations to 2 and 2 unknowns and, and so on. We can in general have n equations and in n unknowns, just an example to give you is for example, we have a distillation column with with for say 40 trays, then and 2 components we will have the mass fraction of component a, in the liquid mass fraction of component a in the gas phase, and temperature. These are the 3 variables at each of those stage - stages and so, we will end up with 3 multiplied by 40 is 120 variables.

So, we might actually end up with 120 equations, and 120 unknowns. And that would be an example of multi dimensional system in non-linear equations

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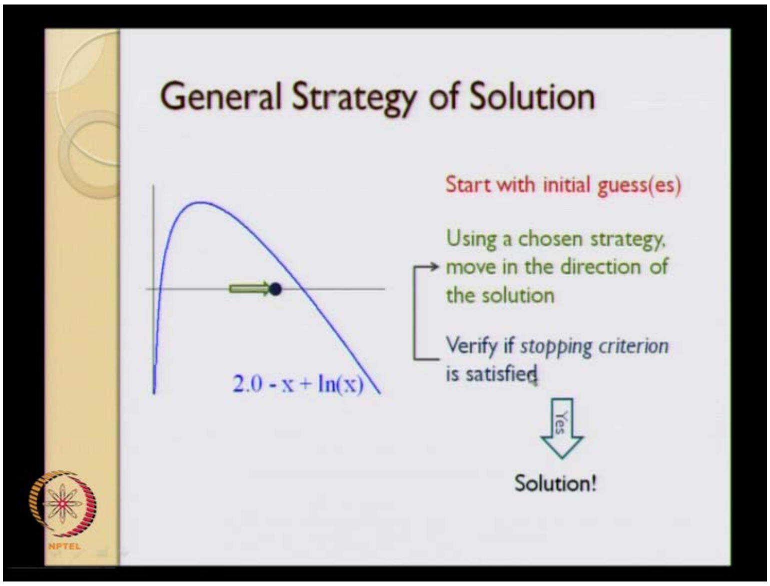

Now, all the methods that we are going to look at will follow a certain general strategy of getting the solution, and the general strategy will involve, we will require require first an initial guess, that is shown by the red dot over here. So, that is going to be an initial guess, we need to either we need to provide an initial guess or the numerical method that we develop has to also incorporate in it a method to get that particular initial guess. Once we get initial guess, we might either have one initial guess or we might start with 2 initial guesses.

So, the first agenda is going to be to start with 1 or 2 initial guesses, and then use a particular strategy in order to move, that particular initial guess in the direction of the solution. In general, at this point we know how the solution, how the curve looks like we know what the solution is exactly going to be. So, we can figure out whether whether or not the particular point is moving towards the solution or moving away from the solution and so on, but in general we do not - if we do not know what does curve looks like or where where the intersection of the curve with the X axis is a priori. We do not know whether the movement is in the direction of the solution or whether it is not.

However, what we need is to choose a strategy which will generate the next the successor moves given a current initial guess, and given the function f either in a closed form like this or some method to generate f of x given X. Now, the next item would be to verify if the stopping criterion is satisfied satisfied, in this particular case the blue dot the that we see over here; it is fairly far away from the actual solution, if you look at. And if you compute the f of f of x at that particular point, at that point so this is going to be f of x, that also we realize is fairly far away from 0. We are trying to solve the problem f of x equal to 0, and we will see that f of x in this particular case is fairly far away from 0.

So, in this case we will conclude that the stopping criterion is not satisfied, and because the stopping criterion is not satisfied satisfied. We will have to go back solve this particular problem, once again using the chosen strategy, again move in the direction of the solution; again verify if the stopping criterion is satisfied. And keep doing this repeatedly, and once the stopping criterion is satisfied; this is going to be the numerical solution, that that we obtain. So, that is going to be a general strategy for whichever numerical method that we choose to use in this particular problem.

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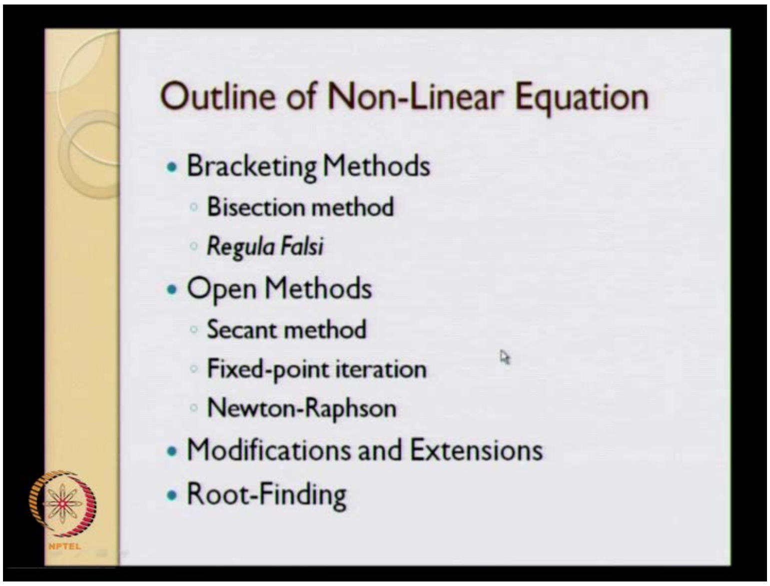

And the general numerical methods, that we are going to use for non-linear equation solving, they fall essentially in 2 different categories - one or one is known as bracketing methods; bracketing methods always will employ 2 initial guesses, and the 2 initial guesses are going to bracket the solution. For example, the one initial guess could be over here, the other initial guess could be over over here.

So, that they are bracketing the solution, and that and we then figure out a way to determine what the next next guess is going to be, and methods that will determine the next guesses. The 2 methods that will cover this particular module: one is called the bisection method, and the next one would be the Regula Falsi method. Then there are open methods which which either are going to start with 1 or 2 initial guesses, and then we will try to move towards the true solution. And those methods that we will cover in this particular course is the secant method, the fixed point iteration, and Newton Raphson method.

So, what I am going to do initially is cover all these methods for a single variable problem, then we will talk about cases where these methods are going to encounter certain issues, and certain problems in this solution techniques. And we will try to look at some modifications, some extensions of these methods. We will also look at extensions of of 2 of these methods specifically the fixed point iteration, and the Newton raphson method we look at their extension to multi variable system. And finally, something that we are going to do also cover towards the last lecture of this module is root finding.

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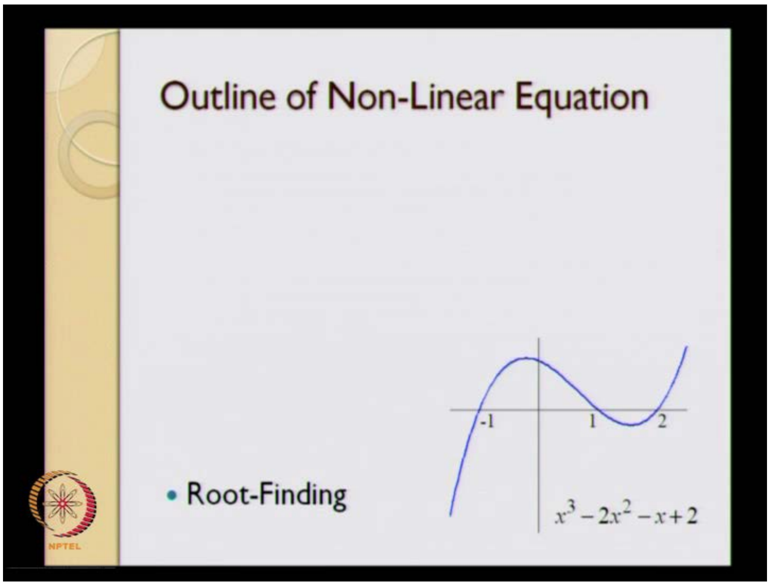

And what we mean mean by root finding is given a polynomial expression expression of this form. For example, this is x cube minus 2 x square minus x plus 2; we want to find out all the places, where this particular curve is going to intersect the X axis. So, for example, this particular equation can can also be written as x minus 1 multiplied by x minus 2 multiplied by x plus 1, that that is the expansion of this particular equation. And so, x equal to minus 1 x equal to plus 1 and x equal to plus 2 are going to be the all the 3 values of this particular of x that are going to satisfy this particular equation equal to 0.

So, the root finding methods are going to rely on some of the techniques that we will covered on the non-linear equation solving front, and its going to be essentially an extension of these non-linear solving methods. In order to find all the roots of a polynomial equation. So, that is the overall overview of what we are going to do in a in non-linear equation solving.

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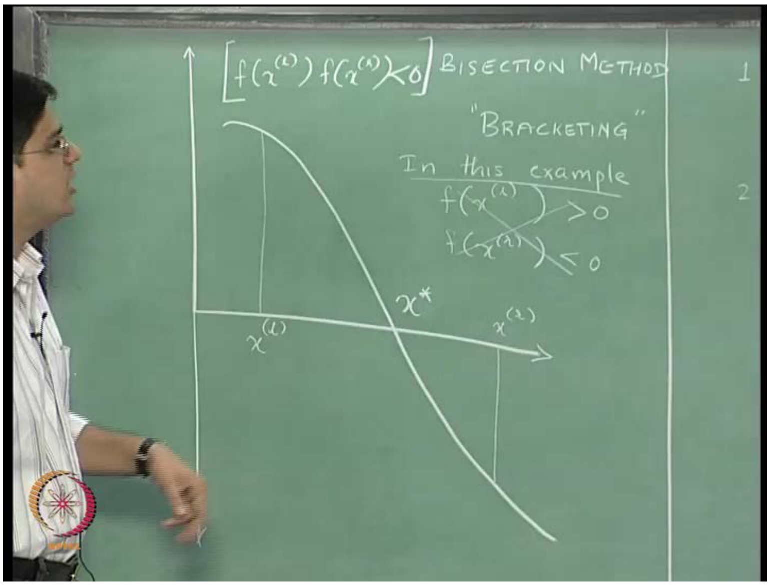

So, let let us look at again the particular curve that that we had previously. So, we will just look at points that go beyond x equal to 1; there was one more solution below x equal to 1 for for that particular curve, we will currently not worry about that. And we will just consider that curve, let us say in this particular form. And this is going to be the solution that we are interested in we will mark that solution as x star - the star representing that its, it is the true solution of the particular problem, I will just write it in larger letters.

Now, the bisection method it is a bracketing method; what we mean by bracketing method, essentially is that we we need 2 initial guesses, such that one initial guess lies to the left of this particular equation, then the other guess lies towards the right of this particular equation. So, the first guess that we are we will have, let say is and we will write this as x l; l to represent that it lies to the left of x star, and the other one we will write this as x r to represent that x r lies to the right of x star, x r lies to the right of x star.

So, this these are the two initial guesses, that we will start of with; since it is a bracketing method, these two guesses should lie on either side of x star. So, now the question is how do we know whether these two guesses lie on either side of x star well that is, because f of x l is going to have a sign which is different from f of x r. So, if… So, in this particular example f of x l is positive, and f of x r is negative in this example. So, if you multiply of course, if if the curve was reversed, if you if you can think of mirror image of this particular curve. In that case, our f of x l is going to be negative, and f of x r is going to be positive, but if we multiply f of x l with f of x r, because we have multiplying a positive number with a negative number, that particular product we know for sure is going to be negative; irrespective of what type of curve that we get. As long as, x l and x r are going to bracket the solution, the product of x l and x r is always going to be negative.

So, rather than looking at this particular criterion to decide whether x l and x r lie on either side of x star, the criterion that we will use to determine that is going to be f of x l multiplied by f of x r should be less than 0.

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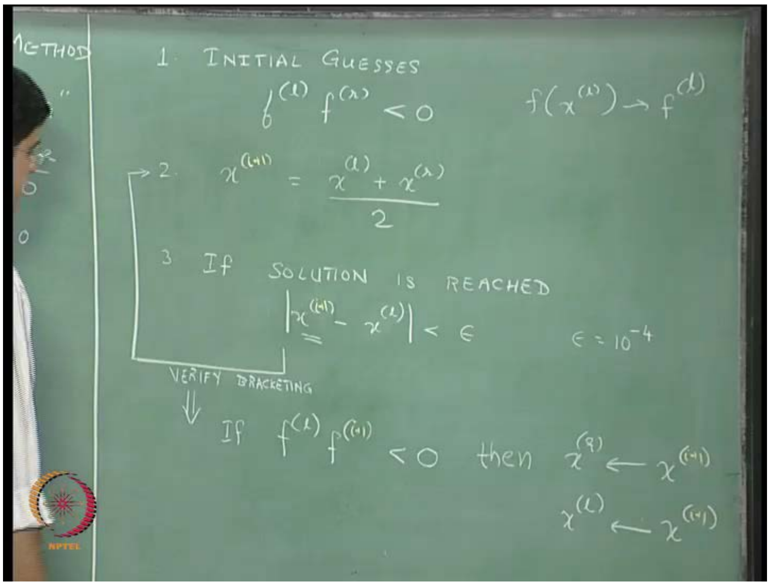

So, the first thing that we need to do is to determine the initial guesses, and the initial guesses are such that f of x l multiplied by f of x r should be less than 0. And from this point onwards I am going to use a shorthand notation f super script l in the bracket to represent f of x l.

So, for f of x l I will just use a shorthand notation f of l, this is going to be our initial guess. The second step that we said is to find out how we will move these from these starting at these 2 initial guesses, how we will get the next solution. So, that we move the initial guess to the next guess. And the bisection method or is also known as the midpoint method, and the idea behind behind my bisection method is quite simple. Is you move from x l, and x r to the midpoint between x l and x r. So, in this particular case the midpoint, let say is lies over here, and that is going to be our x 2. So, our x 2 is going to be our x l plus x r divided by 2.

So, that is going to be the midpoint of of the particular expression of of the particular line connecting x l, and x r. So, that is going to be our x 2.The next question is to check whether, if the solution is converged. In this particular example that I have drawn pictorially, what we see is that the solution is not converged, because we have not really reached the true solution or the other way to look at is that we are still far away from our f 2 our f of 2 x 2 is still far away from 0.

So, there are 2 ways to determine whether the solution is reached or not; one method to determine is whether f of 2 minus f of l, whether that is less than some predetermined value epsilon or the other method is whether, I am I am I am sorry, this should be x of 2 minus x of l. The previous solution, and the current the new solution, whether they are less than a predetermined value epsilon or the other way other criterion, that we can use is whether the function f of x 2, whether or not that is less than certain other value delta.

We can use either or both of these methods, some of the more advanced numerical techniques the that are available in various packages, such as mat lab and so on. They will use both these criterion criteria, and check both these criteria and determine whether the solution is reached or not. In general as we had done in module 2, what we had said is we are interested in finding out how close is our our new solution to the previous solution. And if the new solution is close enough to the previous solution, we will say that the solution is reached.

So, from this point onwards, I am just going to talk about the first criteria is whether the new solution is close enough to the old solution. And this epsilon value, we need to take we that is something that we need to predetermine is how exact we need our solution to be, for example one one way to choose this particular epsilon is this epsilon should be equal to let us say 10 to the power minus 4. For example, in in in this particular case of 2 minus x plus ln of x, the solution that we that we get are around 0.1, and the third, second solution that we get is of the order of 3.

So, what this particular criteria for the solution equal to 3 means, the is that the solution needs to be accurate in essentially up to the 4th decimal place. Beyond that we do not care about with respect to the solution that that we are trying to get. So, if the solution is reached; our x 2 is going to be the the actual solution. If it is not reached, we will increment I, and we will go back over here. And repeat repeat this particular situation.

Now, the question is before going back over there, we now need to ensure that x 2, and the other solution is going to still bracket the the the true solution.

So, in this particular example now, because x 2 lies on the same side of x star as x l our new x l, we are going to be replaced it with x 2. And now x 2, and x r are going to bracket this particular solution So, what before we go to the next step, what we need to do is verify bracketing, and what that essentially means is if f of l multiplied by f of 2 is less than 0; if f of l multiplied by f of 2 is less than 0. Then, what that means is l x l, and x 2 are lying on the either sides of the solution.

So, we replace x r with x 2. (No audio from 27:40 to 27:50) However, if f f of x l multiplied by f of x 2 is greater than 0, then we will replace x l with x 2. So, let us again look back at this particular figure, that the that we had; f of x l is positive, f of x 2 is positive, the product of these 2 terms is going to be positive. Therefore, we are going to replace essentially x l with x 2, because f of x l is because x l then is on the same side of x 2.

So, now what have what we end up doing is we this is no longer our x l, this becomes our new x l So, now we will go back, and we will then use solve this particular expression with x 3 equal to x l plus x r divided by 2. So, we will now have x 3 equal to x l plus x r divided by 2, whether or not the solution is reached; we will check that criterion by subtracting x l from x 3, and taking that absolute value verifying whether that is less than our predetermined value epsilon or not.

If it is less than the predetermined value epsilon, our solution is reached; if not we will we will multiply f l with f 3, verify if it is less than 0; if it is less than 0, then x r will be replaced by x 3. If it is not less than 0, then x l will be replaced replaced by x 3.

So, let us again go back to this particular figure, and see what what we get. So, the midpoint for this particular chord is over here, and this is our x 3. When we consider f of x 3, and multiplied with f of x l, we get the product f l multiplied by f 3 is less than 0; if f l multiplied by f 3 is less than 0, then we retain a x l, but we replace x r with x 3, and that is what I will I will do over here is replace x r with x 3.

So, this essentially becomes our new solution x r, and we repeat that over and over again until, we get convergence. And in order to repeat that, we will I will in this particular algorithm, I will replace all the purple parts essentially with the term i plus 1, and we will keep repeating that, until we get we basically get convergence.

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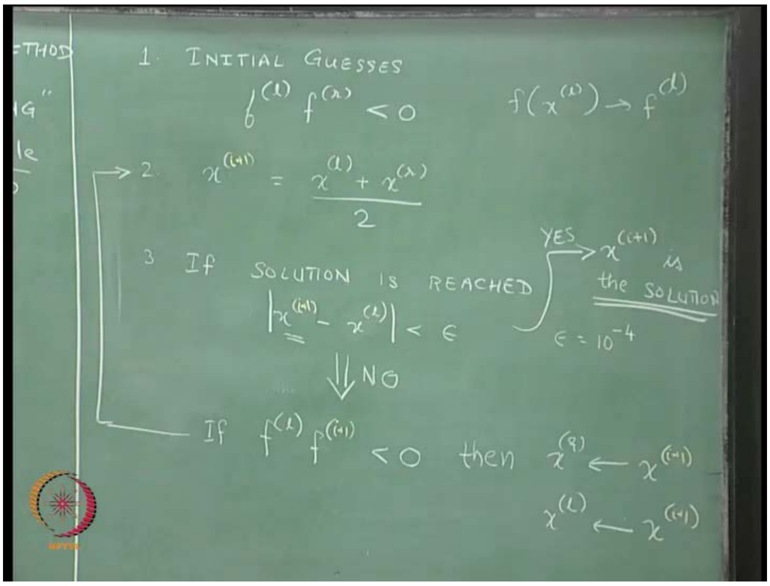

So, and I will just erase this part, and say (No audio from 31:19 to 31:30) and if say yes. (No audio from 31:33 to 31:5o) Essentially this is going to be our algorithm for the bisection method. Start with initial guesses figure out a way to get the new solution x i plus 1, in the bisection method this particular this particular guess is just going to be an (( )), the mid-point of the line connecting x l, and x r. Check whether the solution is reached, if the solution has not reached go back to the step 2 and repeat it, over and over again until the solution is reached to a particular particular pre specified criteria.

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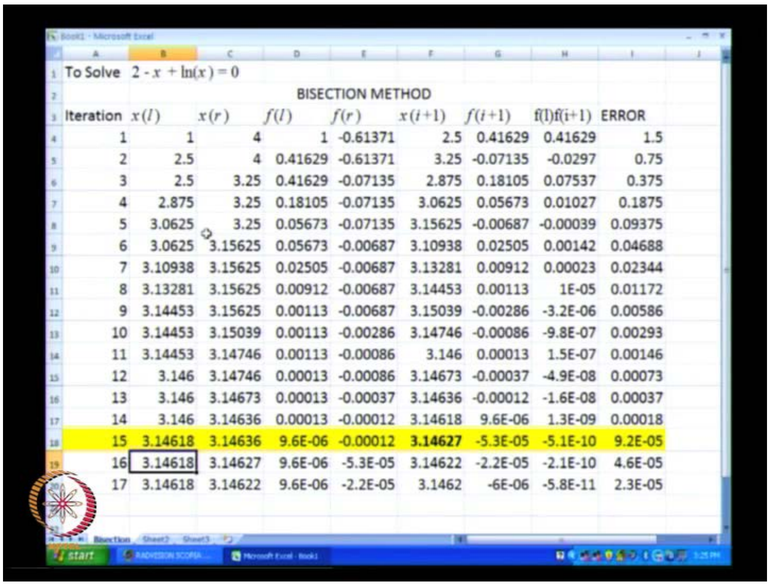

To Solve $2 - x + \ln(x) = 0$

BISECTION METHOD

Iteration	$x(l)$	$x(r)$	$f(l)$	$f(r)$	$x(i+1)$	$f(i+1)$	f(l)f(i+1)	ERROR
1	1	4	1	-0.61371	2.5	0.41629	0.41629	1.5
2	2.5	4	0.41629	-0.61371	3.25	-0.07135	-0.0297	0.75
3	2.5	3.25	0.41629	-0.07135	2.875	0.18105	0.07537	0.375
4	2.875	3.25	0.18105	-0.07135	3.0625	0.05673	0.01027	0.1875
5	3.0625	3.25	0.05673	-0.07135	3.15625	-0.00687	-0.00039	0.09375
6	3.0625	3.15625	0.05673	-0.00687	3.10938	0.02505	0.00142	0.04688
7	3.10938	3.15625	0.02505	-0.00687	3.13281	0.00912	0.00023	0.02344
8	3.13281	3.15625	0.00912	-0.00687	3.14453	0.00113	1E-05	0.01172
9	3.14453	3.15625	0.00113	-0.00687	3.15039	-0.00286	-3.2E-06	0.00586
10	3.14453	3.15039	0.00113	-0.00286	3.14746	-0.00086	-9.8E-07	0.00293
11	3.14453	3.14746	0.00113	-0.00086	3.146	0.00013	1.5E-07	0.00146
12	3.146	3.14746	0.00013	-0.00086	3.14673	-0.00037	-4.9E-08	0.00073
13	3.146	3.14673	0.00013	-0.00037	3.14636	-0.00012	-1.6E-08	0.00037
14	3.146	3.14636	0.00013	-0.00012	3.14618	9.6E-06	1.3E-09	0.00018
15	3.14618	3.14636	9.6E-06	-0.00012	3.14627	-5.3E-05	-5.1E-10	9.2E-05
16	3.14618	3.14627	9.6E-06	-5.3E-05	3.14622	-2.2E-05	-2.1E-10	4.6E-05
17	3.14618	3.14622	9.6E-06	-2.2E-05	3.1462	-6E-06	-5.8E-11	2.3E-05

So, this is going to be our algorithm; now what will do is just go back to Microsoft excel, and try to look at the example of 2 minus x plus l n x example, we will we will take a look at, if you do not have an access to Microsoft excel other alternative that you can use is a Google documents. You can basically, if you have a g-mail account, you can go to Google dot com; and use the Google document spread sheet. And essentially solve the same problem that we are solving over here.

So, let us let us look at this now, I will just make it of zoom to a full screen; if possible there will be. So, the problem that we are interested in solving is the equation 2 minus x plus l n x equal to 0. And we know one of the solution lies between 1, and 4; essentially that solution and some being something like 3.14 or something. So, let us type bisection method over here. what we have is we will start with 2 initial guesses: One initial guess will be x equal to 1, the second initial guess is x equal to 4. We will just call it iteration number 1. So, what we will now do is calculate f of x l, and f of x r.

So, we will just write that down f l, and we will write this as f r. So, f l is nothing but f l is equal to 2 minus x l; x l is in this particular column. So, 2 minus x l plus log ln of x l. So, what we have done is what we have over here is 2 minus B 4, B 4 is this particular cell, B 4 is nothing but x l plus log of B 4; which means log again of x l. We press enter, and this is the value of f of l that is being computed, and then we can just take our cursor to the bottom of this cell, and drag this cell to the right. And we will we will be able to compute our f of r, and if you click on the function key f 2, we will be able to see the equation for f of r. So, f of r now is equal to 2 minus C 4. So, C 4 is essentially x of r plus ln of x of r. So, the plus l n of x of r. So, that is essentially our f of r. So, f of r also has now been computed.

Next, what we what we need to do is to compute x of i plus 1, and the way we will compute x of i plus 1 is let us let us first write the heading x of i plus 1, and x of i plus 1 is nothing but x of l plus x of r divided by 2. So, x of i plus 1 is nothing but average of x of l, and x of r. What we next are going to do is then to calculate f of i plus 1 also. So, we will calculate f of i plus 1, I will write that down over here f of i plus 1; and again f of i plus 1 equals 2 minus x of i plus 1 plus logarithm of x of i plus 1. And that is what we have computed. The final thing that we also need to compute is the product f of l multiplied by f of i plus 1.

So, that is what we need to compute. So, we will just do this, and let us write that down over here. It is f of l multiplied by f of i plus 1, the font I will use this write times new roman. So, that we are able to see this very well So, the product f of l multiplied by f of i plus 1 is nothing but this multiplied by this. As we know that if the product of f of l multiplied by f of i plus 1 is a positive number; that means, essentially x of i plus 1 lies on the same side of the solution as x of l, as a result we are going to replace x of l with x of i plus 1.

So, if… So, what we will write over here is if f of l multiplied by f of i plus 1 is greater than 0, then x of l is going to be equal to x of i plus 1, else x of l is going to be equal to the previous x of l; that is what we have written over there. And the same way, if this particular term. So, if f of l multiplied by f of i plus 1 is greater than 0; that means, that x of r is going to be retained essentially the same same value, else x of r will be replaced replaced by x of i plus 1. So, this is what what we have.

So, this is our second iteration result. And then we compute f of l as before, so we just go over here, and drag it downwards. So, now, instead of computing f of l, we have ended f f of x l at this point, we have now ended up computing f of x of l at the new x l value. Likewise, we can we will just be able to drag this, and this should work. And then, we will drag x of i plus 1, and that is our new guess x for x x x of i plus 1, and then both of these guys we will just be able to drag.

So now, what happens is that the f of i plus 1 at x i plus 1 equal to 3.25 is a negative value; if it is a negative value, now that means, f of l is positive value, f of r is negative value. Because f of i plus 1 is negative value, it is going to be x r that is going to be replaced now by x of i plus 1. So, what we will do now is just take this entire row, and just drag it downwards and that should be enough. So, what what is going to happen is the procedure is going to be repeated at each and every row, and the stopping criteria is the error. Error is as we had discussed earlier, error is nothing but absolute value of the difference between x of i plus 1 minus x of l.

So, that is our error, and we just take this and drag it downwards. So, this is how our error changes with each iteration; a for the first iteration the error is 1.5, the second iteration error is 0.75, the third iteration error is 0.375, so on and so forth. Some of you might actually notice that every time we increase one iteration of the bisection method, we essentially are reducing the error by half.

So, in the first iteration error is reduced to 1.5, from 1.5 error reduces to 0.7 5, from 0.7 5 error reduces to 0.375 and so on and so forth. The stopping criteria if you recall that we had a we had decided was that the error the absolute value of the error should be less than 10 to the power minus 4. This is 1.8 10 to the power minus 4 which is greater than 10 to the power minus 4 whereas, this particular value is less than 10 to the power minus 4.

So, this is going to be essentially up our approximate solution. And so, what we will do is we will just color this yellow. So, our approximate solution essentially is going to be excuse me, essentially have approximate approximate solution is going to be 3.1463, that is that is the solution that we have gotten through this this procedure. So, to recap what we did was we started off with a guess value of x l, a guess value of x r; the first thing that we did is computed f of x l, next we computed f of x r, next we computed x of i plus 1. The x of i plus 1 was quite simple, it is nothing but an average of the value x l and x r, what we did next was computed f of x of i plus 1, f of x of i plus 1 is nothing but 2 minus x of i plus 1 plus log ln of x of i plus 1, that is what we did over there.

Next, what we had to decide was whether x of i plus 1 replaces x l or whether it replaces x r; in order to do that we had to compute the product f of x l multiplied by f of x of i plus 1. So, the product f of x l, and f of x of i plus 1, in this particular case is positive. Now, if this product is positive; that means, x l and x i plus 1 lie on the same side of the solution, because x l and x i plus 1 lie on the same side of the solution its x l, that is replaced by x of i plus 1. So, x l the original x l was one, we replaced it by 2.5, x r remained the same as before. f l is what we have computed using the new value of x of l, f r is essentially remains the same as as we had in the previous iteration.

Again we repeat the same procedure, we get x of i plus 1 which is nothing but an average of these 2 numbers, remember remember this x of i plus 1 was the average of these 2 numbers; this x of i plus 1 was the average of these 2 numbers, this x of i plus 1 going to be average of this these 2 numbers, So on and so forth. Again we computed f of i plus 1. Now, when we compute f of i plus 1, what we see is that the f of i plus 1 is essentially negative, because f of i plus 1 is negative; the x of i plus 1 lies on the opposite side of x l.

So, x l remains the same, but it is x r that we changed, and we updated to the newest value x of i plus 1. So, we keep repeating that procedure until the error that we get is reduced to below the the threshold value of error that we have decided. So, that is essentially how our bisection method is is going to work. So, I will just write this as bisection.

So, this is essentially our bisection method, what happens is that we started off with x l, and x r at 1 and 4 reduced the value from 1 and 4 to it is mid-point, we just 2.5 from the midpoint it went to the other midpoint between 2.5 and 4 which is that 3.25. Then we find the mid-point between these 2 points which is 2.875 so on and so forth. Each time replacing the particular solution x l or x r which is on the same side of the x star that is the true true solution. We keep replacing that, and keep repeating this over and over again. And that essentially or bisection method, and the solution of the bisection method in excel.

The next method that we talked about was the Regula Falsi method. The Regula Falsi method is also a bracketing method; that means, in Regula Falsi again we are going to start with 2 solutions.

The 2 solutions are going to bracket, the 2 guesses I am sorry are going to bracket the 2 solution as before. So, again going back to the overall algorithm that we had written for the bisection method, where we again start with the 2 initial guesses, again we compare we verify that the 2 initial guesses lie on either side of the true solution. We verify that by taking the product f of x l multiplied by f of x r, if that is less than 0, this x l and x r are indeed admissible points for bisection or admissible points for our Regula Falsi method. But in Regula Falsi method this particular method changes, and how that changes we will look at the graphical representation of the Regula Falsi, and then we will go on.

The third part, the next part remains the same; if the solution is reached, we stop we will take x i plus 1 as the solution. If the solution is not reach, then we will either replace x l with x i plus 1 or we will replace x r with x i plus 1, and we will continue further. So, this part again all these parts remains the same, its only this part where we are going to change.

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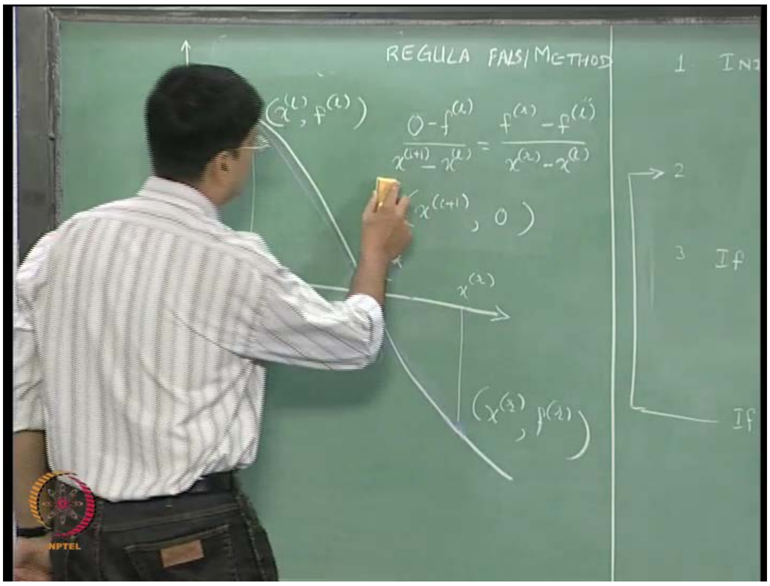

So, let us go back to the the figure that we had previously drawn, we will again start with the same x l, and same x r, and I will erase the points that we had generated using our bisection method.

So, in bisection method, what we did was we looked at this particular segment and reduced it by half, and and we proceeded. The other alternative to do this is, essentially to draw a straight line connecting these points. Draw a straight line connecting these points, and find out the point at which that particular straight line intersects the X axis. So, we will take this and this, and we will draw a straight line perhaps, the straight line will go.

(No audio from 47:38 to 47:55)

So, this is a straight line that we are going to connect between these two points, and the point where the straight line intersects the X axis, we are going to call that as our x 3 sorry our x 2. And then, we will keep repeating this procedure again and again till we we reach our desired solution.

So, now the the question is, what is really this particular point; the point over here is the x coordinate of that point is x l, and the y coordinate is f l. Since, we are no longer using the bisection method, but we are using the Regula Falsi method, I will just write that down over here. So, this particular point is x l, f l this particular point is x r, f r. So, the line connecting x l, f l with x r, f r, (( )) we can write is y minus f l divided by x minus x l is going to be equal to x r sorry is going to be equal to f r minus f l divided by x r minus x l.

This is the this it is straight forward to get this particular expression, the this is just a straight line joining the 2 points x l, f l; x r, f r. Now, the point of intersection of this particular straight line with the X axis is obtained by putting y equal to 0 over here. So, the point of intersection of this this particular curve with this particular sorry line with the X axis is going to be going to give us our new initial guess.

So, this particular point that I have shown over here is nothing but the point (x i plus 1,). This particular point lies on on this curve. So, it satisfies this particular equation. So, when we substitute these values over here, what we are going to get is essentially 0 minus f l is going to be divided by x i plus 1 minus x l is going to be equal to the right hand side.

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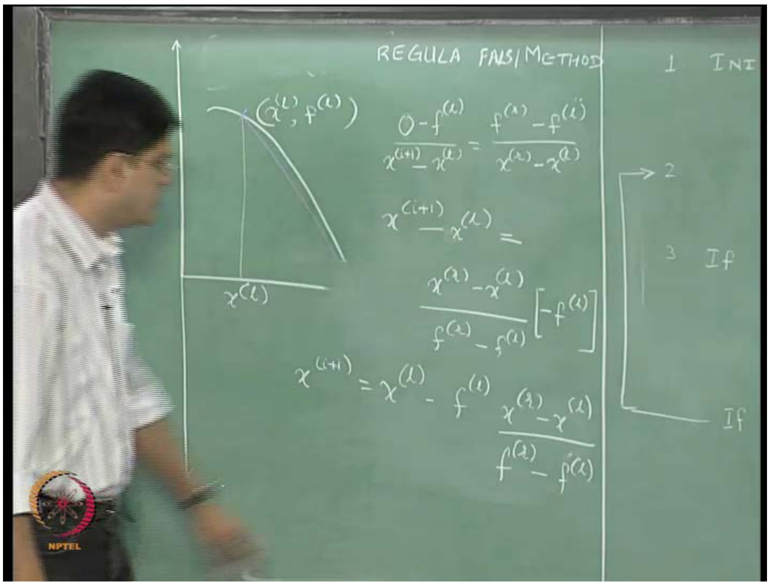

So, I will I will just modify that over here. So, what we will get is 0 minus f l divided by x i plus 1 minus x l is going to be equal to the right hand side. And we can just rearrange this equation, in order order to get the particular the solution for x i plus 1; and that solution is essentially going to be x i plus 1 is going to be… So, we will just invert that, and take f of x l on to the other side, we will have x i plus 1 minus x l is going to be equal to x r minus x l divided by f r minus f l multiplied by minus f of l. Or we we can also write this as x i plus 1 equal to x of l x of l minus f of l multiplied by x r minus x l divided by f r minus f l.

So, instead of generating x i plus 1 equal to x l x l plus x u divided by 2, x l plus x r divided by 2; we have just generated at using this particular this particular method. So, we will just replace the this step number 2 in our bracketing method over here, with the result generated just now.

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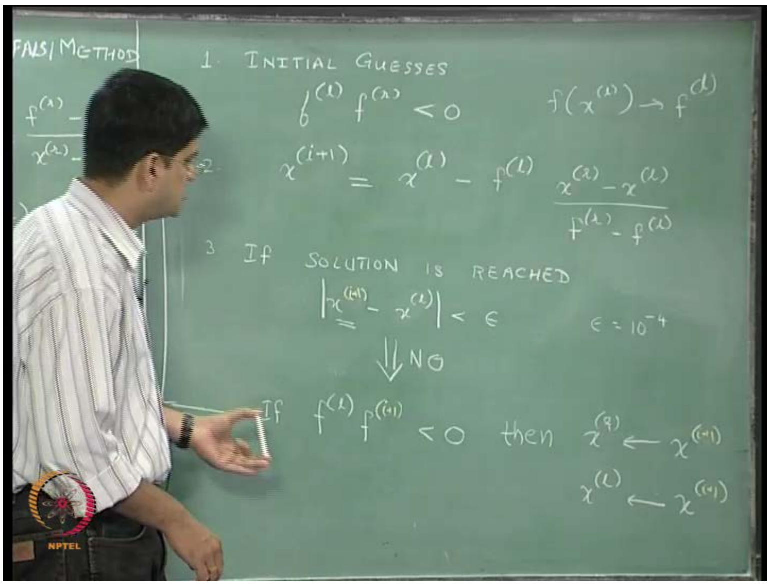

So, we will say x i plus 1, it is going to be equal to x l minus f of x of l multiplied by x r minus x l divided by f r minus f of l. So, this is the only thing that has changed, when we go from a bisection method to a Regula Falsi method; we will keep in mind f of l is nothing but f of x of l, f of r is nothing but f of x of r. And this is what will will will get as the overall algorithm.

So, we will we will choose f f l we will choose f r, we will then use this particular expression in order to get the new value of x i plus 1; compare x i plus 1 with x l, if it is less than epsilon the solution is reached. If it is not less than epsilon figure out, whether x x i plus 1 lies on the same side as as x l or whether it lies on the opposite side of x l; if it lies on the opposite side of x l, our x l will be replaced by x i plus 1, if it lies on the same side of x l over x. If it lies on the opposite side of x l, x r will be replaced by x i plus 1, if else it is x l that will be replaced by x i plus 1. So, that we ensure at each point that the 2 solutions are definitely the 2 guesses are definitely going to bracket our true solution x star.

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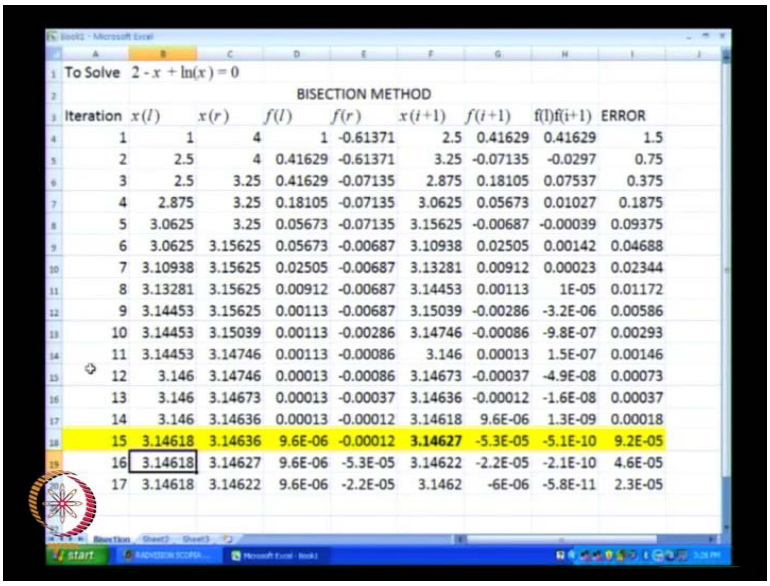
To Solve $2 - x + \ln(x) = 0$

BISECTION METHOD

Iteration	$x(l)$	$x(r)$	$f(l)$	$f(r)$	$x(i+1)$	$f(i+1)$	f(l)f(i+1)	ERROR
1	1	4	1	-0.61371	2.5	0.41629	0.41629	1.5
2	2.5	4	0.41629	-0.61371	3.25	-0.07135	-0.0297	0.75
3	2.5	3.25	0.41629	-0.07135	2.875	0.18105	0.07537	0.375
4	2.875	3.25	0.18105	-0.07135	3.0625	0.05673	0.01027	0.1875
5	3.0625	3.25	0.05673	-0.07135	3.15625	-0.00687	-0.00039	0.09375
6	3.0625	3.15625	0.05673	-0.00687	3.10938	0.02505	0.00142	0.04688
7	3.10938	3.15625	0.02505	-0.00687	3.13281	0.00912	0.00023	0.02344
8	3.13281	3.15625	0.00912	-0.00687	3.14453	0.00113	1E-05	0.01172
9	3.14453	3.15625	0.00113	-0.00687	3.15039	-0.00286	-3.2E-06	0.00586
10	3.14453	3.15039	0.00113	-0.00286	3.14746	-0.00086	-9.8E-07	0.00293
11	3.14453	3.14746	0.00113	-0.00086	3.146	0.00013	1.5E-07	0.00146
12	3.146	3.14746	0.00013	-0.00086	3.14673	-0.00037	-4.9E-08	0.00073
13	3.146	3.14673	0.00013	-0.00037	3.14636	-0.00012	-1.6E-08	0.00037
14	3.146	3.14636	0.00013	-0.00012	3.14618	9.6E-06	1.3E-09	0.00018
15	3.14618	3.14636	9.6E-06	-0.00012	3.14627	-5.3E-05	-5.1E-10	9.2E-05
16	3.14618	3.14627	9.6E-06	-5.3E-05	3.14622	-2.2E-05	-2.1E-10	4.6E-05
17	3.14618	3.14622	9.6E-06	-2.2E-05	3.1462	-6E-06	-5.8E-11	2.3E-05

So, now let us go back to Microsoft excel, and try to see how we can solve the same problem, 2 minus x plus l n x equal to 0. This time using essentially not the bisection method, but the Regula Falsi method, as we had seen a few a few moments back, the difference between bisection method, and Regula Falsi method is only in the way x of i plus 1 is computed; remaining everything remains the same.

So, because of that I will for just to save sometime, what I will do is I will just copy this particular worksheet by clicking on move, and copy; I will just create a copy on this worksheet before sheet 2. So, now what we have is a copy of by the worksheet in which we use the bisection method, I will call this Regula Falsi; and in Regula Falsi again we are interested in solving the problem 2 minus x plus ln of x equal to 0.

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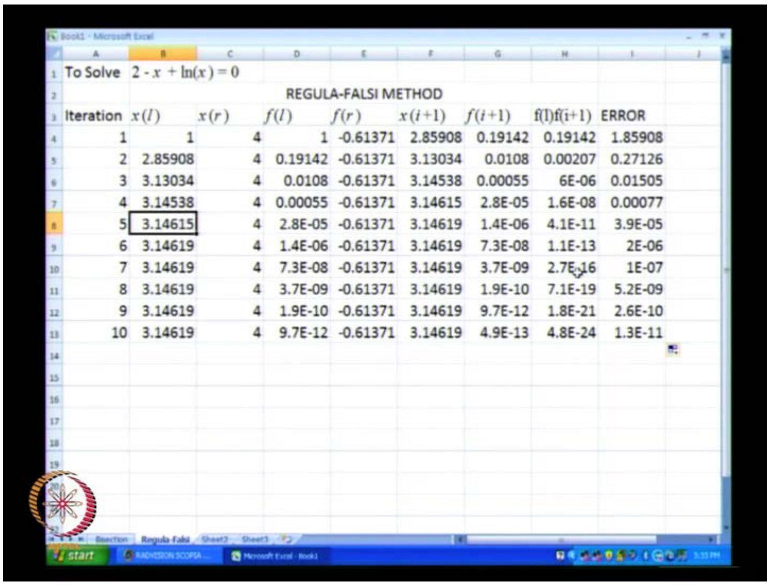

This time using r e Regula Falsi method. So what I will do is (( )) all these values I will just delete them. So now, we have our original worksheet, the first row of the worksheet that we we had used for the bisection method, and that that is what we will use essentially for our Regula Falsi method. So, our x l again the same way as before we start with x l equal to 1, x r equal to 4; we compute our f of x of l in the same manner as before, we compute our f of x of r in the same manner as before. Now, what changes is how x i plus 1 is computed, and the way, so what I have done is just deleted this x of i plus 1, and because this x of i plus 1 does not have any value in it over here, that is why we are getting an error, when excel computes its solutions.

So, x of i plus 1, if you recall was equal to x of l minus we had x of l minus f of l multiplied by x l minus x r multiplied by x l minus x r divided by f l minus f r. This is our x of i plus 1. So, again what what we have done is I will just write down the equation x for x of i plus 1; x of i plus 1 is nothing but x l minus f of l multiplied by x of l minus x of r divided by in brackets f of l minus f of r. This is what our just to make it easy to read, I will just make it into times new roman. So, in now now let us look at the function that we get over here f x of i plus 1. So, what we need x of i plus 1 is nothing but x l, which is the column B row 4 that is B 4.

So, x l is B 4 minus f of l which is D 4 multiplied by B 4 minus C 4 right; B 4 minus C 4 which is x of l minus x of r divided by f of l minus f of r, f of l is in basically D 4, and f of r is in D E 4. So, what we have is B 4 representing x of l minus D 4 representing f of l multiplied by B 4 minus C 4, which represents the difference x of l minus x of r divided by D 4 minus E 4, which represents the difference f of l minus f of r. This is what we have over here, once we get x of i plus 1, we will compute f of x of i plus 1 in the same manner, again we have not changed it from now what we had done in the bisection method. Remember when I was showing it on the board all we did was we replaced the way in the second part, the second step in that is the way we compute x of i plus 1 given f f of given x l and x r; that is the only thing that changes.

So, f of i plus 1 is nothing but 2 minus x of i plus 1 plus ln of x of i plus 1. We get the product f of l multiplied by f of i plus 1, because this particular product is positive; that means, essentially x x of i plus 1, lies on the same side of as x of l, and we replace x x of l with x of i plus 1. So, what we are now going to do essentially is really just take these guys, and just drag them below; and this is going to be our new x of i plus 1. And likewise, we will have a new value for error as well.

So, this is the result at iteration 2, and if you want to get the results at iterations 3, 4, 5, 6, 7, 8 so on and so forth. We just highlight the entire row, go to the right bottom edge of that row, and just drag it downwards, and let us drag it undo until set an iterations. So, this is what happens in 10 iterations using the Regula Falsi method, with we started with 1, 4; the next value of x of i plus 1 is 2.8859, and so on and so forth. And as you can see our stopping criterion was that the error should fall below 10 to the power minus 4.

So, this is the point, where the error has fallen below 10 to the power minus 4, and this is going to be the solution that we get. And again we will highlight this solution, and we really do not need any of any of this data those, that is redundant data.

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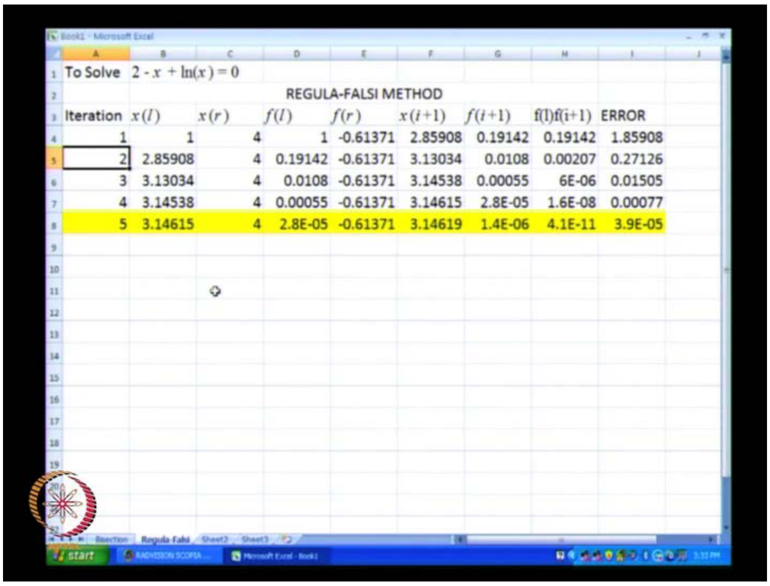

So, when we are using the Regula Falsi method, now what we actually have is basically in about 5 iterations; we have gone from the initial solution of (1,4), and we have finally, reached the actual solution 3.14615. That particular solution is accurate within the desired accuracy, that we had selected that is 10 to the power minus 4. So now, what we will do is just take a take a few minutes before finishing off this lecture, take a few minutes to just recap what we have done in this particular lecture.

So, what we have done in this lecture is initially we gave an overview of what we mean by non-linear equation solving; non-linear equation solving is nothing but finding out the points at which any curve f of x intersects the X axis, that is the interpretation - the geometric interpretation when we look at this problem in one-dimensional sense. So, that is the geometric interpretation of solving the non-linear equations. We then talked about the various methods of solving the non-linear equations. The methods came in two categories; the first category was bracketing method, in which we start with 2 solutions, and the 2 solutions are going to bracket our actual solution x star.

At each point of time, we will ensure that the that the 2 guesses are going to lie on either side of our true solution x star; that is what we we did in both the Regula Falsi method as well as the bisection method; the way bisection method worked is we take an initial segment. For example, in the example that we took we said took the segment between x of l equal to 1, and x of r equal to 4; and at each iteration we half the overall length of segment four. So, the length of segment from 3 at went to 1.5 from 1.5 at went 2.75, 1.75 it went from 2.375, and so on and so forth. And we slowly converge on the true solution x star; that is what we did in the bisection method.

Then we covered the Regula Falsi method - in the Regula Falsi method, the overall procedure was the same, except the way we found the the solution x of i plus 1; the next guess of the solution x of i plus 1. And the way we did that is we joined the point x l, f l, and the point x l, x r, f r; we joined it with a straight line; and found the point where that straight line intersects the X axis that became our new guess.

So that is how, we we we covered covered the two bracketing methods, bisection method, and the Regula Falsi method. In the next lecture, we are going to cover the open methods; specifically the fixed-point iteration, the Newton rap son’s method, and the secant method. Thank you, and see you in the next lecture.