Computational Techniques
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Module No. # 03
Lecture No. # 01
Linear Equations

Hello and welcome to module 3; here we are talking about computational techniques to solve linear equations. So, what I am going to do in this particular module is consider linear equations and linear systems, and go over some of the methods to solve these linear equations.

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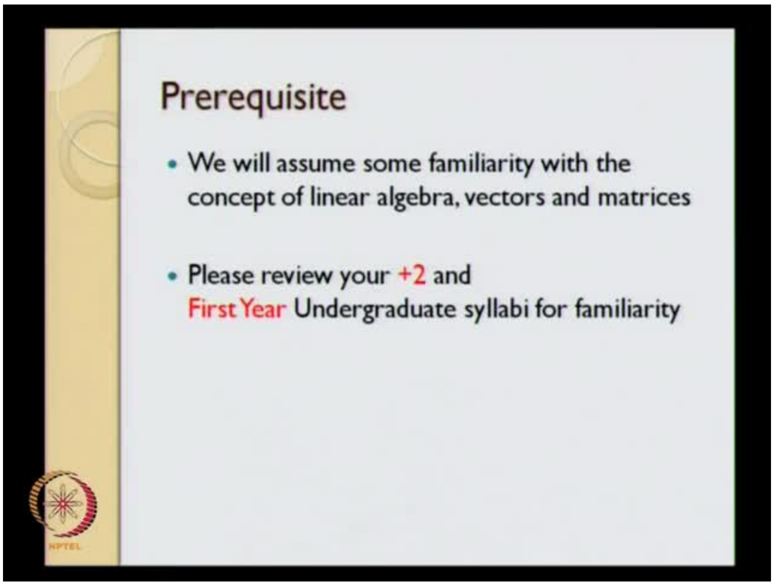

So, we will first go and motivate ourselves, what linear equations are; before doing that we will just talk about some of the prerequisites. I will assume some familiarity with the concepts of linear algebra, vectors and matrices.

For example, I will assume that you will know you know what how to multiply 2 matrices or what vectors are, how what are the addition rules for the vectors and matrices, what are the multiplication by scalar rules for vectors and matrices and all these things I am going to assume. We will just do a quick recap of that, but it is just going to be a few minutes of a short recap; just to orient ourselves with the terminology that I will use in this particular module. If you are not familiar or little bit rusty with these concepts, then I would request you to please, review your plus 2 syllabus and your first year undergraduate syllabus; specifically look at the linear algebra vectors and matrices parts of those courses.

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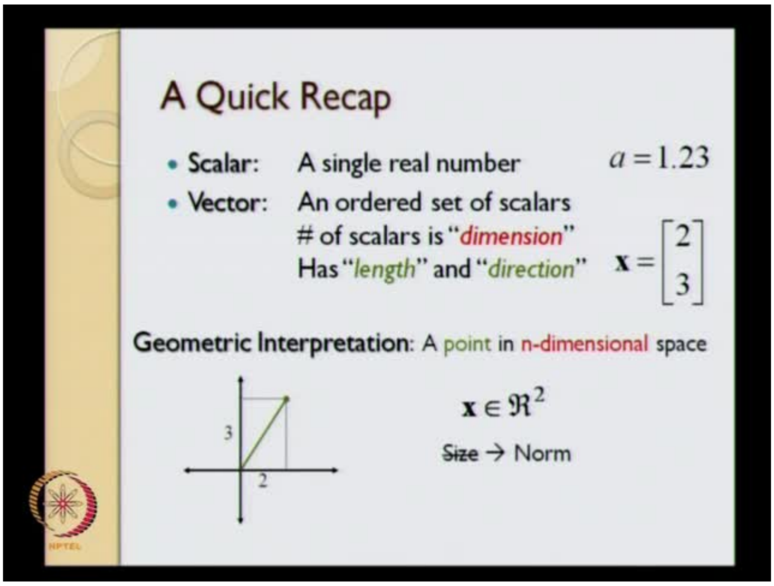

So, those are essentially going to be prerequisites for this course. What I will do in the next few minutes is just go over some of these concepts - what do we mean by scalars vectors, what are matrices and so on. So, quick recap; so the question, what is a scalar? A single real number in this particular case is what I define as a scalar.

So, an example over here is, a equal to 1.23, this is going to be scalar; so it is just a single number and we are restricting ourselves only to real number. And a vector on the other hand can be considered as an ordered set of scalars, what we mean is, that it is not just one single number, but several numbers. In this particular case, we have two scalars; number 2 is a scalar, number 3, itself is a scalar and an ordered set these two scalars is a vector; these vector we have represented as x over here.

Now, the number of scalars that go in representing a vector typically gives the dimension of that vector. For example, in this particular case, the vector is two-dimensional; now, these vectors can either be row vectors or column vectors. This particular case, we have basically 2 rows and a single column; the other way of representing this is x equal to 2, 3 which basically means that it is going be a vector in a single row that is a row vector.

However, consistently when we talk about vectors in this particular module, what we would mean is that the vector consists of n rows and one single column; so this is how we will typically represent a vector.

So, if for example, if a vector is five-dimensional, there will be five numbers stacked on top of each other; so 12345 that would be essentially a five-dimensional vector in our case and it would not be in in the horizontal direction, the vector that we represent.

So that is what terminology we will follow for this particular module. Now, another concept associated with vector is the length of the vector; for basically a distance of the vector you can call that. And the second property is going to be the directionality of the vector; so, it has both length associated with it and a direction associated with it. Geometric interpretation of a vector is, you can consider a vector as a either point in an n- dimensional sub space n-dimensional space sorry So, this particular point is going to be a vector that is 2, 3 that means that the dimension in the first direction is going to be 2 and dimension in the distance in the second direction is going to be equal to 3. So, one way to look at it is a vector is going to be a point that in an n-dimensional space, but a more appropriate way or a more common way to look at it is a point that is connected by an arrow or a line to the origin.

So, when we say a vector 2 3, what we what image it typically conjures up is basically of a line going from the origin to a point 2, 3 and as shown by this light green line over here; so that is going to be the geometric interpretation.

In this particular example, the dimension of this vector is 2 and we represent this as, x is an element of the R 2 space; R representing real numbers and 2 is represents the dimension of that particular space. And just a moment back I said that, we considered that the vector has a length or a size of the vector.

So, instead of talking about length, we will actually talk about the norm of a vector and the length of a vector is nothing but what is known as a Euclidean norm of a vector. For example, in this particular case, the length of this vector is going to be equal to 2 squared plus 3 squared square root of that number.

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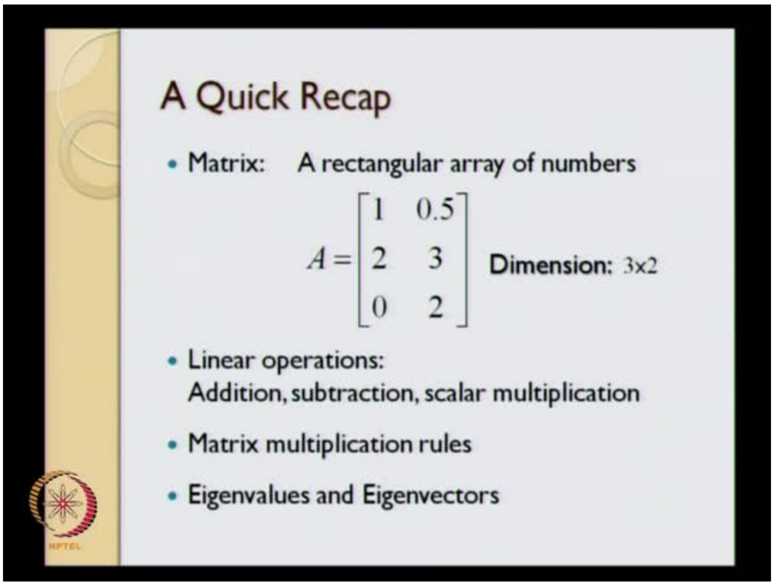

So, the length of this vector is going to be square root of 4 plus 9 or square root of 13. And then there are several linear operations that we can do on this particular matrix and that is addition, subtraction, scalar multiplication these are the various linear operations that can be that can be done on a matrix or bunch of matrices. And we should be familiar with the rules of addition and subtraction; you can add 2 for addition of 2 different matrices, the dimensions have to be the same, that means you can add a 3 by 2 matrix to another 3 by 2 matrix; you cannot add a 3 by 2 matrix to a 2 by 2 matrix for example. Same thing with subtraction, you can multiply any matrix of any size with a scalar and so if we have to if we were to consider alpha A, alpha A is going to multiply each element of the matrix A.

Then there are matrix multiplication rules; again, I would not go into the matrix multiplication rules, but for the 2 matrices to we for us to be able to multiply 2 matrices those matrices have to be commutative, that means that the number of columns of the first matrix should be equal to the number of rows of the second matrix.

So, in this particular case, we cannot multiply A multiplied by A, because it is a 3 by 2 matrix; so when we are we cannot multiply 3 by 2 matrix to another 3 by 2 matrix, but we can multiply A with A transpose in which case it is going to be a 3 by 2 matrix multiplied by a 2 by 3 matrix and the result is going to be a 3 by 3 matrix.

So, these are again the matrix rules multiplication rules and you should be familiar with the matrix multiplication rules; if not, it is a good idea at this stage to stop this video and perhaps, go and revise some of the matrix multiplication rules. And the final thing is going to be that, I will assume that some familiarity with these Eigen values and Eigenvectors. We will go over, what we mean by Eigen values and Eigenvectors, but we will not necessarily discuss, how to get Eigen values and Eigenvectors and so on and so forth, that is, what I will assume you already know based on your previous plus 2 syllabus and your first year undergraduate syllabus.

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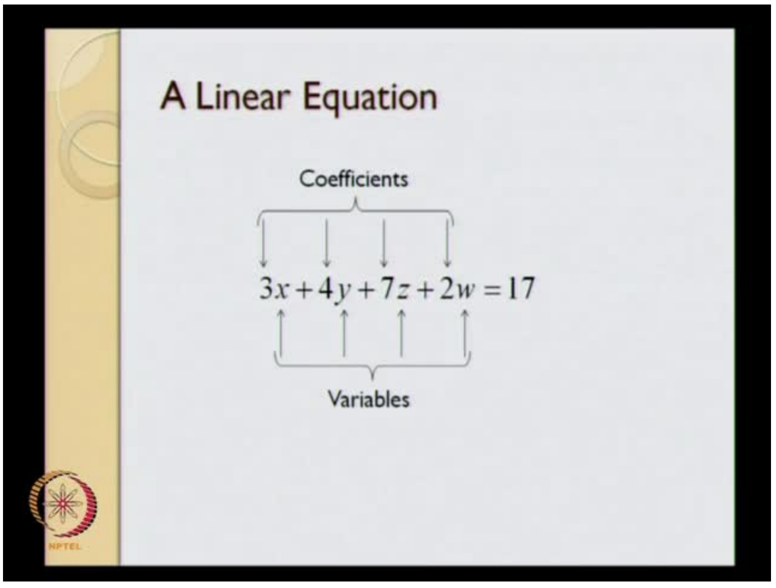

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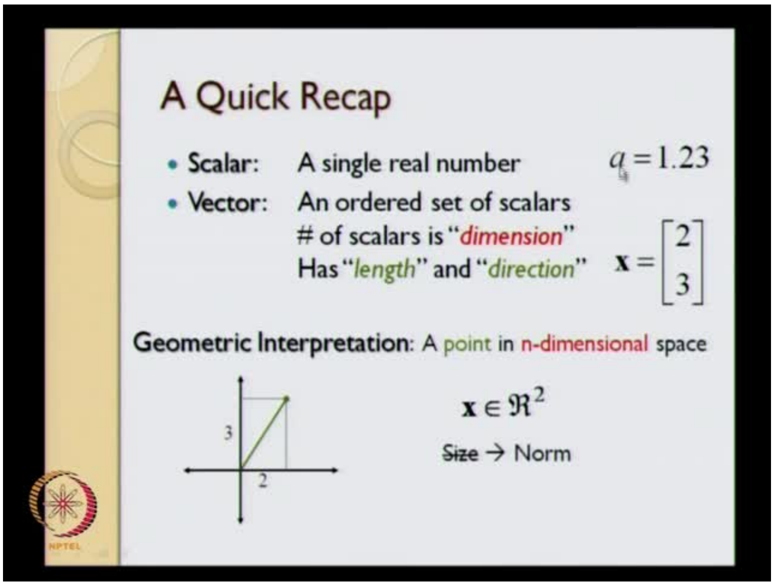

So, this really is a quick recap, and what we mean by a linear equation, is an equation of this type; for example, 3 x plus 4 y plus 7 z plus 2 w equal to 17; so x y z and w are the various variables. (Refer Slide Time: 08:50) So, since, I have used a notation of this form, this notation will represent a scalar and a bold phase notation will represent a vector. So, in this particular case, we have four variables x y z and w; there are four coefficients 3, 4, 7 and 2; so this represents the left hand side of the equation and the right hand side of this equation is 17. So this is a linear equation that we have.

Now, if we want to try to solve this linear equation, there are four unknowns so that we need at least four equations in order to get a unique solution. For this particular case, though if we have four equations that does not guarantee a unique solution or that does not even guarantee occurrence of a solution. And based on the past knowledge, I think most of us over here will be familiar with the fact that for n equations and n unknowns to have a unique solution, the rank of the matrix has to be equal to n. Now, what that means and what exactly we mean by rank of the matrix should be n, where exactly does the statement come from, those things we are going to discuss in the next few minutes.

So, this was over all a recap of the things that we have done so far, and what I am going to start now is really start with the main meat of the module that I wanted that I intend to cover in this particular lecture and in the subsequent 5 lectures over here.

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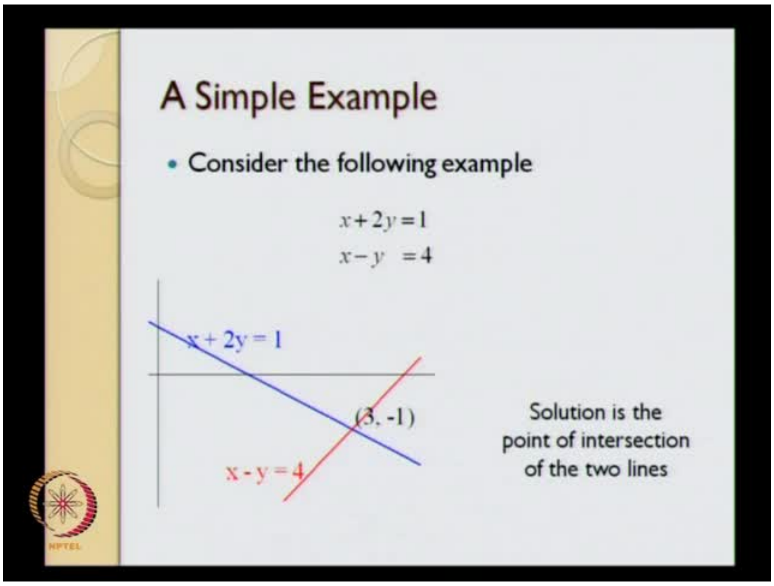

So, let us consider a simple example; the simple example that we are going to consider is x plus y equal to 1 that is one equation and x minus y equal to 4 that is the second equation; these two equations in 2-dimensions represent a line. In case of 3-dimensions, for example, if you had x plus 2 y plus 3 z equal to 5; in that particular case, that equation would have represented a plane in that particular three-dimensional space. So any equation…

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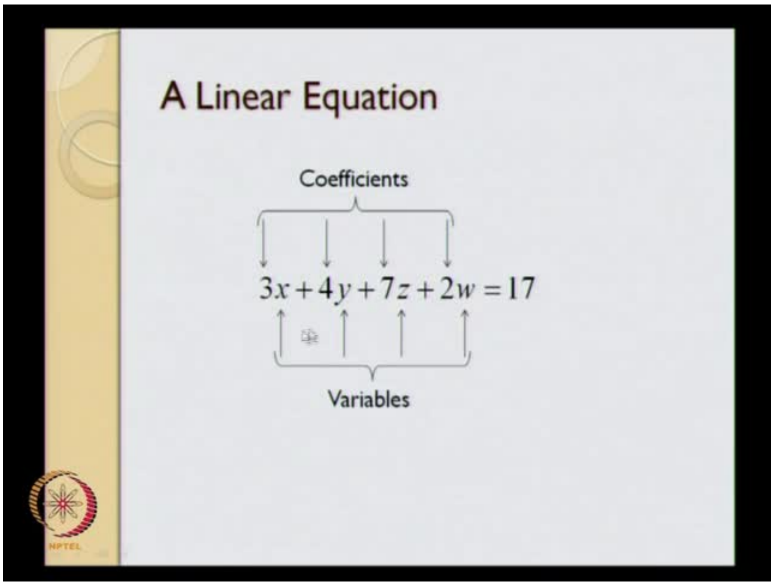

So, for example, this particular equation it is because we have four variables; so this particular equation is going to reside in a four-dimensional space and this will represent a 3-dimensional surface or a 3-dimensional hyper surface as its technically called in a four-dimensional space.

So, what happens is, when we write down an equation of this sort, out of four variables 1 degree of freedom is lost, why? Because all these four variables have to satisfy one particular condition; so, when one of the degrees of freedoms is lost by writing down an equation of this sort, what this will represent? It will represent a three-dimensional hyper surface in a 4-dimensional space.

So, we have one equation x plus 2 y equal to 1 that is represents this blue line over here and we have another equation x minus y equal to 4 and that represents the red line over here and the point of intersection of these two lines is the 0.3, minus 1 and that point is the solution of these two equations.

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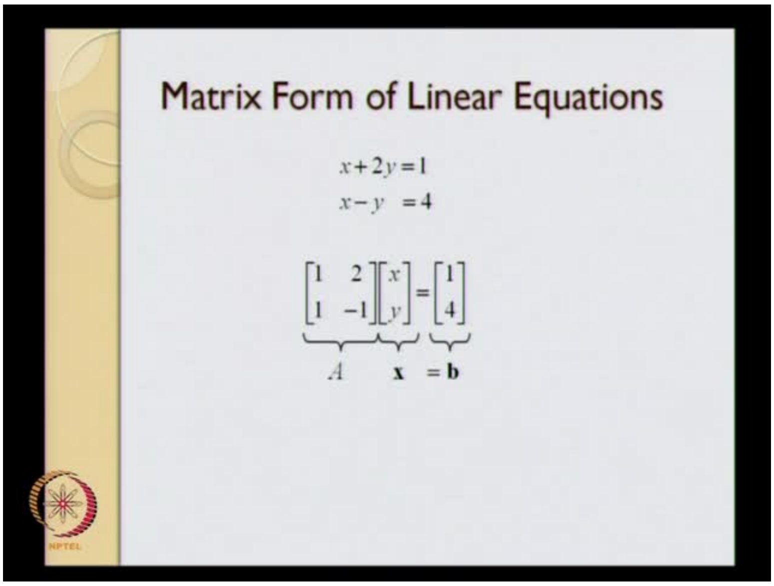

So that is the point that satisfies both these equations. Now, we can represent these equations in the matrix form, in this particular form. So, what we have is the first coefficient goes of the first equation goes over here; the second coefficient of the first equation goes in row 1 column 2 and so on and so forth. If you had multiple variables, if the first equation the coefficients will go on the first row of the matrix A; the right hand side will go on the first row of the right hand side of the equation and that is the vector b. Likewise, we have 1 multiplied by x plus minus 1 multiplied by y equal to 4; so that becomes our second equation.

We might have n number of equation in m number unknowns; if that is the case, we will have n number of rows that is the number of rows will be equal to the number of equations, and m number of columns that means the number of columns will be equal to the number of variables. So that is the matrix form of linear equation.

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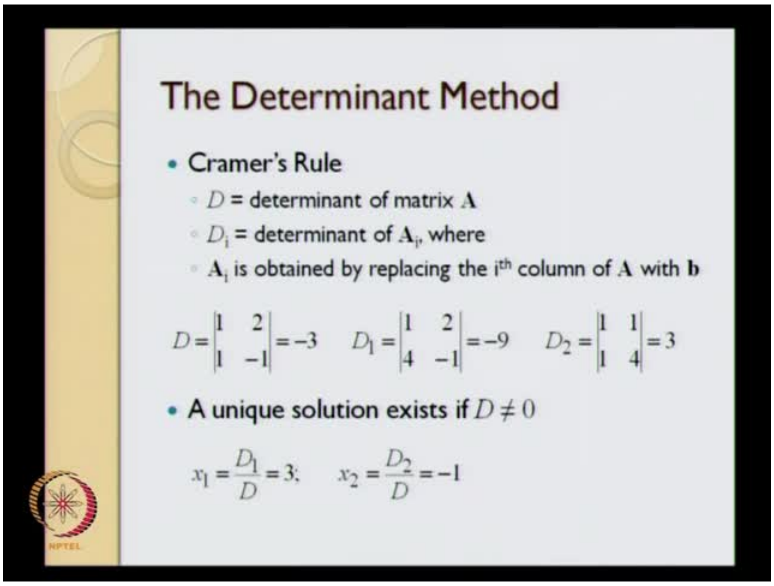

And we will talk about the determinant method to solve the linear equations. This method is not used in general for anything beyond say 3-dimensional equations and the method again is very popular as Cramer's rule; it is I am quite sure most of first of us would have would be aware of the Cramer's rule. What we do is we define several determinants of the matrix; the original matrix A that was 1 2 2 minus 1; we will take that determinant of that matrix and that determinant will call it as D and that value of the determinant is minus 1 minus 2 that is equal to minus 3.

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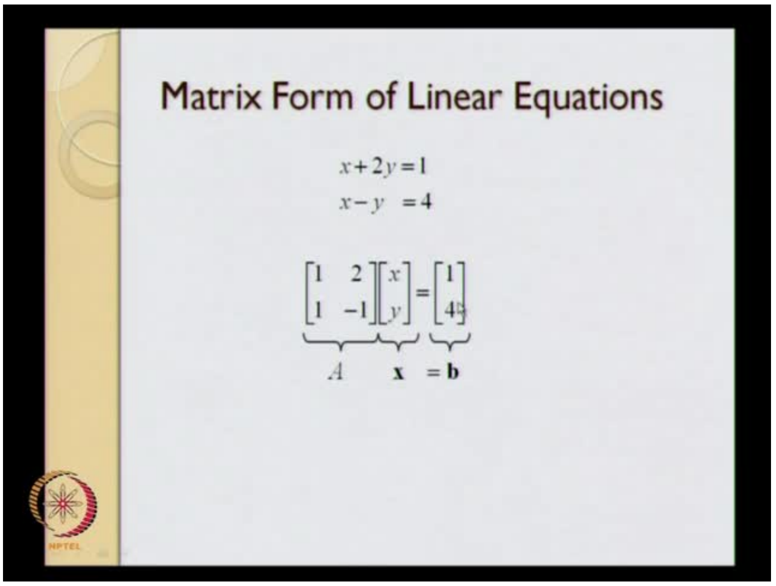

Then we will construct determinant the matrix sorry; we will construct matrix A 1 and A 2; A 1 is constructed by replacing the first column of this particular matrix A with the vector b; so we will replace 1 1 with 1 4 and we will call that matrix as A 1, and our matrix A 2 is going to be, when we replace 2 minus 1 with 1 4.

So, we will construct matrix A i by replacing the ith column of A with the elements of vector b; we will then take the determinant of A; take the determinant of all the matrices A i. And so in this particular case, the determinant of A we have represented as D, determinant of A 1 we have represented as D 1, and determinant of A 2 we have represented as D 2. As you can see, A is 1 1 2 minus 1, D 1 is 1 4 2 minus 1 and A 2 is 1 1 1 4. In A 2, we have replaced 2 minus 1 with 1 4 and in A 1, we have replaced 1 1 with 1 4 and we get these values of determinants. And a unique solution exists if determinant D that means the determinant of the matrix A is not equal to 0; if this determinant is equal to 0, then we will either have no solution or we will have infinite number of solutions.

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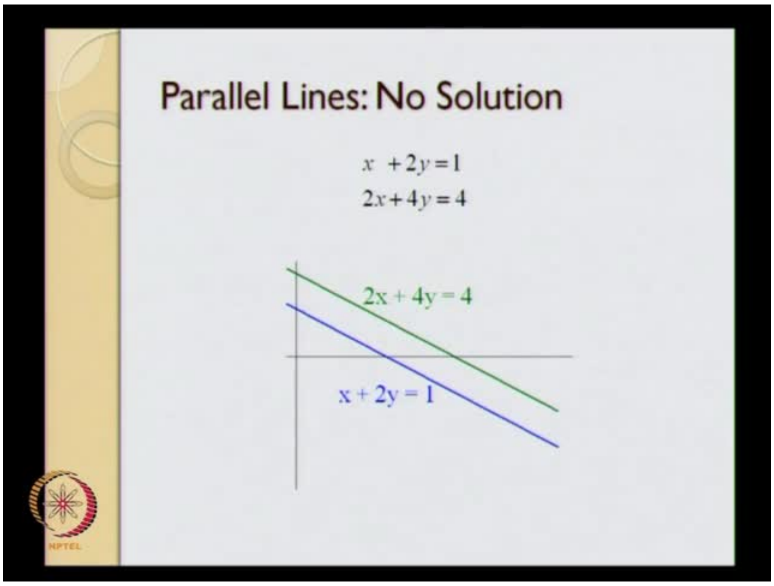

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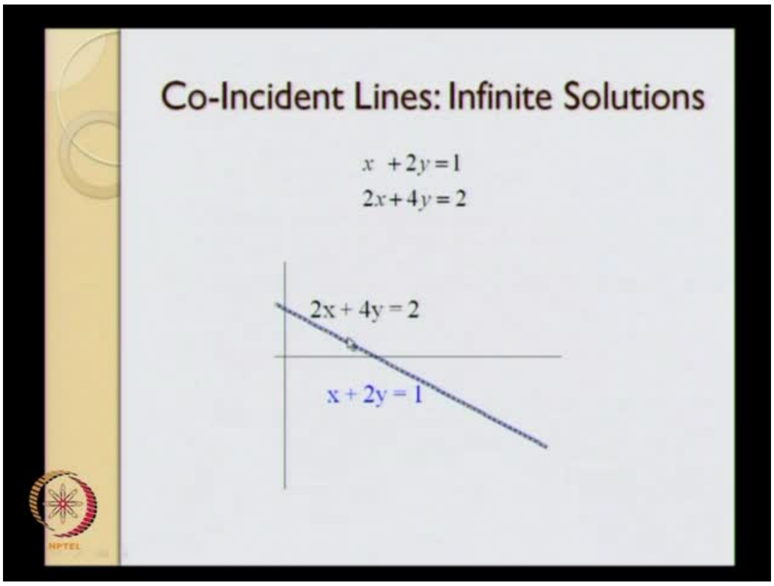

Let us take up couple of examples of no solution; so the first equation was 1 plus 2 y equal to 1. If the second equation is written as 2 x plus 4 y equal to 4, what this represents is, this represents two parallel lines; the first line is x plus 2 y equal to 1; the second equation is 2 x plus 4 y equal to 4. Now, because these two lines are parallel to each other, there is no point of intersection; so these two equations do not have any solution. But if we were to write the second equation as 2 x plus 4 y equal to 2 keep in mind, when we write is as 2 x plus 4 y equal to 2 what we are doing essentially is just multiplying the first equation by 2 and if we were to do that both these lines will be coincident.

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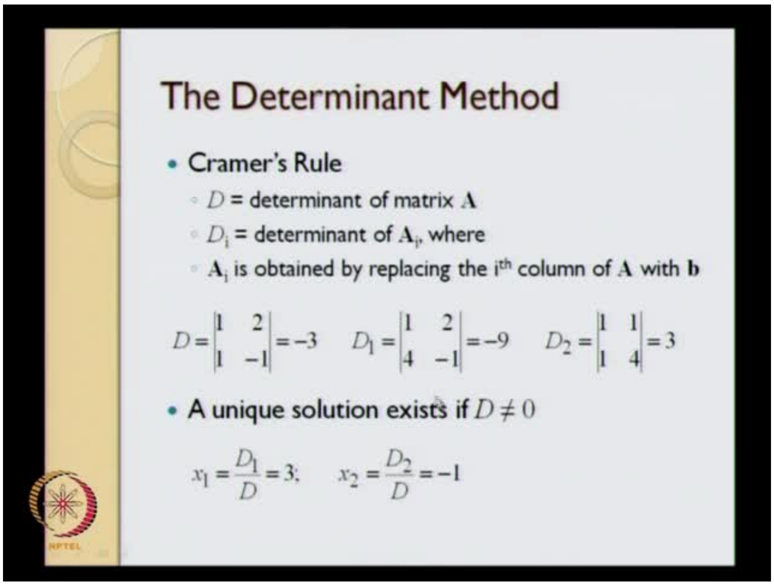

So, if these two lines are going to be coincident, we will have infinite number of solutions; in both these cases, the matrix A is going to be 1 2 2 4 and if we take the determinant of the matrix A, it is going to be 4 minus 4 equal to 0. So, in either case if the determinant of the matrix A is equal to 0, we do not have a unique solution and if the matrix, if the determinant of matrix A is not equal to 0, then we will have only one solution.

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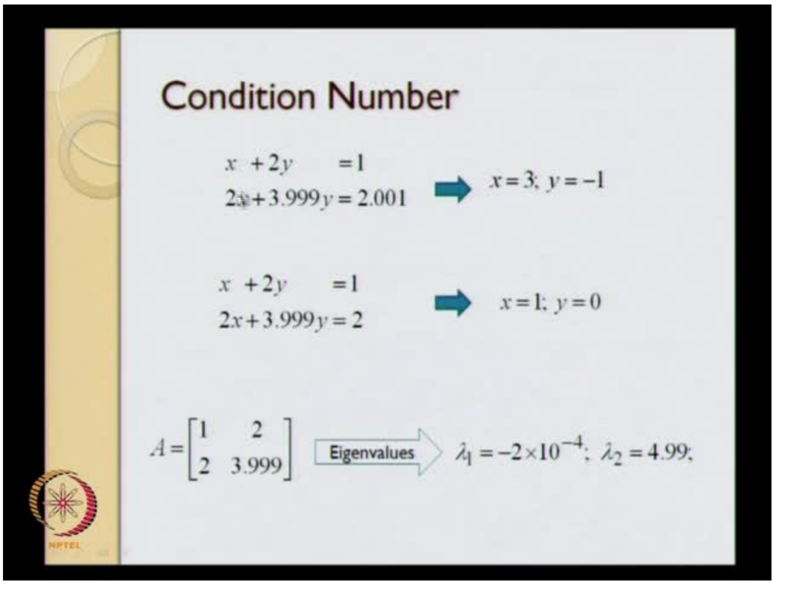

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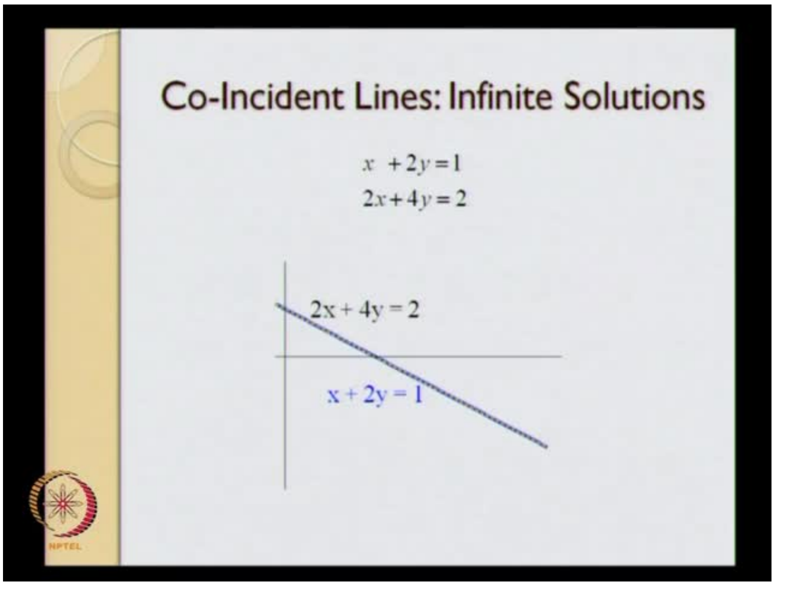

If we do not have a unique solution, we can either have no solution as in this case or we can have infinite number of solutions as in this particular. And in this particular example, we have x plus 2 y equal to 1 as the first equation and the second equation is 2 x plus 3.999 y equal to 2.001. So, if you go back to the previous slide, what I have done is instead of writing the second equation as 2 x plus 4 y, I have written the second equation as 2 x plus 3.999 y.

When we do that we can see that, what that actually means is that the second line is not exactly coincident on this line or not exactly parallel to this line; it is sloped at a very small angle to this particular line. As a result of this, what happens is that small changes in any of those numbers associated with matrix A or with the vector b to the right hand side, a very small change in any those numbers can result in large change in the solution.

For example, the first set of equation is x plus 2 y and equal to 1, and 2 x plus 3.999 y equal to 2.001 and the solution we get is x equal to 3 y equal to minus 1. But now, if the right hand side instead of being 2.001, if it was just equal to 2, so the equations remain very similar to each other; the only thing that has changed is this particular number has changed from 2.001 to 2 and we see that the solution has changed from x equal to 3, minus 1 to x equal to 1, 0. So, a very small change in one of these numbers or more than one of these numbers is going to cause a very large change in the solution; such type of systems are known as ill conditioned systems or poorly conditioned systems and whether the system is well conditioned or poorly conditioned is given by the ratio of the highest Eigen value to the lowest Eigen value.

So, the matrix A in this particular case is 1 2 2 3.999 that is the matrix A; if we take the Eigen values of this particular matrix, the smallest Eigen value is minus 2 into 10 to the power minus 4, and the largest Eigen value is 4.99; the ratio of these two Eigen values is approximately 25000.

So, this particular ratio of two Eigen values is very large number; as a result of this what we see is relatively small changes in any of these elements of the matrix of the vector b or any of the elements of the matrix A is going to result in a very large change in the solution.

So, these are typically very difficult problems to solve and if you are going to use any kind of software in order to solve this problem; for example, if you are going to use matlab, then if the ratio of Eigen values becomes of the order of 10 to the power 9 or 10 to the power 10, you will get an error saying that the matrix is ill conditioned and the solution may not be accurate. What that really means is that the ratio of the highest and the lowest Eigen values is very large; as a result small errors, which might creep in because of in the previous module we had talked about round up errors. So, if errors creep in because of n rounding off of numbers, the solutions can be very inaccurate; for example, if this particular number 2.001 was rounded off to 2, we can see that the results ha would change so drastically.

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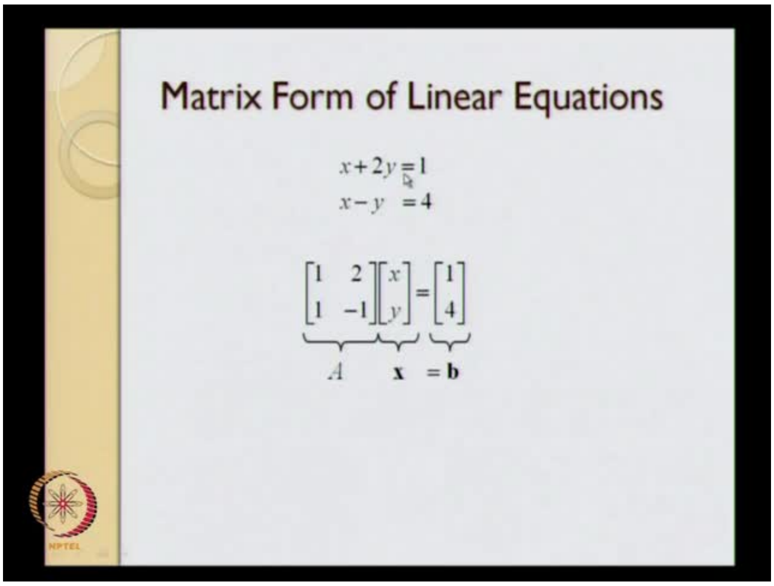

So, whereas if this was well condition system that means if we go up and in this particular example, if we were to change this particular equation as x plus 2 y equal to 1 and if we were to change this equation to x minus y equal to say 4.001, the solution from 3, minus 1 would actually change to 2.999 or something like that and minus 1.001.

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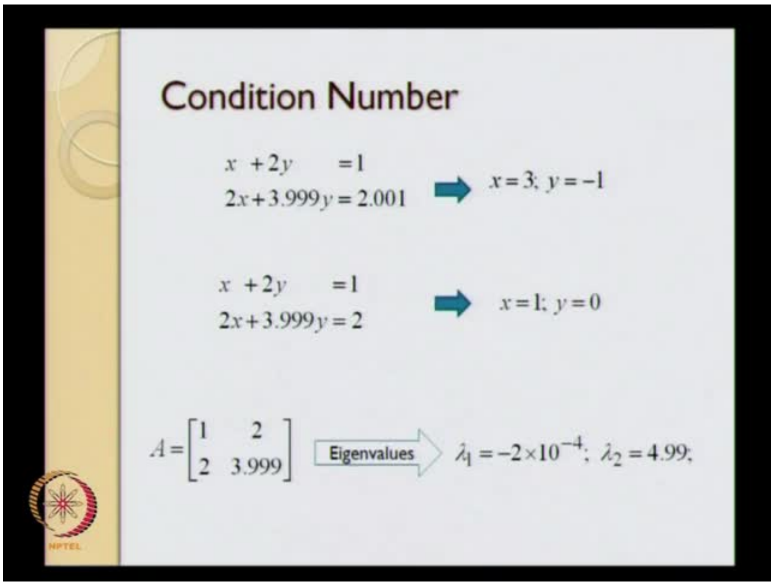

So, a fairly small change in this particular equation would cause relatively equally small change in the solution; whereas in this particular purely in poorly conditioned examples, a small change in any of this numbers of matrix A or vector b are going to cause large changes in the solution x y.

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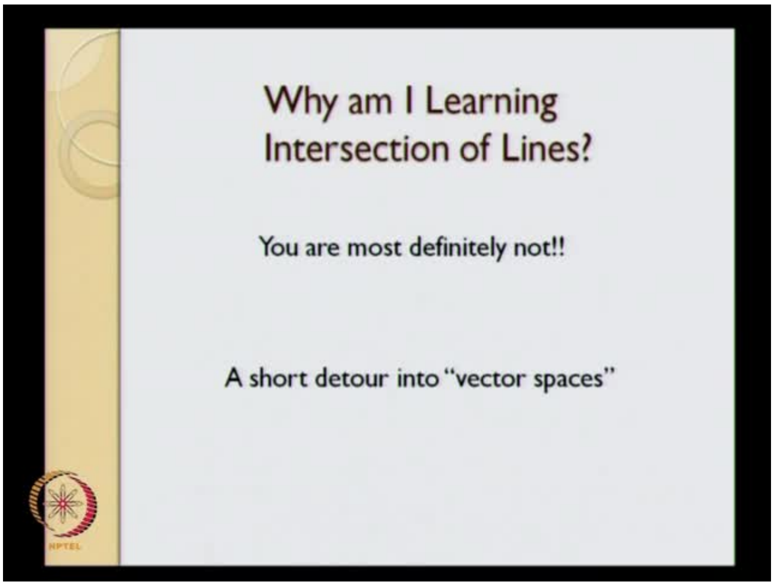

Now, the question is that, so this was a question; when I use to teach the class, one of the students are actually asked; so I will try to answer this up front, why am I learning intersection of lines in my third year of undergraduate chemical engineering? And the answer to that is we are most definitely not learning intersection of lines in fact in module 4, we will use solution of linear equations quite extensively in order to solve non-linear equations. Module number 7 and 8 we will use a solution of linear equations in order to solve ordinary differential equations and partial differential equations; we will look into that examples when we come to module 4, module 7, module 8, and so on.

We will also look at inversion of matrices and inversion of matrices uses some of the same concepts that we are going to learn for solving equations. Matrix inversion or finding inverse of matrix will be useful in module 5, where we are going to talk about basically interpolation and regression.

So that is another place, where linear algebra is going to be useful and what I will do is to take a short detour into vectors and vector spaces, and so on to actually show that this particular field of linear algebra is an extremely rich field.

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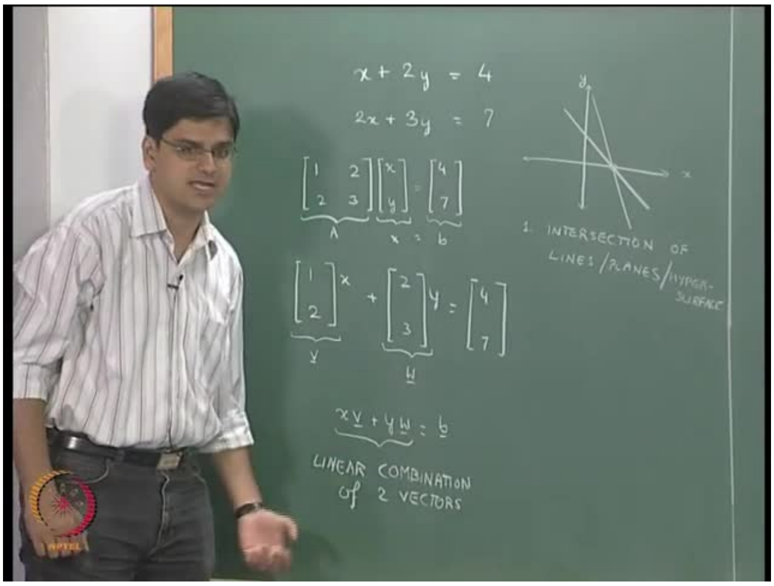

So, let us look at the two equations x plus 2 y equal to 4, and 2 x plus 3 y equal to 7; so we want to let us say solve these two equations. What we have seen in the power point slide so far is the interpretation of this equation as a straight line and this equation as another straight line and the solution is nothing but the point of intersection of these two lines. So, what we had done over there is, this we call as the axis x, this we call as the axis y and these two are going to be two straight lines; let us say somewhat like this and this particular point is of intersection of these two lines is going to be the solution of these two equations.

So, one interpretation of the solution of these two equations is what is known as what we called as intersection of lines or planes. In 3-dimensional case, it is going to be intersection of planes; in n-dimensional case it will be intersection of hyper surfaces. So that is one interpretation of solution of these two equations. The way we put we had put this particular equation in the matrix notation is we had taken 1 2 2 3, that means the coefficients over here in the matrix A and then we had this x y as the vector x and on the right hand side, we had a vector 4 and 7 that would be the vector b; so this was our matrix A, this was our vector x and this was our vector b.

So that was one way of representing this particular set of equations; so this is indeed the matrix notation of this particular equation. The other way of thinking abou this particular problem again is that this column of the matrix A is going to multiply with the element x and this column is going to multiply with the element y.

So, this particular equation again, we will be able to write this as 1 2 multiplied by x plus 2 3 multiplied by y equal to 4 7 and to check whether what we written is correct or not, we can just expand this particular equation and we will be able to get x plus 2 y equal to 4 that is the first row; so the first row is this particular equation and 2 x plus 3 y equal to 7, so we do get the second equation again.

So, when we represent in this particular form, this is alternative way of representing these equations. This representation, this representation and this representation are we have are one in the same thing; we have just tried to put in different forms in order to motivate and in order to understand what a different perspective of look looking at this particular set of equations.

So, now, this if we notice is in the form of a vector; this we notice is also in the form of a vector; let us call this particular vector as a vector say v bar; let us call this particular vector as a vector w bar. So, this equation we will be able to rewrite them at this particular equation as x times v bar plus y times w bar equal to b bar.

So, over here, what we had was these two equations over here; what we have on the left hand side is, it is a linear combination of two vectors. So, the left hand side is… So, the aim of solving linear equations is then going to be to find two scalars x and y, such that the linear combination of v and w is going to result in another vector b.

So, one interpretation of solving these equations was intersection of two lines; another equation, another interpretation of these solving two equations is to find the scalars, such that the linear combination of the two vectors is going to be the result vector on the right hand side. Keep in mind we have put these vectors in the consistent form that I had spoken about some 15 minutes back, that means each vector we are going to represent as n number of scalars ordered in one single column.

So, we will have n rows and a single column in order to define any vector; we are going to use that as a consist for consistency of notations from this point forward. So, what this means is that now, we have a vector v; now this vector v is vector 1 2, and vector w is vector 2 3.

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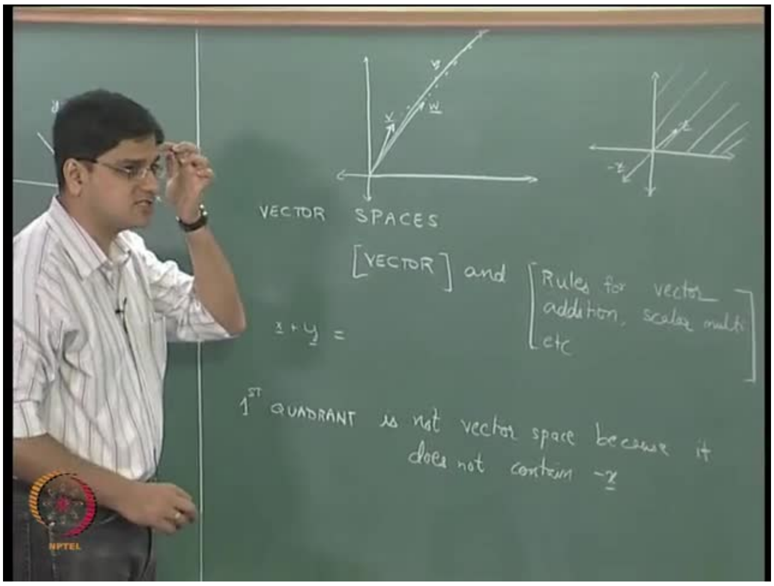

So, vector 1 2 is going to be 0.1 over here and then 2 over here; so vector 1 2 is going to be this and vector 2 3 is going to be a point over here and a point over here. So, this is going to be vector w; so v bar and w bar are the two vectors. Now, what we want to find is, find this scalar x and y, such that the linear combination will result in 4, 7; so 4,7 we basically take 4 units on this side and then 7 units on this side.

So, this is going to be the point 4, 7 and we want to find the linear combination; so basically we are going to complete the parallelogram, such that we will finally reach this particular point. So, we will try to complete a parallelogram in this particular format and this is going to be our b bar and the amount by which v is going to stretch and w is going to stretch is going to be represented by x and y respectively.

I have not really shown this in a very clean way, but the idea is that the b bar vector should be a linear combination of the two vectors on to the left hand side and that leads us to the next concept of vector space.

Vector space is nothing but the space which contains vectors and the rules for vector addition, subtraction, and so on. So, a combination of vectors and rules for vector addition, subtraction, multiplication, scalar multiplication, linear combination of vectors will essentially result in an overall space that means it is a space that satisfies all these rules and contains vectors that is essentially going to be a vector space.

For example, the typical 2-dimensional space that we talk about is a vector space, why it is it a vector space? Because we can have vectors in that 2-dimensional space that means we can have vectors of the form v, w and v bar in the 2-dimensional space and all the rules of addition, subtraction, scalar multiplication, and so on are applicable on this particular 2-dimensional space and I should actually be showing this n on all sides. So, what we mean by the rules of vector addition, subtraction, multiplication, and so on is that if we have two vectors x bar and y bar, x bar plus y bar should be equal to y bar plus x bar scalar multiplication alpha multiplied by x bar plus y bar should be equal to alpha x bar plus alpha y bar; there should be a 0 vector, such that x bar plus 0 bar is going to be equal to x bar.

There is an negative vector minus x bar, such that x bar plus minus x bar is going to equal to 0 bar, all those rules again which we have studied right in the beginning, have to be satisfied in this particular space in order for that space to be called a vector space.

So, for example, if we now talk about say the first quadrant that means instead of instead of, talking about the entire 2-dimensional surface, if we only talk about the first quadrant; the first quadrant is not a vector space, because if we have an x bar in the first quadrant, this is going to be minus x bar. So, first quadrant is not a vector space; the first quadrant is not a vector space, because it does not contain minus x bar; if x bar is in that vector space is in that space minus x bar is not in that space and that is the reason why, the first quadrant is not going to be a vector space. But the entire 2-dimensional space is a vector space, because x bar minus x bar are both present in that vector space; there exits, however another vector to which you can add x bar plus minus x bar; so that we get 0 bar; there exists a 0 vector, such that x bar plus 0 bar is going to be equal to x bar.

All the rules of vector addition are going to be applicable in the entire 2-dimensional space; all the rules of scalar multiplication are going to be applicable in the entire 2-dimensional space; so this is a vector space.

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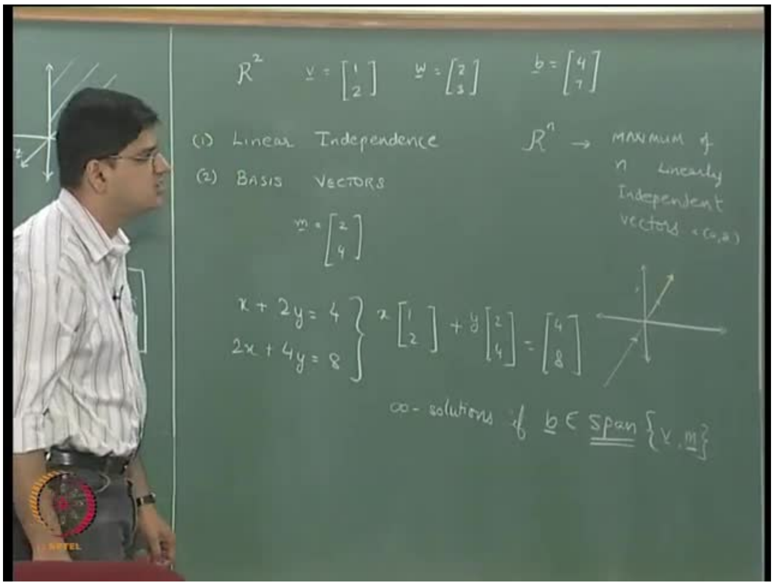

So, now, if we have 2 vectors in; now, in this particular case what we had is we had an R 2 space that means the vectors where 2-dimensional. Our v bar was 1 2 and w bar was 2 3, and b bar was 4 7. So, all of these vectors v, w and b are elements of a 2-dimensional space; now if v and w are linearly independent, then v and w can form the basis for that particular vector space, that what that basically means is that any vector in R 2 can be represented by v and as a linear combination of v bar and w bar.

So, we have talked about two other concepts; linear independence was one concept and the second concept is basis or basis vectors. If we have an n dimensional subspace sorry If we have n dimensional vector space R n, there can be at most n linearly independence vectors.

For example, in an R 2 space, there can only be two linearly independent vectors; there cannot be more than two linear independent vectors, but two vectors in an R 2 space need not always be linearly independent. For example, if we use another vector say m bar equal to 2 4, then m bar and v bar are not linearly independent, because we can multiply alpha v bar, with where alpha equal to 2 and we will actually get m bar.

So, v bar and m bar are not linearly independent; now we had those two equations; the second set of equations that we saw was x plus 2 y equal to 4 and 2 x plus 4 y. Let us say equal to 8 is the second equation; in this particular case, if we were to write this equations in a form similar to this, we will write these equations in a form similar to this. What we will get is x multiplied by 1 2 plus y multiplied by 2 4 equal to 4 8; so one vector is the vector 1 2, the other vector is the vector 2 4 and if we were to plot these two vectors, this is going to be our vector 1 2 and this is going to be our vector 2 4.

Now, these two vectors are actually collinear or parallel to each other, as a result of this they are going to be able to span only this particular line; any point if that we consider outside of that line, we will not be able to represent it as a linear combination of vector 1 2 and 2 4.

So, if we cannot represent this particular point as a linear combination of these two vectors, we cannot get a unique solution; whereas if we have another point lying over here, we will have infinite number of solutions, because you can always find various values of x and y, such that this the two equations are going to be satisfied. For example, we have this as 1 2, 2 4 and 4 8; the point 4 8 also lies on the same line; so this is going to be 0.48.point 4 8.

So, one example of x and y that satisfies the 0.48 point 4 8 is going to be x equal to 0 and y equal to 2. So, if we stretch y to double its length, we are going to reach 4, 8; other possible solution is x equal to 4 and y equal to 0, that means we do not do anything with y, we will shrink it to 0 and we will stretch the first vector 4 times, so that we reach 4 8.

Likewise, you can choose x equal to minus 1; let us say and then y is going to be equal to 2.5. So, in that case, if we have taken the vector x and reflected it on this side, we need to stretch the vector y by 2 and half times and then add it to the vector x, to the to the first vector. So, as you can see over here, there are going to be infinite number of solutions not w bar sorry m bar.

So, if b bar lies in the span, so span is another concept that I am introducing; I am not so sure, whether these concepts have been covered in undergrad material or not. This is a little bit advanced stuff that some curriculums do not really cover, but what we really mean by span is a set of all vectors, which can be obtained by linear combination of these two vectors.

So, in this case, because v bar and m bar actually lie on that same line, as a result of this the span of v bar and m bar is only going to be this one dimensional subspace of the 2-dimensional vector space. So, this line is a subspace on which if b bar lies on this particular subspace, then we will have infinite number of solutions; if b bar does not lie on that subspace, we will not have infinite number of solutions; we will have 0 solutions. And if v bar and w bar were linearly are not collinear, they are spanning the entire R 2 space in that particular case we will get exactly one solution.

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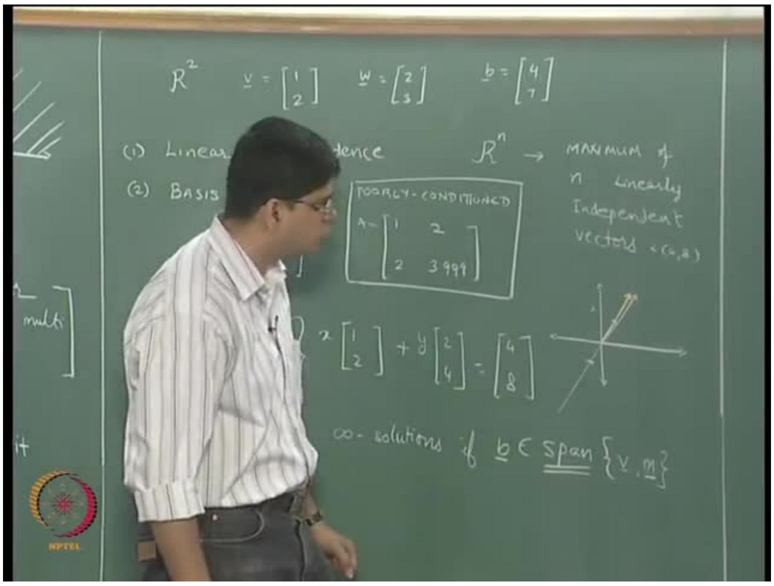

Why what do we see when v bar and w bar are not collinear one? When b bar and w bar are not collinear or when v bar and w bar are linearly independent, what we see is that rank of the matrix v bar and w bar, which is equal to rank of our matrix A should be equal to n in this particular case should be equal to 2 that means if we take determinant of the matrix A, that determinant should be non-zero.

This is same as So, coming to the Cramer's rule that we spoke about what we said, when we discussed Cramer's rule is that the determinant of the matrix A should be non-zero for the system to have unique solution. In this particular case, this is we are going to get unique solution.

Now, the second case is when we have 0 solution and when we have infinite solution; so our v bar was 1 2 m bar was 2 4, if b bar, I will call it as b 1 bar was 4 8, we will get infinite solutions; if b 2 bar was let us say 4 7 we will get no solution.

We know this right, because if this particular guy was 8 that means the two lines lie on one and single line, when we talk about the intersection of lines. In the interpretation as the intersection of lines what 1 2 2 4equal to 4, 8 means is that the two lines lie on top of each other; the other interpretation based on the vector spaces is that the two vectors are actually in the same direction and the third, the right hand side vector if it lies in that same direction, we will have infinite number of solutions; if it does not lie in the same direction, we will have no solution.

So that is the interpretation in the two different forms; so infinite number of solutions now, what we see is that b 1 bar is in span of v bar, m bar. In other words, the rank of matrix should be equal to the rank of matrix A; so the rank of matrix formed by putting this 4, 8 along with v bar and m bar, that means the rank of matrix 1 2 4. In this case, it is the rank of both these matrices this equal to 1, as a result there are infinite number of solutions. If we were to change this to 1 2 4, 2 4 7 we will have 0 solutions because rank of this particular matrix 1 2 4, 2 4 7 is going to be equal to 2 which is not the same as the rank of this particular matrix.

So, this we get the rank condition or the determinant conditions; now, we have obtain in a different way and that different way is talking about vectors and vector spaces this way of thinking about vectors and vector spaces is going to be a more general way of thinking about linear equations and linear systems as you will probably see in some of your mathematics higher level mathematics courses.

Now, what do we what happens, when we have basically, when the matrix A is an ill conditioned matrix? Again, we will look at a 2-dimensional case; multiple dimensional cases are actually much more complicated than looking at 2-dimensional cases, but what we saw in a poorly conditioned matrix, the poorly conditioned matrix A was 1 2, 2 3.999. So, this was poorly conditioned matrix.

So, what happened in a poorly conditioned matrix was that, we had this as v bar and the second vector was very close to the first vector. So, we had the first vector as 1 2 and the second vector was 2, 3.999; so these were the two vectors. As a result, when we had the 0.22 two different 0. points 4, 8 versus 4,7.999 those resulted in very different shrinking and stretching of these two vectors. This was a little bit of a detour to talk about the various aspects of linear algebra, what I will do is now, just give you an over view of what to expect in rest of this module.

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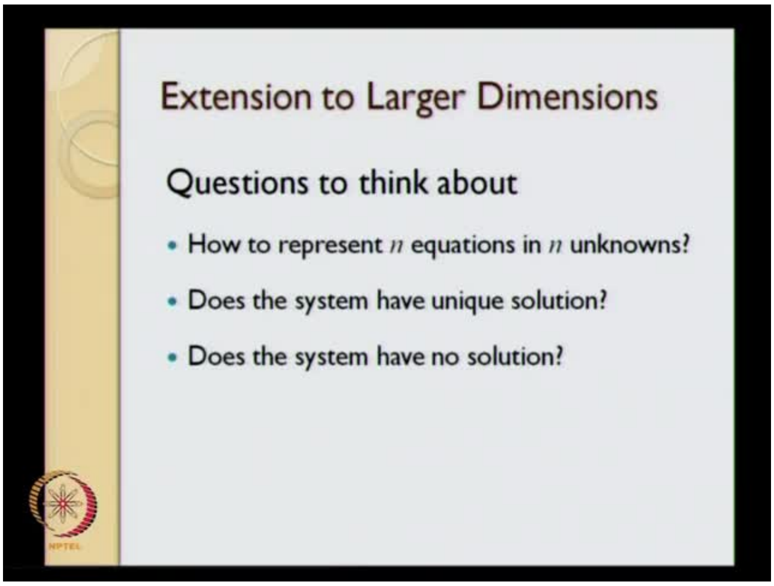

So, what we are going to do in the next lecture, and the lecture so on is, not just restrict ourself to 2-dimensional or 3-dimensional systems, but to start thinking about higher dimensional systems and how to represent n equations in n unknowns, when does the system have unique solution, when does the systems have no solutions so on and so forth. These are the questions I want you to think about before we move on to the lecture 2 of this module is when does a system of n equations in n unknowns have a unique solution, when does system of n equation and n unknown have infinite number of solution, when does the system n and n equations and n unknowns do not have any solution.

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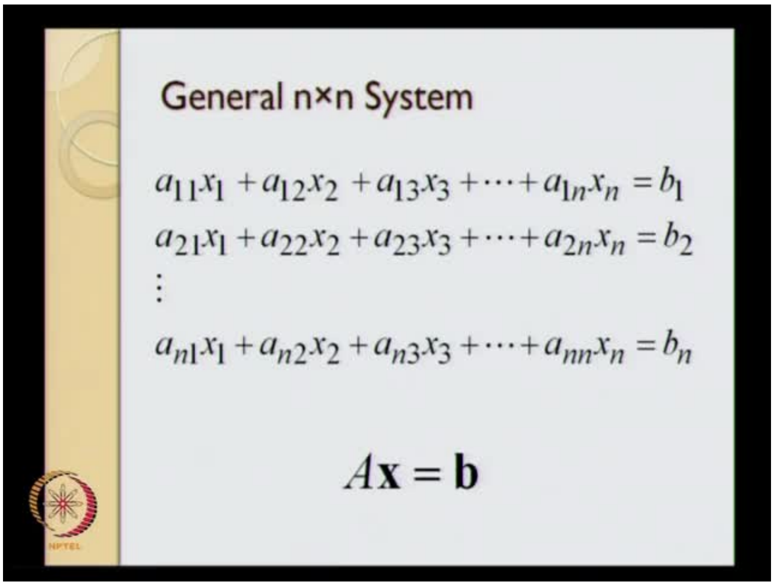

This is the way we will write general n by n system of equations, where x 1, x 2, x 3 and x n are n variables; a 1 1, a 1 2 ,and so on are the n co coefficients for the first equation, a 2 1 up to a 2 n are n coefficients of the second equation and so on up to a n 1 a n 2 up to a n n are n coefficients of the nth equation. And the next question is how do we this equation in terms of a x equal to b in the standard format; that is the second thing that I want you to think about before we start on to the lecture 2 of this particular module. and

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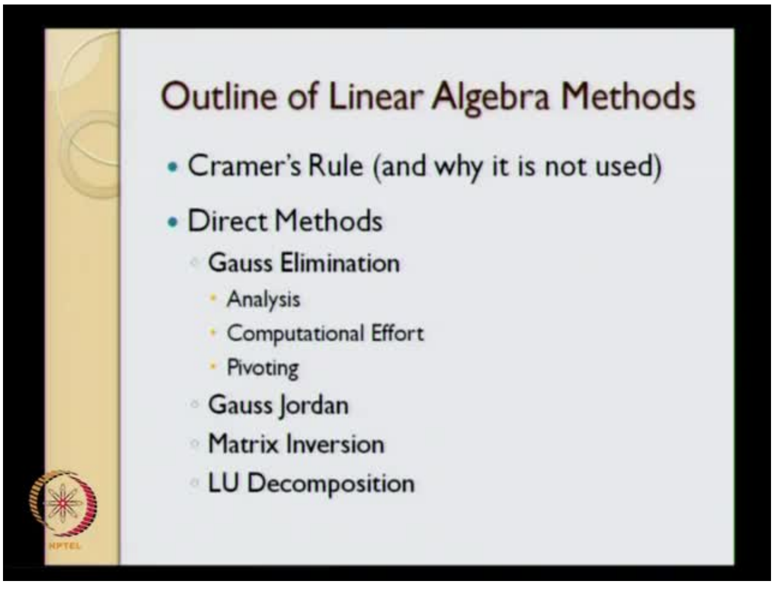

So, these are the couple of things that I want you to think about and we will start of the module by talking about general n by n system of equations, that is, what we will do first. Then we will solve the previous equation that we talked about, that is x plus x plus to 2 y equal to 4 2 x plus 3 y equal to 7. We will try to solve it with hand and try to understand a few things about, how we have solved this equation and that will motivate us into going to some of the numerical methods to solve these equations, and that numerical method that we will talk about the first is the Gauss Elimination method.

We briefly talked about Cramer's rule; I will discuss why it is not used in in the lecture 2 of this module in the very next lecture. After that we will talk about Gauss Elimination and do some analysis of Gauss Elimination.

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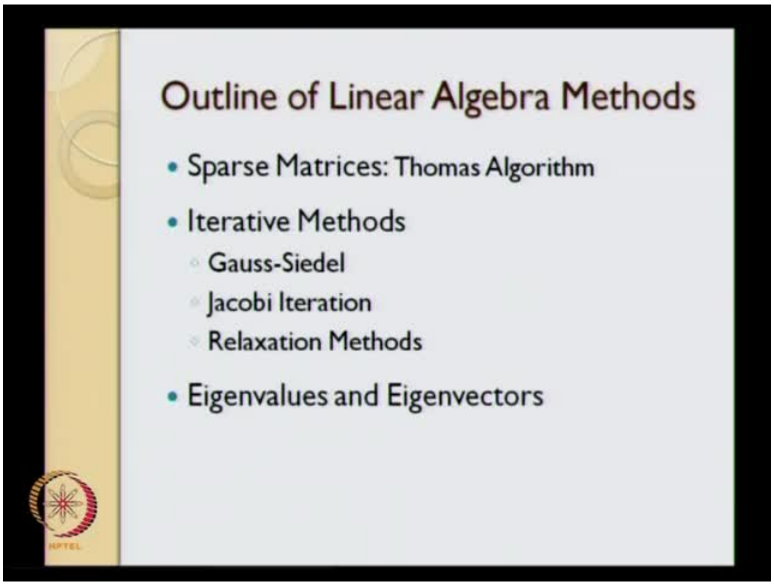

We will talk about Gauss Jordan; we will use the Gauss Jordan method for matrix in inversion and talk about LU decomposition method. Next, we will talk about special type of matrices known as sparse matrices and specifically of tridiagonal matrices and algorithm to solve those tridiagonal matrix equations in an efficient manner. All these equations are what are known as direct methods, then we will talk about what is known as iterative methods and finally, we will talk about Eigen values and Eigen vectors.

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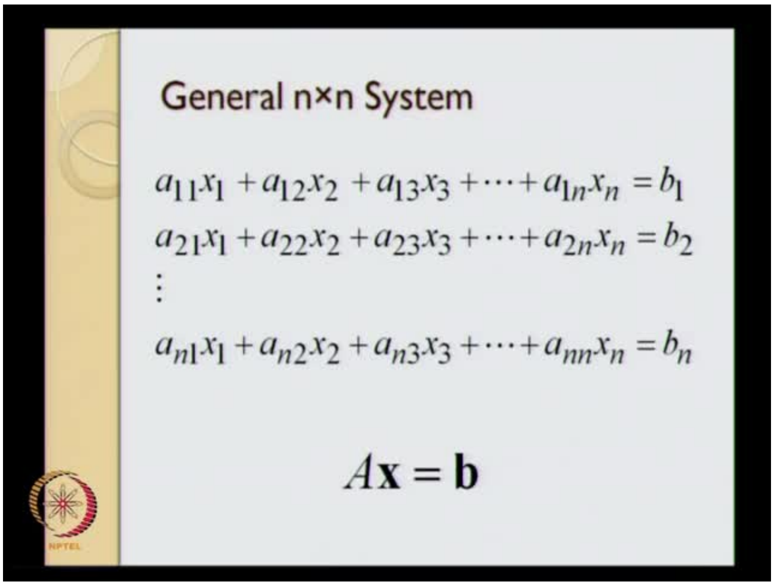

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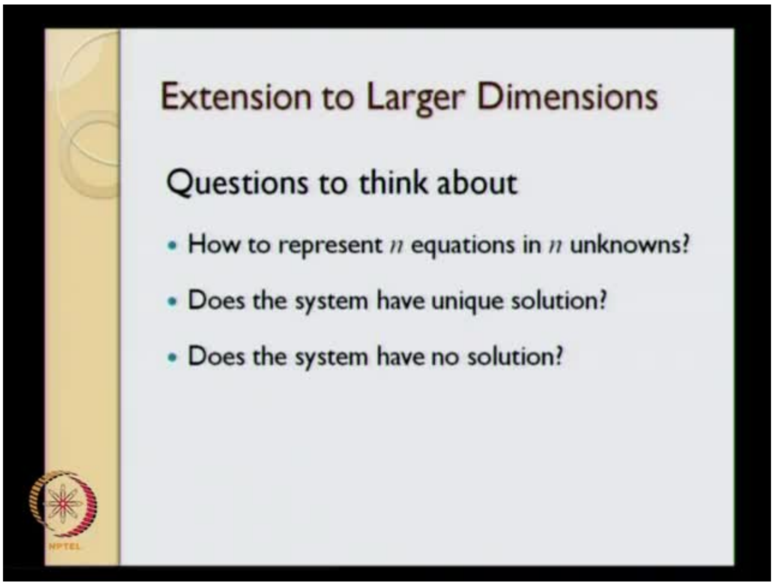

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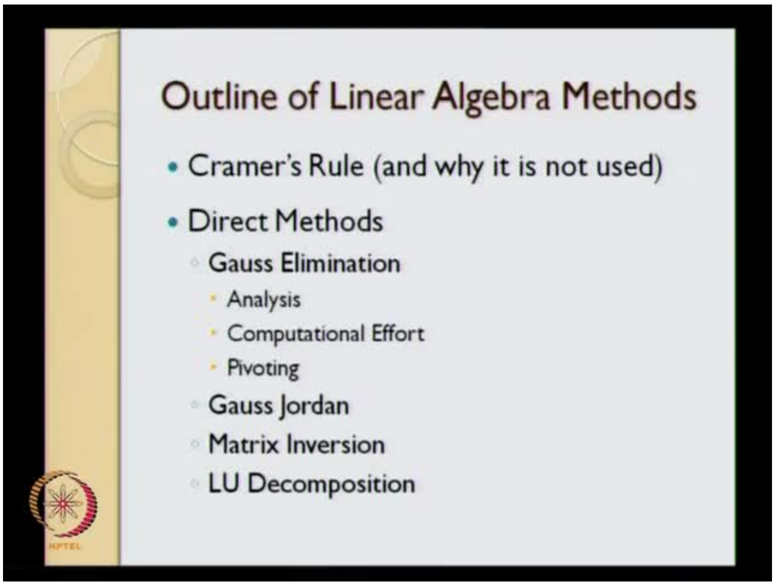

So, in the next lecture I will start off with by trying to answer the question, how do we do we represent a general n by n system of equations, then I will try to answer the questions about when does the system have unique solutions, infinite solutions, no solution and then we will start off with the Gauss Elimination method.