Computational Techniques

Prof. Dr. Niket Kaisare
Department of Chemical Engineering
Indian Institute of Technology, Madras

Module No. # 04

Lecture No. # 02

Nonlinear Algebraic Equations

Hi, and welcome to the second lecture of module 4. In the previous lecture, we introduced non-linear algebraic equations where do they rise in chemical engineering problems, and we started of with bracketing methods. We looked at two bracketing methods, the first one was bisection method and the second one was the regula falsi method. In the bracketing methods, what we use was we used essentially two initial guesses x l and x r, we chose the two initial guesses such that there they lie on the either side of the two solution x star, we verified that by finding out the product f of x l multiplied by f of x r is less than 0, if that is satisfied those are admissible initial guesses.

And then we used ways to actually get the new solution, x of i plus 1. In bisection method, what we did was we started of with a particular length of the chord, we half the chord at each dimensional. In the regula falsi method, what we did was, we got the two points which lied on the curve, we connected those two points with the straight line found out the point where that particular straight line intersects the x axis that became our new guess. What we did in regula falsi method is essentially when we get the new guess we again ensure that that particular new guess bracket at the true solution, and then we proceeded. The methods that we are going to cover in this lecture are the methods which are known as open methods, and they are called open methods, because they do not the the solutions do not have to bracket the true solution that that we have.

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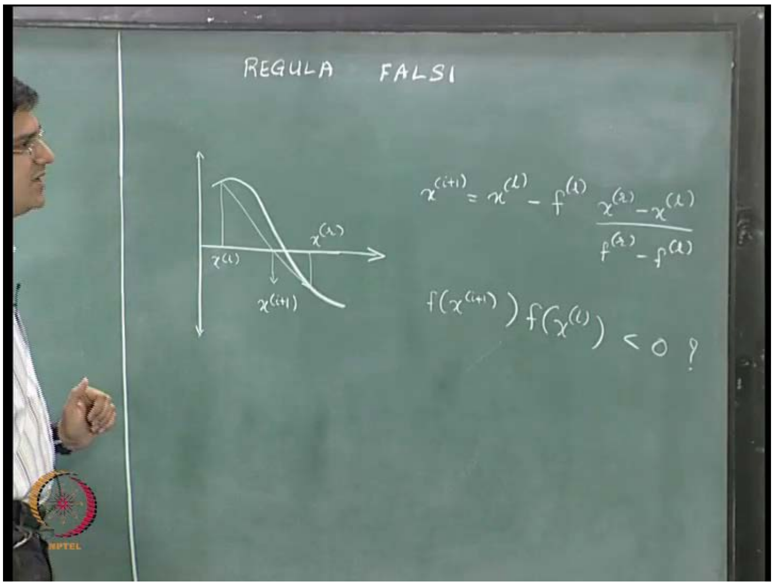

So, what I will do is, I will do a very quick brief recap of what we covered in the regula falsi method, and that is going to motivate our next method which is called the secant method, the next method is called the secant method, and we will use the regula falsi method in order to motivate our secant method. So, let us just recap what we did with with respect to regula falsi (no audio from 02:36 to 02:44).

Let us say, we have a curve of this type and we started of with x r and x l, what we did was we connected this x r and this x l with a line and we found out the intersection of that particular line with the x axis and then we call that as x i plus 1.The expression for x i plus 1, that we developed in the previous module was x i plus 1, was equal to x l minus f l multiplied by x r minus x l divided by f r minus f l, I mean of course, you can interchange these as well, as long as you interchange both the numerator and denominator it really does not matter, the reason is negative, this will be negative of this and the denominator will also be negative, this the negative negative will cancel.

So, this is what we did, in order to get the new guess x i plus 1, in regula falsi then in the next step, that we implemented was we check f of x i plus 1, multiplied by f of x l whether or not it is less than 0, if yes, we replace x l with x r; if no, we replace x if with xl if yes, we replace x r with x i plus 1; if no, we replace x l with x i plus 1. So, in this particular case, we replace x l with x i plus 1. So, that was our regula falsi method and we keep repeating we kept repeating that un until x i plus 1, was close enough to x of l that is, what we did in the regula falsi method.

The secant method works very similar to the regula falsi method, but the only difference is we do not care whether the solutions whether the solutions bracket the true solution or not. So, x i plus 1 is going to be generated in a very similar way, but we drop the the notations x l and x r, we drop the pretense whether x l and x r lie on the either side of the solution, we just start with solution 0 with solution 1 and we keep generating the new solutions and we discard the oldest solution and retain only the latest solution.

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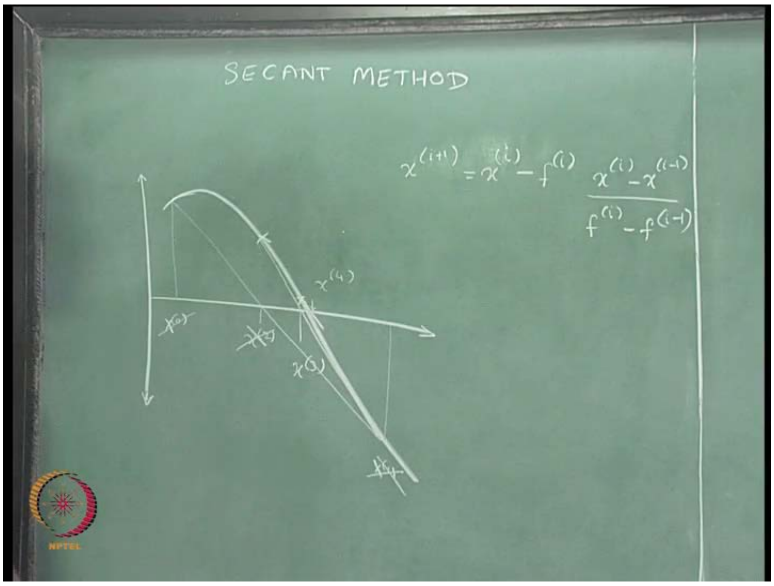

What I mean by that is, we will replace this particular equation with another equation in secant method (no audio from 05:39 to 05:45) and the generation of the new point is going to be x i plus 1 will be equal to x i minus f i multiplied by x i minus x i minus 1 divided by f i minus f i minus 1,if you compare it with, what we have just written a few minutes ago, if you compare this particular expression with the expression for the secant method, the two expressions are one and the same, the only difference is we do not bother whether that x i and x i minus 1 lie on either side of the curve, they can both lie on one side of the curve they can both they can lie on either sides of the curve and it really does not matter matter because the secant method does not depend on bracketing the the solution.

So, let us again draw that same curve that we had drawn previously, we will call this as x 1, we will call this as x 0, and we will connect them with the straight line, this becomes our x 2. So, now we have x 0, x 1 and x 2, what we did in regula falsi method, we we multiplied f of x 0 with f f of x 2 and decided whether x x 2 is replacing x 0 or x 2 is going to replace x 1.We are not going to do that in secant method, we are going to discard the oldest solution and retain the two new solutions. So, we have x 1 and x 2 as the two new solutions and we will look at this particular point we will join these two with with the same curve and this is now our x 3.

Now, up to this point, the regula falsi method and the secant method were used in the same manner, what we we ended up doing was, we discarded x 0 and retained x 2 and our x 1 and x 2 still bracketed the true solution that we got. This is not what we are going to do in secant method, what we realize over here in secant in this method is x 2 and x 3 are lying on the same side of the solution x star. In regula falsi, we would have retained x 1 and x 3; in secant, we do not retain x 1, x 1 is the oldest solution we currently have in the memory; we will discard x 1 and stay with x 2 and x 3.

So, now we will have the two points not bracketing the solution. However, that does not matter, we will still go ahead and draw that particular chord extended and see where that particular chord intersects the x axis, this becomes our x 4. Now, when we have our x 4, we will discard our x 2, will stick with x 3 and x 4 again draw the chord and will keep repeating that until we converge to the desired solution.

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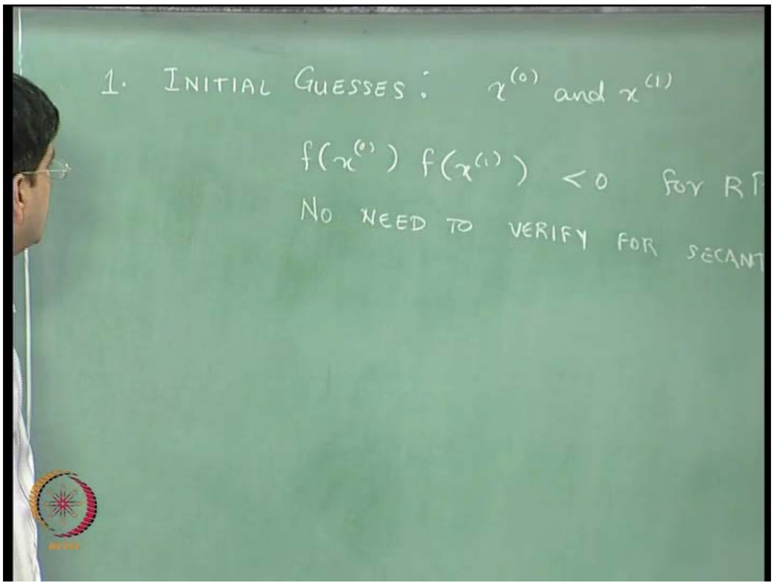

So, looking at the algorithm for secant method, the first step that we do is, initial guesses are to be chosen (no audio from 09:26 to 09:40). In our regula falsi method, what we had to ensure was f of x 0 multiplied by f of x 1 was less than 0 for regula falsi no need to check this (no audio from 10:03 to 10:14). So, that is one, that is the difference between secant and regula falsi, we do not need to check whether x 0 and x 1 lie on either sides of of the the true solution x star.

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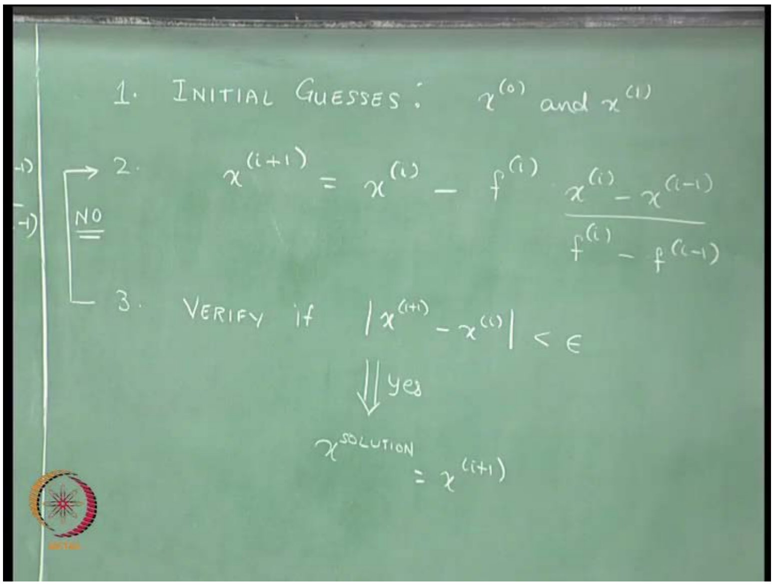

So, that is the initial guess, I will just erase this because since, we do not need this particular criterion, I will just erase this criterion rather than keeping it on the blow board and cluttering the board. The next is to to determine then how we are going to move forward; that means, to use x i minus 1 and x i plus 1 in our x i minus 1 and x i to get x i plus 1(( )) for although, I am repeating myself over here, I will just write it down for the sake of completeness (no audio from 11:03 to 11:15).

This is going to be our x i plus 1(no audio from 11:20 to 11:58). So, initial guesses is x 0 and x x 1, x i plus 1 is going to be determined through this particular expression, we verify whether the difference x i plus 1 minus x i is less than some tolerance epsilon, if it is, then this our numerical solution is equal to f x x of i plus 1, if it is not, we increment i and we repeat this process repetitively till our this particular condition is met.

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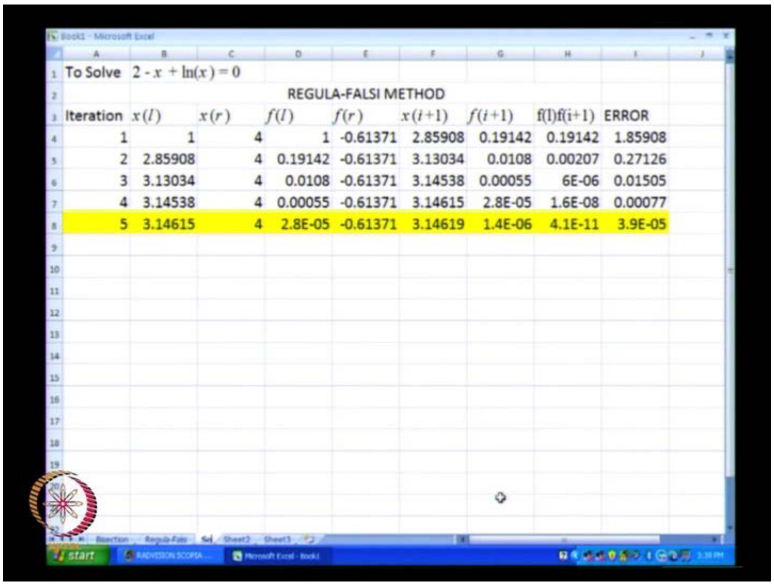

To Solve $2 - x + \ln(x) = 0$

REGULA-FALSI METHOD

Iteration	$x(l)$	$x(r)$	$f(l)$	$f(r)$	$x(i+1)$	$f(i+1)$	f(l)f(i+1)	ERROR
1	1	4	1	-0.61371	2.85908	0.19142	0.19142	1.85908
2	2.85908	4	0.19142	-0.61371	3.13034	0.0108	0.00207	0.27126
3	3.13034	4	0.0108	-0.61371	3.14538	0.00055	6E-06	0.01505
4	3.14538	4	0.00055	-0.61371	3.14615	2.8E-05	1.6E-08	0.00077
5	3.14615	4	2.8E-05	-0.61371	3.14619	1.4E-06	4.1E-11	3.9E-05

So, this is going to be the algorithm for for the secant method. So, let us go back again this time again to excel and try to see what how secant method is going to work. So, now, we have seen, how the secant method works, the way the secant method works is very similar to regula falsi except the difference between the secant and the regula falsi is that every time, we get x of i plus 1 we replace the older solution and not not necessarily we are not necessarily bothered whether the solution that the guesses are going to bracket the true solution or not.

So, this was what we did in Microsoft excel in the previous lecture when we when we had worked with regula falsi what I am going to do, again is just copy the previous lecture essentially just create a copy of the previous lecture. So, that I do not have to redo everything and we will call this as the secant method.

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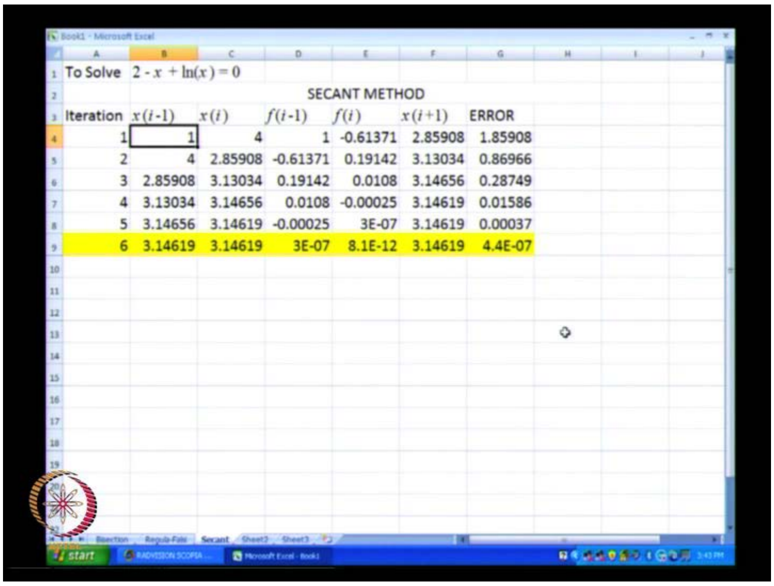

In the secant method, what we are interested is again solving the same problem, 2 minus x plus ln of x using the secant method. And in the secant method, we do not have the concept of x l and x r instead, we will we have x of i plus 1 and x of i minus 1 and x of i. So, x this guy is going to be nothing but x of i minus 1 and the next guy is going to be nothing but x of i likewise, we will have f of i minus 1 and we will have f of i. So, these are the two solutions, the two guesses of the solution that we we are going to start of with and then compute x of i plus 1, f of x of i plus 1 and the product f l multiplied by f of i plus 1 is not needed at all.

So, we will just delete that part and let us not let us go ahead and delete this part also completely. So, that we start with with almost a blank slate we will start with the initial conditions that we started of with. So, we start of with the two initial solutions x of 0 is equal to 1 and x of 1 is equal to 4 we can calculate f of 0 f of 0 again same way as we had calculated in the previous lecture of this module, 2 minus b 4 minus sorry plus ln of b 4 and likewise for f of i is going to be 2 minus c 4 plus ln of c 4, x of i plus 1 as we had said was computed using x of i minus 1 minus f of i minus 1 multiplied by x i minus x i minus 1 divided by f i minus f i minus 1.

So, this is that particular slope term; this is the f term and this is the x term at i minus 1 and it is exactly the same as we have done in the regula falsi method, the computing of x i plus 1 given x i minus 1 and x i. So, we have not bothered rewriting those that particular formula, remember we had just copied that particular formula from our previous case. So, again we do not need f of i plus 1 and the error at and the product f of i plus 1 multiplied by f of i at this stage, we will just move the error over here.

So, all this is all that, we need for the secant method, in iteration 2 when when the iteration number when i is equal to 2, x i minus 1 is nothing but x of 2 minus 1 which is x of 1 and if you notice x of 1 at the previous iteration was this particular value. So, we will do equal to this term. So, that is what our x of i minus 1 is going to be when i equal to 2, x of i when i equal to 2 is the x of i plus 1 when i equal to 1 because when i was equal to 1, x of i plus 1 is nothing but x 2 when i equal to 2, x of i is nothing but x 2. So, we will write this as equal to x of i plus 1.

And now, what we do to compute f of i minus 1, we just drag this over here to compute f of i, we just drag this over here and to compute x of i plus 1, we just drag this and the same way with respect to the error, we are going to drag. And now, we select this entire row and drag it again may be for our total of 10 iterations let see and yes. So, that is the 10 iterations. So, that is, what we have, we now, have gotten and we see that it has converged essentially in in 6 iterations. So, we will go ahead and delete this part.

So, our error criterion was that, the error should fall below the epsilon value was 1 e minus 4, the here the error is 3 e minus 3.7 e minus 4, as a result of which we repeat this for one more iteration and we get 4 e minus 4 and the solution that we get over here, is essentially 3.14619 that is going to be the solution using the secant method. So, now, if you compare what I want to do is, I just want to compare the results of the regula falsi method with the results of the secant method. So, what I will do is, I will just copy what we had done with the regula falsi method in our previous lecture and just paste it over here just for comparison, I will click on paste special, we only want the values we do not want the computations. So, we will just paste the values over here.

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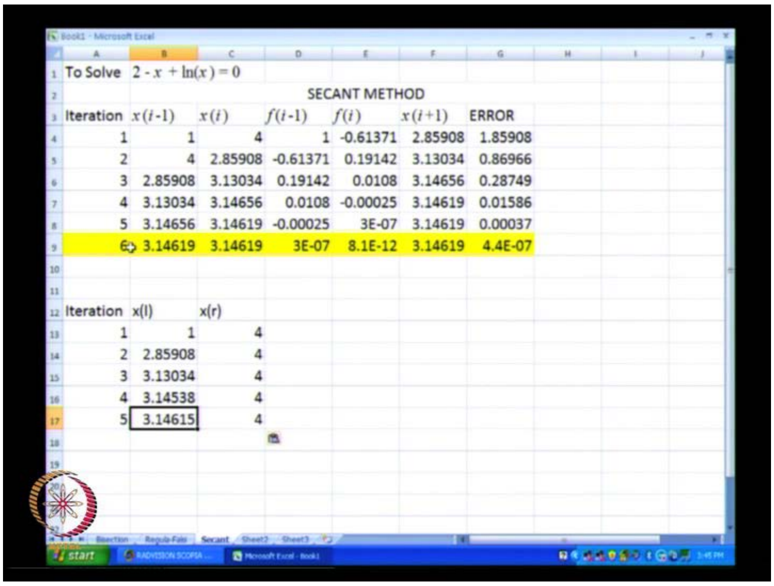

So, what we started of how what happened in the regula falsi method is that, at all times, the news guesses of the solution that we generated, were all generated to the left of the true solution x star. As a result of this x r that is the right hand solution remained the same at 4.0 all the time and it was only this solution that we kept changing with the number of iterations, but if you see what is happening in the secant in the secant method the first solution was dropped, second solution was moved back and the new solution came over here. In the third iteration, it was the second solution was the drop was dropped third iteration moved here and the new solution moved over here.

In the fourth iteration, the third solution was dropped the the post previous solution was moved here and the new solution came over here and so on and so forth. So, what we did in the secant method is to retain only the last two solutions not the left and right solution and that is essentially where the secant method differs from the regula falsi method. In regula falsi method, we essentially kept the solution such that they bracket both the solutions at any given time. So, that is essentially, the way the secant method works.

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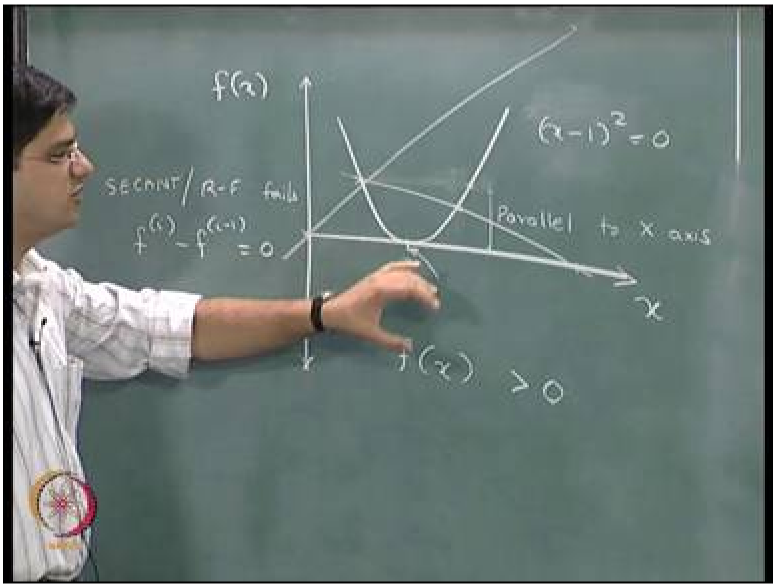

Let us say, that our f of x was of this particular form (no audio from 20:32 to 20:39). We are plotting f of x against x. The an example of this this particular equation is for example, if we have x square minus 4 equal equal to 0. So, this is going to be an an example of that. So, actually this this would not be. So, let us let us just look at the case, where x square equal to 0 rather than x square minus 4 equal to 0 and this particular guy is 0 and this curve is going to be essentially tangential to this particular axis, if we look at any point on this curve except for for the origin at all the other points on this particular curve our basically our f of x is always going to be positive, why just look at x square equal to 0, we could also look at say x minus 1 square equal to 0.

So, in that particular case, the 0.11 comma 0 is the true solution is is the true solution of this particular equation, but any point you take along this particular curve, our f of x is going to be greater than 0 for any point which is not equal to x equal to 1. So, if we will try to use any of the bracketing methods, we will not get any admissible solution because no matter which two points you take on this particular curve, our f of x l multiplied by f of x r is always going to positive going to be positive for this particular example. So, no matter what method we are going to use, we will not be able to get starting initial guesses for bracketing method.

However, when we are looking at for for example, a secant method as the secant method does not have any problem in an example of this type, the reason why it does not have a problem is that it does not matter that f of x l and f of x r is going to be positive or not in either cases the secant method is going to work. There is one problem; however, for this this kind of a curve for the secant method is that, if the line joining f of x l and f of x r happens to be parallel, (no audio from 23:20 to 23:27) if it is parallel to the x axis then there is no there is no point at which this particular line is going to intersect the x axis and our secant method is is not going to work. In other words, what do we really mean by the line being parallel to x axis it is nothing, but f i minus f i minus 1 equal to 0.This is when secant or the regula falsi method fails.

However, the secant method will not fail, if we have even a slight difference between x i and x i plus 1. So, if we have a situation of this type, we can always connect this point, we will essentially get this particular point as the next point and then we will be able to connect this we will get this as the next our next point and so on and so forth and then we can continue and we will end up reaching the solution eventually. So, this is this is what we will we will end up end up getting and using a secant method. The secant method will still we will still be able to employ to a problem of this sort; however, the regula falsi method, we will not be in opposition to employ in a in an example like this.

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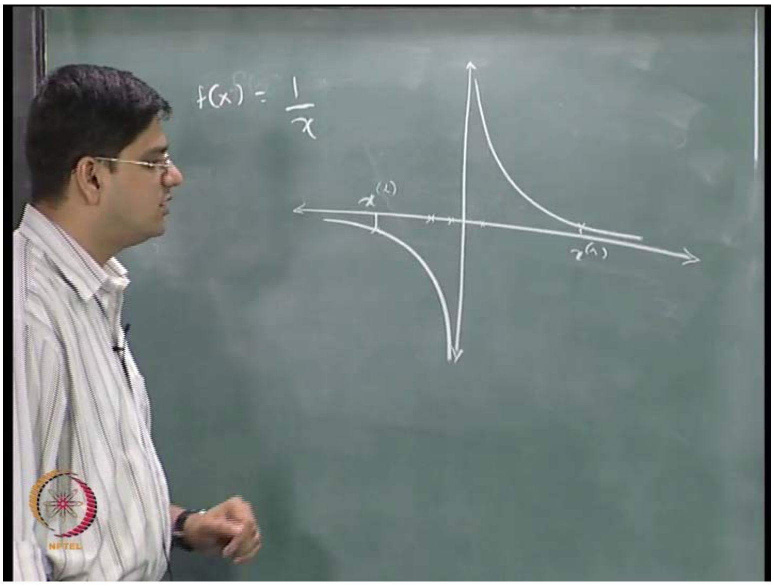

Now, let us look at another example (no audio from 25:04 to 25:14) f of x equal to 1 divided by x. In this particular case, the curve looks like this; it is it is a hyperbolic type of a curve and essentially when x is very close to 0 on the positive side we will reach. So, for example, if x is 0.001 we reach plus infinity, when x is going to be equal to 0.00 minus 0.001 we will reach minus 1000 and as we go closer and closer to the y axis we will go towards infinity.

Now, this particular example does not have any solution because this curve does not intersect the x the x axis except essentially at at infinity. However, if we are going to use a bracketing method especially a bracketing method such as the bisection method, these two are going to be admissible solution this is going to be our x l because and this one is going to be our x r, the reason is the product f of x l multiplied by f of x r is going to be negative. So, these are going to be our admissible solutions.

So, eventually as we reduce this particular. So, the next point as perhaps going to be over here, this is going to our new x r, then the next point will end up being over here, this is our new x l, this is going to be our new x l and so on and so forth and we will eventually reach the origin, as we approach the origin what we will find is that, f of x l tends to minus infinity; f of x r tends to plus infinity.

So, although the solution does not exists for this particular problem, we will be able to figure out that this particular curve has a discontinuity close to the particular point at at 0 whereas, such properties of the curves, we will not be able to obtain using any of the open methods that will that we look at, this is kind of cheating a little bit because in this particular example, we know a priori, that the solution does not exist and we are tricking our problem and we are essentially making a trick to say that, bisection type of a method is able to figure out that the solution does not exist whereas, method like say a secant method or the Newton-Raphson's methods or the other methods that we are going to consider here.

Those methods will not be able to figure figure out that the solution does not does not exist for for this particular problem. So, this is in general about bracketing and and open methods and specifically about the secant method, the next method that we are going to that we are going to consider is going going to be fixed point iterations (no audio from 28:30 to 28:55)

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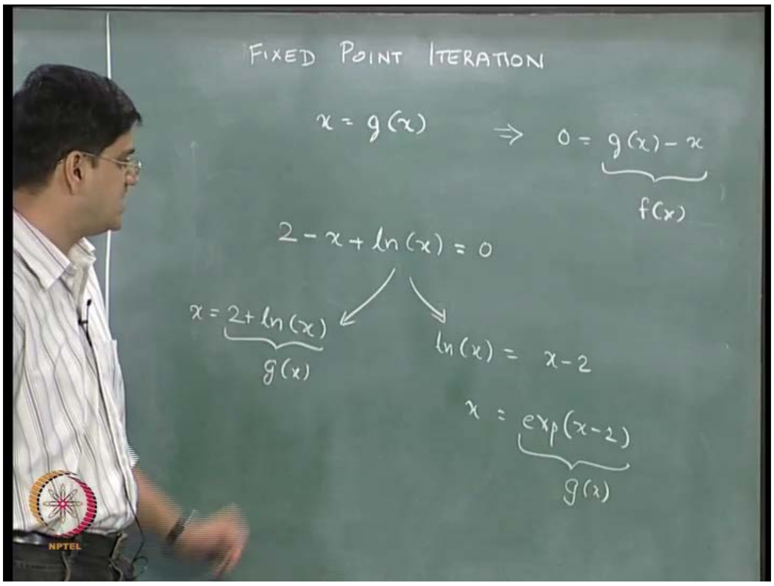

The fixed point iteration method is will be used to solve any general problem of of the form x equal to g of x. This is nothing but recasting of the original problem in in this particular form, if this problem we were to write write down as 0 equal to g of x minus x, this guy becomes our f of x and, f of x equal to 0 is is going to be the form that we have looked at so far. Now, the question is whether x is equal to g of x is going to be unique and the answer of course is no, if we look at the previous the example that we have been looking at in in this particular module that was 2 minus x plus ln of x equal to 0, one way we can write this as x is equal to 2 plus ln x.

In that particular case, this becomes our g of x, the other way to write write this this as is ln of x is going to be equal to x minus 2 or x is going to be e to the power x minus 2. So, this this is going to be the other possibility of g of x. So, x equal to 2 plus ln x is also x is also one way of writing this and this is also the other way of writing this. So, there is nothing unique about which which way which way, we we can choose in order to write this particular this particular expression expression.

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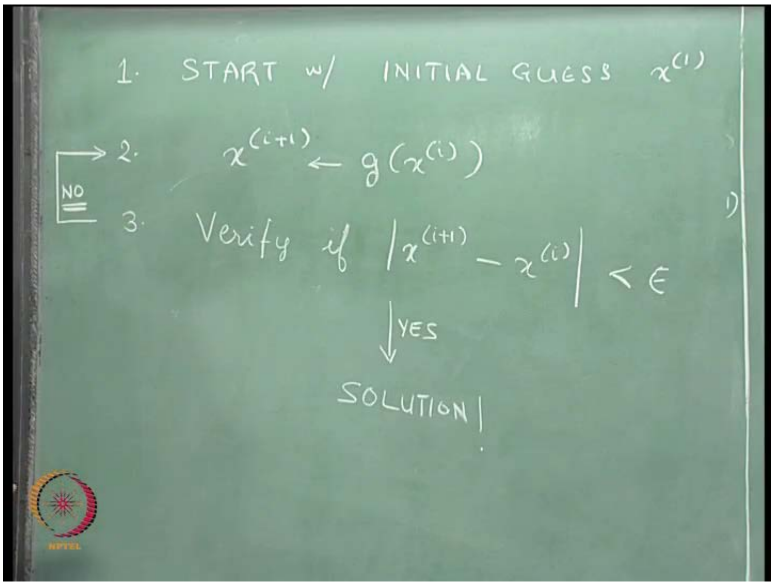

So, now what do we do in in the fixed-point iteration is is as follows, we will use that particular equation as an update equation. So, (no audio from 31:06 to 31:14) first point is start with one initial guess, (no audio from 31:18 to 31:32) we will write x i plus 1 equal to nothing but g of x i, I will use this left phasing arrow to signify that the value computed as g of x i is going to be assigned to f to x i plus 1.

(No audio from 31:56 to 32:28)

So, we start with initial guess, we use g of x of i and assign it to x i plus 1, verify whether x i plus 1 minus x x i is less than epsilon, if it is indeed the solution is reached, if it is not, the solution is not reached and we keep continuing this particular method over and over again. One of the method, one of the problems actually with the fixed point iteration method is that, it is not guaranteed to converge under all conditions, often times will encounter a situation in which x i plus 1 keeps diverging away from the solution as you know as we try to solve this particular problem, what we will do is, we will just now look at again the we use excel in order to look at how the fixed point iteration is going to work and then we will look at essentially the graphical interpretation of the fixed point iteration method.

So, far what we have done in the previous lecture and this lecture is looked at the bisection method, regula falsi method and the secant method all of these methods essentially used two initial guesses as the as for for finding the solution. Now, we will go fixed point iteration, fixed point iteration method works essentially with only one initial guess and let us, just increase the sizes over here yes and we will also increase the font size. So, that it is easier for all of you guys to see and the problem that, we are interested to solve is the same problem as before.

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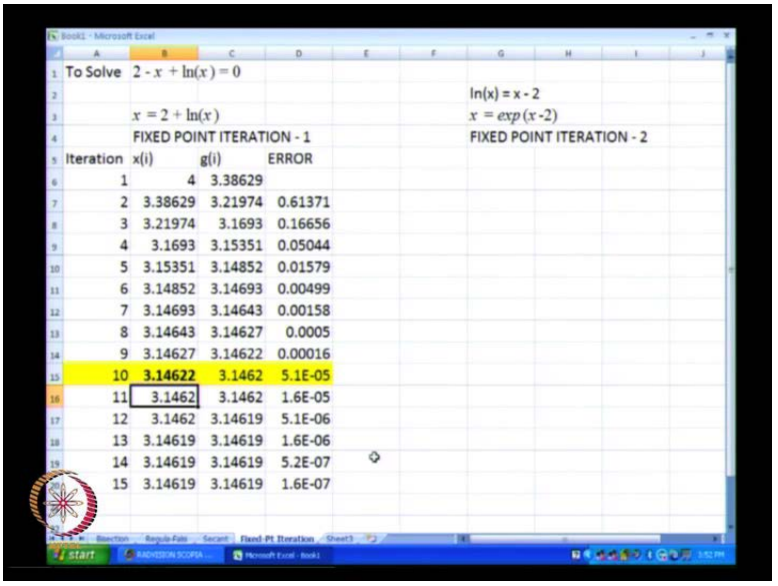

So, we will just copy this. So, this is the problem that, we are interested in solving and we will solve it in two different ways. I will call those two ways as FIXED POINT ITERATION-1 and FIXED POINT ITERATION- 2. So, we will just format the cells and merge the cells and we have fixed point iteration-1 sorry that should be fixed point iteration-1 and copy this and call this as fixed point iteration number-2.

The difference between fixed point iteration-1 and fixed point iteration-2 is, what we will do is in fixed point iteration-1 is take minus x to the right hand side and write this equation as x equal to 2 plus ln of x. So, we will write this equation as x x equal to 2 plus ln of x. So, that is the the way, we are going to do the first fixed point iteration and the second fixed point iteration, I will just insert one row here. So, that it is it is clearer to visualize.

In the second fixed point iteration, we take 2 minus x on to the right hand side and take the overall exponent of that. So, we will get x equal to e to the power x minus 2. So, this we we can write it as ln of x is going to be equal to x minus 2. So, we have taken x minus 2 on to the right hand side and now, we take an exponent and that will give us essentially x equal to e to the power x minus 2 (no audio from 36:38 to 36:46) and we write this as exp, exponent of x plus 2 and just change the font to our favorite Times New Roman. So, that is easier to see.

So, the two ways of do solving using the fixed point iteration is 1 to write this particular equation as x equal to 2 plus ln x, other is to right that equation as x equal to exponent of x minus 2 and same way as before iteration the first the initial guess is x of i g of x of i and x of I, we will let us say we will start of with 4 over here g of x of i is going to be nothing but 2 plus ln of x of i and this is what we get, at the second iteration x of i plus 1 is nothing but g of x of i. So, this is going to be equal to this guy.

And as you see implementation of fixed point iteration is so much simpler in excel compared to any of the other methods and that is all we need essentially, we do not need anything else and we just keep drag dragging it may be let us say, up to say fifteen iterations and we will also plot the error and the error is nothing but absolute value of the difference between the current solution and the previous solution. And our stopping criterion is essentially where the overall error ends up being less than ten to the power minus four, ten to the power minus four was the tolerance value that we had used and as before we will just highlight the final solution over here.

So, the final solution again we get is 3.1462, keep in mind that the various solutions that we have gotten from say, the secant method is again 3.14619. Since our error criterion was ten to the power minus four, it is essentially up to that particular digit that we can be confident of the solution not to the digit after that. So, we can confidently say that the solution is in in fact, 3.1462, we cannot say anything about the last digit in that solution and that is what, if we just keep comparing with all the other methods that is, what we will we will see 6.1462 6.1462 6.1462 and 6.1462 is what we get as as the solution using all of these methods. So, this was one way of essentially using the fixed point iteration.

Now, let us let us look at what what happens if we start with let us say a very large value 10, if we start with a very large value for fixed point iteration, we we are able to again converge to that same same va solution 63.1462. Let us say we start with a value of say two, again we find that when we start with value of two, we have reached again 3.146.

Let us say what happens when we start with a value of one, again we reach the same value as 3.146 and what happens if we start with value of let us say 0.2 again we are reaching 3.146.

So, you see what is happening is that, when we write down the solution as x equal to 2 plus ln x, no matter what initial guess we use for for this particular method. We are getting our solution to essentially go to the value 3.1462, that is one of the thing that happens with fixed point iteration using the fixed point iteration in this particular form, we will not be able to get to the solution which we got I think the other other solution was 0.15, we will not be able to get to that solution even if we started 0.1 what we will get we will get an error, I have written this as 0.1 and I click enter and I get an error, the reason why get an error essentially is that, the next guess of x i plus 1 is a negative value and a log of a negative value is not defined. So, let us go back to what where we started of with we started of with the initial guess of x i equal to four and with the guess of x i equal to four, we get the convergence in ten iterations.

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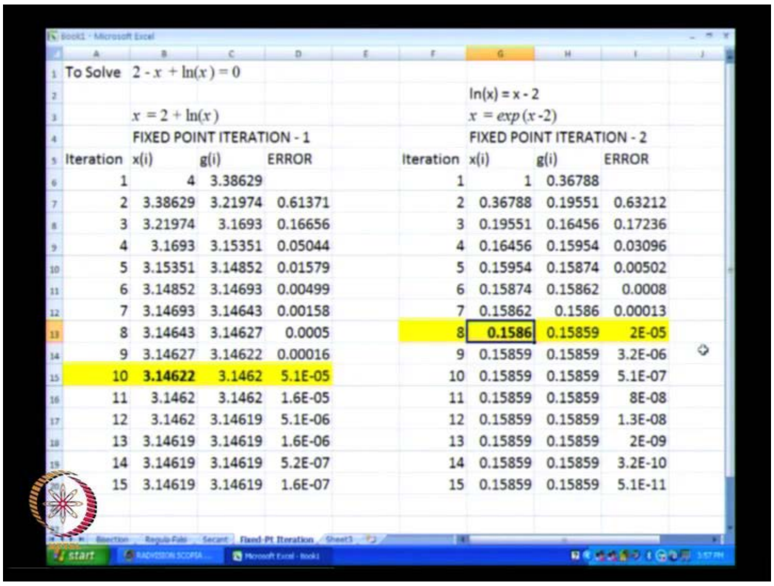
To Solve $2 - x + \ln(x) = 0$

	$x = 2 + \ln(x)$					$\ln(x) = x - 2$ $x = exp(x-2)$		
	FIXED POINT ITERATION - 1					FIXED POINT ITERATION - 2		
Iteration	x(i)	g(i)	ERROR		Iteration	x(i)	g(i)	ERROR
1	4	3.38629			1	1	0.36788	
2	3.38629	3.21974	0.61371		2	0.36788	0.19551	0.63212
3	3.21974	3.1693	0.16656		3	0.19551	0.16456	0.17236
4	3.1693	3.15351	0.05044		4	0.16456	0.15954	0.03096
5	3.15351	3.14852	0.01579		5	0.15954	0.15874	0.00502
6	3.14852	3.14693	0.00499		6	0.15874	0.15862	0.0008
7	3.14693	3.14643	0.00158		7	0.15862	0.1586	0.00013
8	3.14643	3.14627	0.0005		8	0.1586	0.15859	2E-05
9	3.14627	3.14622	0.00016		9	0.15859	0.15859	3.2E-06
10	3.14622	3.1462	5.1E-05		10	0.15859	0.15859	5.1E-07
11	3.1462	3.1462	1.6E-05		11	0.15859	0.15859	8E-08
12	3.1462	3.14619	5.1E-06		12	0.15859	0.15859	1.3E-08
13	3.14619	3.14619	1.6E-06		13	0.15859	0.15859	2E-09
14	3.14619	3.14619	5.2E-07		14	0.15859	0.15859	3.2E-10
15	3.14619	3.14619	1.6E-07		15	0.15859	0.15859	5.1E-11

So, that was the fixed point iteration number one, using x equal to g of x written like this, the other possibility to write x equal to g of x is to write write it in this particular form, in this this case, g of x is e to the power x minus 2. We will start with x i equal to four and see what happens with g i, g i is nothing but e x p of x minus 2 and that is the value that, we we get over here, at the second iteration x i plus 1 equal to g of x of i and then g i plus

1, we can just drag that over here and error is nothing but absolute value of the difference.

So, this is what we have and all we need to do is drag it and let us drag it again for fifteen iterations and again you will see something funny that is happening as soon as I release the mouse button and what what what is what happens is this, is if we start with an initial guess x i equal to four, I will end up diverging. So, this what what we get is, we get our x guess to go towards infinity ten to the power we started with four from four we went to seven from we went to two hundred and nineteen from two hundred and nineteen we went to one e ninety four and essentially from one e ninety four we would have gone to infinity, that is what happens if if you start with a poor initial guess like four in in in this particular case.

Let us see what happens, if we half this particular value from four to two and this is, what we get. So, we start with x i equal to two, we get go to g i equal equal to one from one we go to 0.36 and so on and so forth and we reach the other solution. So, starting with x i equal to two, again we are reaching the the other solution and not this particular solution. Let us start with x i equal to three and see where we get, again with x i equal to three, we are getting to the solution 0.15. In fact, if we start with x i very close to the other to the solution, let us say 3.14, even in the case, when we started with 3.14 which was very close to the true solution 3.146 what we find over here, is that our solution still goes to the other solution 0.1586.

So, in the fixed point iteration using x equal to e to the power x minus 2 converges to the solution 0.1587 and not to the solution 3.1462 no matter how close to 3.1462 we start and if you start with say 3.2, we we will divert. So, if we start beyond the second solution, our fixed point iteration is diverging using x equal to e to the power x minus two. So, just to finish this off starting with x i equal to one, our stopping criterion is that the error should fall below ten to the power minus four. So, this is going to be our solution 0.1586.

So, that is the solution using fixed point iteration, the method to what what we just solve essentially that the fixed point iteration under certain conditions that converges to the solution on the certain other conditions, the fixed point iteration is not going to converge to the solution. The particular problem was chosen such that, we will not have big problems with the fixed point iteration method. In the next lecture, what we will do is we will cover essentially how the fixed point iteration converges and under what conditions is the fixed point iteration not going converge.

Now, what will go on to is, the next example, the next numerical technique for solving non-linear algebraic equation and that numerical technique is known as the newton raphson method, newton raphson method is arguably the most popular a technique for solving non-linear algebraic equations, the reason for it is popularity, we will cover in in the next , it is essentially gotten to do it is it is properties of convergence, the biggest reason for it is popularity of course, is because it is easily extendable to multiple dimensions.

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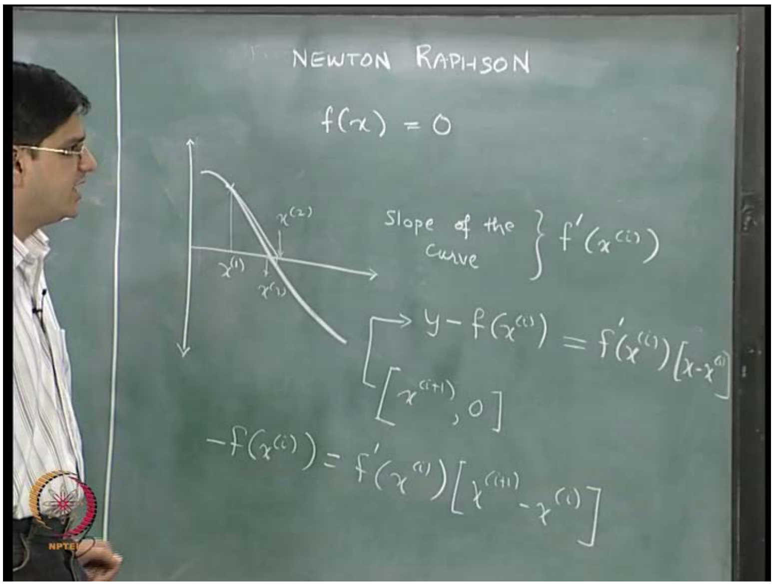

So, the final method (no audio from 46:32 to 46:38) that we are going to cover in algebraic equations solving is the Newton-Raphson's method. So, again we are interested in solving f of x f of x equal to 0 and in the Newton-Raphson's method, what what we do is essentially we take we look at the the curve. So, this is what we did in the secant method is we had for example, two points and we try we connected those two points by a straight line, line at the point at which that particular straight line intersected the the x axis is what our new guess essen essentially was. So, this was our x i plus 1 or rather let us say this was our x 2 then x 2 and x 1 we connected with with with a straight line again and where where x 2 and x 1 we connected with the straight line and this became our x 3, then x 3 and x 2 we connected with a straight line and this become became our x 4.

So, what happened in secant method is, x 2, x 3, x 3, x 4 when when we started doing that this started coming closer and closer to each other. So, when let us consider the case where x 2 and x 3 are going to be very close to each other. So, this particular line segment that joints x 2 and x 3 is nothing but a slope of the curve close to the the point x 3. So, the question that we ask ourselves is, instead of calculating the the slope of the curve by using the previous solution x 2 and current solution x 3 what else can we do in order to get the slope of that curve and the straight forward answer what else we can do to get the slope of the curve is nothing but it is first differential with respect to x. So, that is what we do essentially in in the newton raphson method.

So, what happens in the Newton-Raphson's method is, we do not start with two solutions, but just like fixed point iteration, we will start with with only one solution. So, let us say this not one solution, we start with one initial guess this is our x 1, we will obtain the tangent to the curve at point x 1 and draw that particular line and find the point at which the tangent intersects the x axis that particular point is our x 2 at f of x 2 again we find a tangent to that curve find where it intersects the x axis that becomes our x 3 and so on and so forth.

So, what we obtain is essentially we find a slope of the curve at x i, the slope of the curve of f x i is nothing but f dash of x i. So, the slope of the curve is f dash of x i the line that passes through that particular point with this particular slope is nothing but y minus f of x i is going to be equal to f dash of x i multiplied by x minus x I, again this is the slope, this is line with slope f dash of x i that passes through the point f of x i comma x i comma f of x i. So, at the point at which this particular line intersects the x axis is nothing but the point x i plus 1 comma 0.

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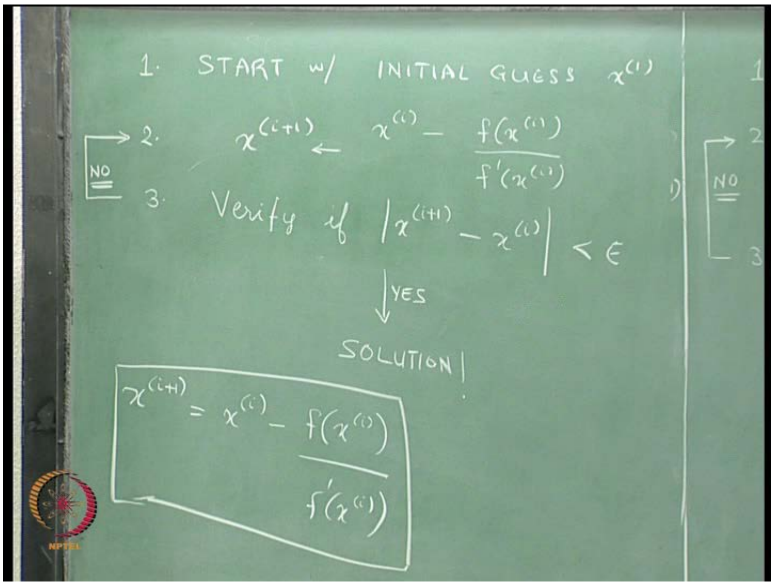

So, we substituted that, in this particular equation and we will get minus f of x i equal to f dash of x i multiplied by x i plus 1 minus x i. So, we can divide throughout by f dash of x i and take x i on to the other side and the the resulting equation that we will we will get is x i plus 1 is going to be equal to x i minus f of x i divided by f dash of x i. So, this is how we are going to use Newton-Raphson method to come up with the next solution x i plus 1. So, let us go to the algorithm that we developed for the fixed-point iteration, we start with a single initial guess x i, in fixed point iteration what we did was, we assigned g of x i to x i plus 1, we are not going to do that in Newton-Raphson instead what we are going to do in Newton-Raphson method is, we are going to use this particular equation and assign it to x i plus 1.

So, our x i plus 1 is going to be nothing but x i minus f of x i divided by f dash computed at x i, verify if x i plus 1 minus x i is less than epsilon, if it is less than epsilon, it is the solution; if it is not less than epsilon, we will repeat until we get convergence. So, this is going to be our algorithm for the Newton-Raphson method and the Newton-Raphson method alike the fixed point iteration method will start with just one initial guess and not two initial guesses and we will just increase the font sizes before. So, that it becomes easy to view.

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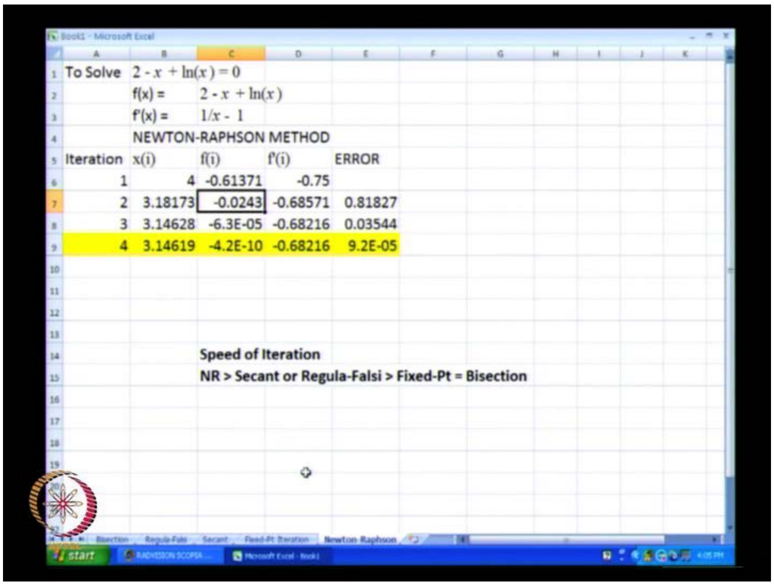

So, we wanted to solve (no audio from 53:33 to 53:38) this particular equation f of x equal to 0. So, we will write, this is our f of x and let us differentiate f of x and write this as f f dash of x is going to be equal to. So, f dash of x is going to the. So, f dash of ln of x is going to be 1 divided by x minus 1 (no audio from 54:17 to 54:23) and just make this times new roman. So, now for Newton-Raphson's method we need f of x we need f dash of x and that is that is essentially what what will be sufficient for us to start with our Newton-Raphson's method and I will just write down Newton-Raphson (no audio from 54:46 to 54:52). So, iteration we have as before, we will have x i, f i and f dash i, we will have x of i, f of i and f dash of i

(No audio from 55:13 to 55:28)

This is what what, we get, this the iteration index, we start with 1 and x i as before, we will start with 4 and f of i is nothing but 2 minus x i plus ln of x i and f dash is nothing but 1 divided by x i minus 1. So, this is our f i, this is our f dash of i. In the second iteration, our x of i plus 1, I will just write that down over here that equation down which we had just just computed on the board, it is x i minus f of i divided by f dash of i, again the Newton-Raphson's is also fairly simple to implement in in relatively simple one dimensional problem like this. So, x of i is nothing x of i plus 1 is nothing but x of i minus f of i divided by f dash of i and so, this is going to be our new solution x of i i plus 1, we will just drag this and we will drag this also, in order to get f i and f dash of i and then we will compute the error as well and error is nothing but the absolute value of the difference.

So, this is the error and now, we are ready as before we are now ready to just drag and drop that particular row, we will just take this and may be will draw drag it say for ten iterations and see what where we go and we really did not need all these ten iterations, we have reached our desired solution in essentially in just four iterations over here All right. So, we started of with x of i equal to four, f of x of i we computed f dash of x of i, we computed x of i plus 1 was equal to x of i minus f of i divided by f dash of i, this is how x of i plus 1 was computed, again for x of i plus 1 we computed f, again we computed f dash when from this, again we we kept computing x of i plus 1 each and every time for at at each iteration and we find that the solution converges reaches the desired tolerance value of ten to the power minus four in four iterations only.

So, what we can now do is just just compare go back and compare the the number of iterations that various methods have taken, this particular problem I spent some time in order to select a problem which showed the the kind of features that are expected from these numerical techniques this this problem shows those features. So, bisection method, it took fifteen iterations to get to the solution. Regula falsi and the secant method took five and six iterations respectively, the number of iterations required by regula falsi and secant methods are usually approximately the same as each other because the underlying method of solving the particular problem to get x i plus 1 is not very different between regula falsi and secant method.

Fixed point iteration again took about ten iterations or eight iterations in order to converge, Whereas Newton-Raphson's took actually just four iterations to converge. In general, the speed of iteration which will cover in our our next lecture goes Newton-Raphson's is faster than secant or fixed point which in turn is faster secant or regula falsi which in turn is better than fixed point which is approximately equal to bisection. We will cover this aspect later on in our lectures, that this is what the speed of iteration follow Newton-Raphson's is better than the secant method which is better than the fixed point iteration method in the open the various open methods. So, this is essentially what we have intent to cover using excel, what we will do in the the next lecture is cover some of the other aspects of.

So, this is what we have done using excel we have covered bisection, regula falsi, secant method, fixed point iteration method and the Newton-Raphson method using excel, the way we have tried to use excel is to bring out some of the salient features of each of these methods, that are going to be of interest to us in discussing this methods going going forward. We will talk in the next lecture about, what we mean by the speed of convergence, how the convergence happens for three of these methods we will take a bisection method the regula sorry the bisection method the fixed point iteration method and Newton-Raphson method and and try to analyze this these three methods in the next lecture, what we have covered. So, far in our lectures is is the single variable case of solving f of x equal to 0 using these these five different methods.