Computational Fluid Dynamics

Prof. Sreenivas Jayanti
Department of Chemical Engineering
Indian Institute of Technology, Madras

Module No. # 06

Dealing with complexity of physics of flow

Lecture No. # 35

Derivation of the Reynolds averaged Navier Stokes equations

Identification of the closure problem of turbulence

Boussinesq hypothesis and eddy viscosity

In the last lecture, we have seen the basic idea of time averaging and the decomposition of any time, any flow parameter into a time average quantity and a fluctuating quantity. Let us just recall what we have derived and based on that, use this concept to come up with the time average form of the Navier Stokes equations.

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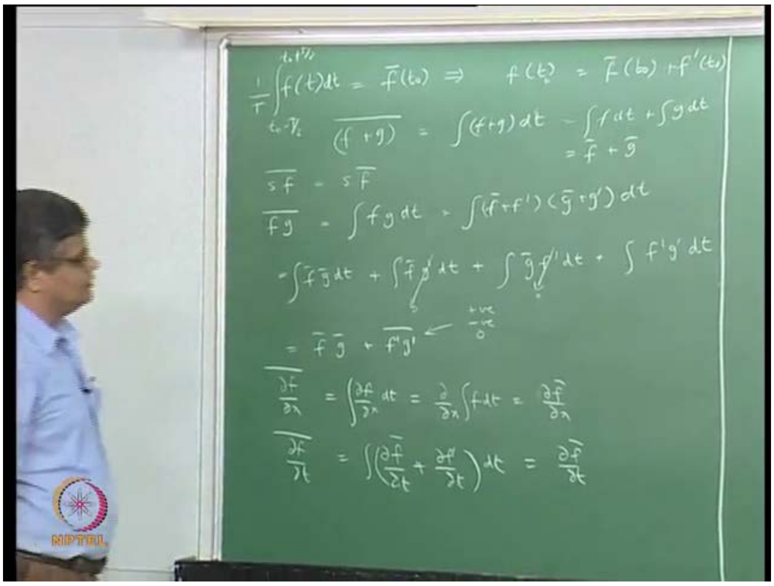

We have said that if f is a quantity, which is a function of time, for example, the f here may represent a particular velocity component at a particular spatial location as a function of time and we have said that in turbulent flow, this is going to show a very rapid variation. Therefore, if you were to take an integral of this around a point t 0 over a time interval of capital T and you divide by the total time period, then this gives us the time average quantity of f at this particular time t naught (Refer Slide Time: 01:26). And, the idea is that capital t the time period that we have taken here for averaging is too large; is very large compared to the typical time period of turbulent fluctuations and at the same time it is very small compared to the typical time system transient that we are looking at. The nature of turbulent flow fluctuation is such that it is possible to come up with a time period, which satisfies both the conditions. That is, the system transients that we are usually interested in; that is, a start up of particular chemical reactor over a period of minutes or hours is very long compared to the typical time period of oscillation fluctuations, which are produced in turbulent flow.

So, this enables us to do this decomposition of based on this, we can write f at a particular time t as t naught; as f bar at t naught plus f prime at t naught. So, you have a time average quantity and a fluctuating quantity and using this definition, we can show that f plus g over bar. So, this is integral of f plus g d t; we have forgotten to put d t here (Refer Slide Time: 03:05), over this interval which you can break up into f d t plus g d t which gives us f bar plus g bar and similarly, we can show that a scalar times f where s is a scalar which is constant and f is a fluctuating parameter. The time average of this is s times, the time average of this and we can also show that the product of 2 fluctuating parameters f and g is integral of f g d t. So, that is integral of f bar plus f prime times g bar plus g prime d t and we can show that we (Refer Slide Time: 04:00) can write this as 4 integrals: f bar g bar d t plus integral of f bar g prime d t plus integral of g bar f prime d t plus integral of f prime g prime d t and each of this integration is carried out over this time frame, over this time period capital T during which f bar and g bar are by definition constant. Therefore, this comes out of the integral when you integrate this thing goes to 0 and similarly, this goes to 0 leaving us with f bar g bar plus f prime g prime bar and we have said earlier through a homely example that this quantity here is not necessarily 0 (Refer Slide Time: 05:00). It can be both; it can be positive or negative and it can be even 0 only. When it can be 0? Only when f and g are independent the fluctuations arising out of f are not related in any way to the fluctuations related to g.

And, we can also show by the same extension of the same ideas that the time average of a derivative which often appears in our equations is nothing, but dou f by dou x d t integral and assuming that f is a continuous function of both x and t we can write this like this. So, that becomes dou f bar by dou x and given that we can even write this also as and. So, this will become dou f bar by dou t. So, that all derivatives with respect to the time average quantity are derivatives of that time average quantity whether spatial or temporal and the only thing that brings us difficulty extra terms is the product of 2 fluctuating quantities and this g can be a derivative like this or it can be just value like this

So, using these rules we can come up with the time averaging of the Navier Stokes equations and we will start with the time averaged form of the continuity equation we will take the case of incompressible flow in which case the continuity equation is nothing, but dou u k by dou x k equal to 0 and this is equal to dou u by dou x plus dou v by dou y plus dou w by dou z equal to 0.

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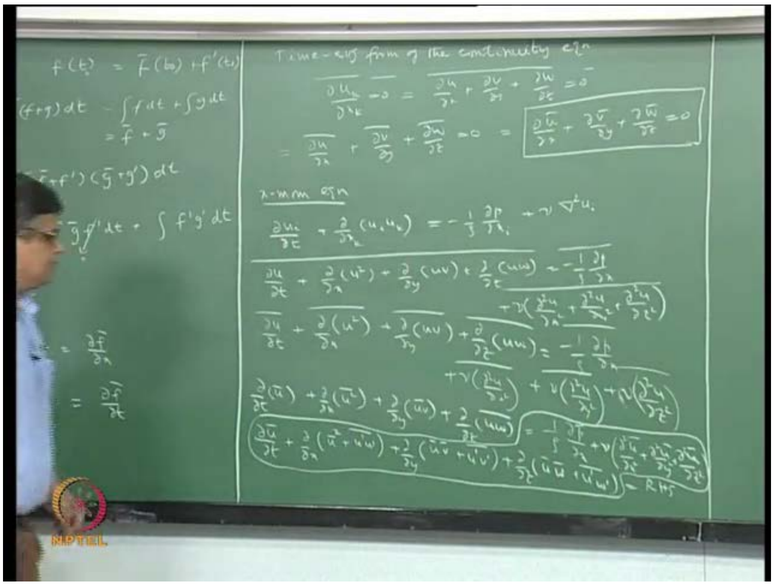

And we can time average both the left hand side and right hand side. This time average of 0 is obviously 0 and we can say that this quantity is like time average of f plus g plus h. Therefore, it is time average of this quantity plus time average of this quantity plus time average of this. So, we can write this as dou u by dou x bar plus dou v by dou y bar plus dou w by dou z bar equal to 0 and we know that dou u by dou x the time average of this is nothing but dou u bar by dou x dou v bar by dou y plus dou w bar by dou z equal to 0. So, this is the continuity equation which is time averaged. And, we can see that the continuity equation in instantaneous in terms of instantaneous quantities is dou u by dou x plus dou v by dou y plus dou w by dou z and the time average form involves the spatial derivatives of the time average quantities (Refer Slide Time: 09:25). If the flow is steady then, u bar v bar w bar will not be functions of time and that is the definition of steady flow in turbulent quantities and therefore, if it is, if the flow is steady in the turbulent flow sense then we do not have to solve for u prime and that makes the… that makes life easier.

Now, let us take the continuity equation - the x momentum equation. Again, we take the incompressible form. Therefore, we can write this as dou u i by dou t plus dou by dou x k of u i u k or u k u i is equal to minus 1 by rho dou p by dou x i plus nu del square u i we are neglecting the effect of gravity because if you have constant density it does not play much part. So, we can expand this and write this as dou u by dou t plus dou by dou x of u square plus dou by dou y of u v plus dou by dou z of u w equal to minus 1 by rho dou p by dou x plus nu times dou square u by dou x square plus dou square u by dou y square plus dou square u by dou z square. This is the x momentum equation expressed in terms of the instantaneous quantities u v w and p and each of which is fluctuating and each of which can be decomposed into u bar plus u prime v bar plus v prime w bar plus w prime and so on (Refer Slide Time: 11:30).

So, now we want to have a time average form of the Navier Stokes equation. So, in order to do that just as we have done for the continuity equation, we time average the left hand side and we time average the right hand side and when we do that we see that there are 1, 2 3, 4, terms here and those 4 terms is like time averaging of the 4 terms which are being added together. So, that is equal to the sum of the time average quantities. So, this is equal to the left hand side becomes equal to dou u by dou t bar plus dou by dou x of u square over bar plus dou by dou y of u v over bar plus dou by dou z of u w over bar equal to on the right hand side we have minus one over rho dou p by dou x over bar plus nu dou square u by dou x square over bar dou square u by dou y square plus nu dou square u by dou z square over bar.

So, now we can see that we have derivatives here and we have derivatives and product of 2 quantities this is like f g in this case g is also equal to u in this case g is equal to v g equal to w. So, we can write this as partial derivative with respective t of u bar plus dou by dou x of u square over bar of u v over bar u w over bar equal to minus one over rho is constant. So, this is dou p bar by dou x and here plus nu is again constant we can say that this is the… like the derivative, what is applicable for the first derivative is also applicable for the second derivative. This becomes dou square u by dou x square dou square u bar by dou y square plus dou square u bar by dou z square.

So, the right hand side has become, is already expressed in terms of time average quantities of p bar u bar and u bar here now for the left hand side quantities we have now the derivatives this is this remains a same. So, we can write this as dou u bar by dou t and here we apply the product rule that is f g over bar is equal to f bar g bar plus f prime g prime over bar and if you do that here we get dou by dou x of u bar square plus u prime u prime over bar plus dou by dou y of u bar v bar plus u prime v prime over bar plus dou by dou z of u bar w bar plus u prime w prime over bar and that is equal to the r h s ok.

So, now if we look at this equation here (Refer Slide Time: 15:23) this is the time averaged form of the x momentum equation and we see that in this time average form we have the temporal gradient of u bar spatial gradient of u bar spatial gradient again with respective to y of u bar v bar and all these things and on the right hand side again, we have that is lost in that line here. So, this is gradient of the mean pressure gradient and gradients of the mean velocity that are coming in to picture. So, in that sense, as a result of time averaging we are getting equations expressed in terms of the similar derivatives to what we have here of the time average quantity. So, we can write this in this succession way and say that this is equal to the time average form of this is equal to dou u i bar by dou t and here this is dou by dou x k of u i prime u i bar u k bar plus u i prime u k prime over bar equal to minus 1 by rho dou p bar by dou x i plus nu del square u i bar.

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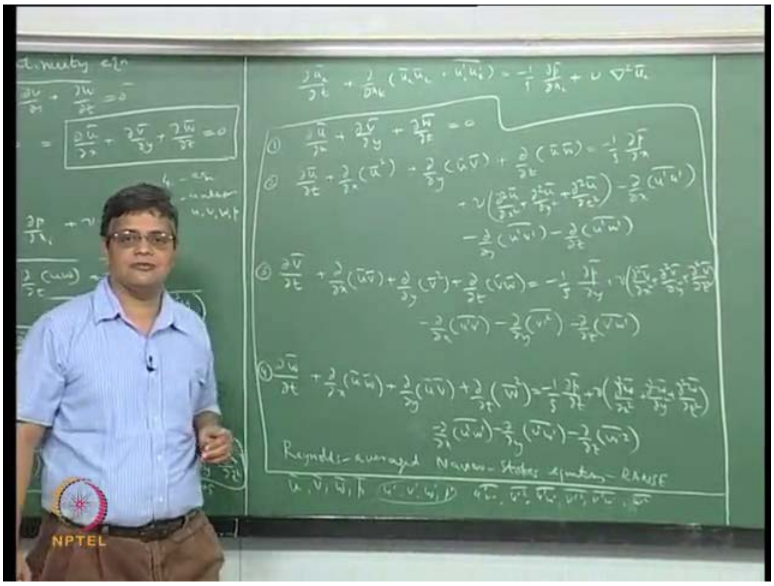

So, we have a time averaged momentum equation which is, in which the variables are the time average quantities of this and what do we have here? If you look at this, if you look at the continuity equation and the 3 momentum equations which are expressed here we can write this down for for the sake of better understanding and we already have this. So, we can… what I am doing here is that I have here dou by dou x of u bar square and then I have dou by dou x of u prime u prime bar; I take these time average of the fluctuating component terms on to the right hand side and retain these things the derivatives of the time averaged primary quantities on the left hand side. So, which is what I am doing here plus dou by dou z of u bar w bar equal to minus one by rho dou p bar by dou x plus nu times dou square u by dou x square plus dou square u bar by dou y square plus dou square u bar by dou z square and then I have the left over term here minus dou by dou x of u prime u prime bar minus dou by dou y of u prime v prime bar minus dou by dou z of u prime w prime bar. So, this is my time averaged x momentum equation and I can also write by inspection the time averaged y momentum equation in the same form will be like this dou by dou z of v bar w bar equal to minus 1 by rho dou p bar by dou y plus nu of dou square v bar by dou x square dou square v bar by dou y square plus dou square v bar by dou z square and the left over terms from this product terms (Refer Slide Time: 20:15).

So, that is minus dou by dou x of u prime v prime bar minus dou by dou y of v prime square bar minus dou by dou z of v prime w prime bar and finally, the z momentum equation in the time averaged form will be like this dou by dou x of u bar w bar plus dou by dou y of u bar v bar plus dou by dou z of w bar square equal to minus 1 by rho dou p bar by dou z plus nu del square w bar by del x square dou square w bar by dou y square dou square w bar by dou z square and the 3 terms left over from the product quantities that is minus dou by dou x of u prime w prime bar minus dou by dou y of v prime w prime bar minus dou by dou z of w prime square.

So, these are the time averaged form of the Navier Stokes equations (Refer Slide Time: 21:50). So, we can call these as Reynolds averaged Navier Stokes equations; usually referred to as RANSE or rans type of things and how many equations are there? We had 4 equations and we have 4 time averaged form of these equations and how many variables did we have to start with? We had originally, if you take these 3 variables coming here u v w and the generalized momentum equation is like this and we have again u v w here u v w here and p.

So, we have originally 4 equations and 4 unknowns which are u v w and p now we have 4 equations and what are the unknowns here we have u bar v bar w bar p bar these are obviously, the variables of interest we would like to know what these things are as a function of x y z t and of course, we have u prime, v prime, w prime, p prime, which are not of interest and we do not to know much about these things. We do not need to know about this because there is rapidly fluctuating and they are not appearing in person in these equations directly. They are appearing, we can see from these equations these are not variables because we do not have anywhere here this u prime; we only have u prime square over bar. So, that is a different processed quantity it is not the same as this, but of course, if you had u prime as a function time then you could actually do the averaging and then come up with the same thing, but we say that these are not variables directly which are appearing in these equations, but the variable that are actually appearing in these equations in addition to these are u prime, v prime bar, u prime square bar u prime w prime bar ok.

So, the 3 things and again we have 3 here out of which this we have already put here. So, we have v prime square bar v prime w prime bar and once if you come to the time averaged z momentum equation, this is u prime w prime bar which we have already put here; v prime w prime bar we have already put here. So, we are left with w prime square bar. So, our time averaged Reynolds averaged Navier Stokes equations contain 4 variables which are of interest to us and 6 additional variables in terms of quantities in terms of the rapidly fluctuating quantities u prime, v prime, w prime, p prime which are of not interest to us ok.

So, we would say that we are not interested in this, but it is necessary that we know these things because only then the time average form of the Navier Stokes equations is complete. if we say we neglect these things then we are leaving out those terms and they are 0. We said that f prime g prime bar here may be positive or negative or 0 and it is only 0 if only the fluctuations in f and g are not related. At this stage, can we say whether these are related are not related? Even if one were to say that the fluctuation of u prime v prime and w prime are not related independent even then we would still have this quantity here which is non zero because, it is a u prime square we know that turbulent flow has fluctuation. So, it must have fluctuations in u and v and w ok

So, here we have square of the fluctuation of the same quantity. So, this cannot be 0; if it is turbulent flow and similarly this cannot be 0 if it is turbulent flow and this cannot be 0 that if it is turbulent flow. So, that means, in addition to the 4 variables that are of interest to us at least 3 of these are non zero if we have turbulent flow. If the flow is truly turbulent and therefore, is exhibiting fluctuation, so, that means, that we are faced with the problem that we have the same 4 number of equations, but now we have at least 3 more variables which are coming into picture and in the general case there are 6 more variables. So, we have 4 equations and 10 variables. So, we are backed to the earlier specification and the momentum equation where we started out with 4 equations the continuity equation and the 3 momentum balance equations and we had u v w and 9 stresses - the sigma i js. So, we had at that time we had to go through the Newtonian assumption - Newtonian fluid; assumption of linearity between stresses and shear rates to come up with additional equations.

Thereby, in which we try to express the stresses in terms of pressure and viscous stresses where the viscous stresses are linearly related to the deformation rates and so on and so forth. We are back to the same position of that time of what is known as the closure problem that we have more number of variables than the number of equations that are available. So, if we want to do the time averaging of the governing equations to get rid of those high frequency oscillations, we will have an insoluble problem because there are simply more number of variables and we know that these cannot be neglected they are not 0. Now, can they be neglected? If you neglect them then the resulting equations are of exactly the same form as the equations for lamina flow accept that you have replaced the u with u bar and v with v bar and so on.

So, in that sense the equations are mathematically not different from the equations that we have for lamina flow and therefore, the solution would also be not different from lamina flow and that would mean that you had the same friction factor as you had for turbulent flow and same heat transfer coefficient for as per lamina flow, even though flow is turbulent. So, and we know that is a gross under estimation or over estimation in the general case. So, that means, that you cannot neglect the effect of these things and say that we have only… we have time average quantities (Refer Slide Time: 29:44). So, the effect of these has to be considered now the second thing is that can we say that this is the difference between the lamina like and the turbulent like is coming only from these squared quantities; can we make the simple assumption that these u prime v prime v prime w prime and u prime w prime those things are 0? So, that we have only 3 quantities, if you do that then let us take the case of fully developed turbulent flow between 2 infinitely long types it is like the (Couette flow) kind of assumption (Couette) flow with a pressure gradient.

So, you have 2 infinitely long pressure; infinitely long plates which are separated by a constant distance and you have steady in flow and therefore, the flow is driven by a constant pressure gradient; even though it is in the turbulent - in the turbulent flow idea of the steadiness. So, in such a case we can see what are the time average form of the Navier Stokes equation. Tell us if the flow is 1 dimensional and if there is no time averaged vertical velocity and time averaged velocity in the depth direction, then the continuity equation will tell us nothing but dou u bar by dou x equal to 0 and that is true because, we have assumed that the flow is fully developed.

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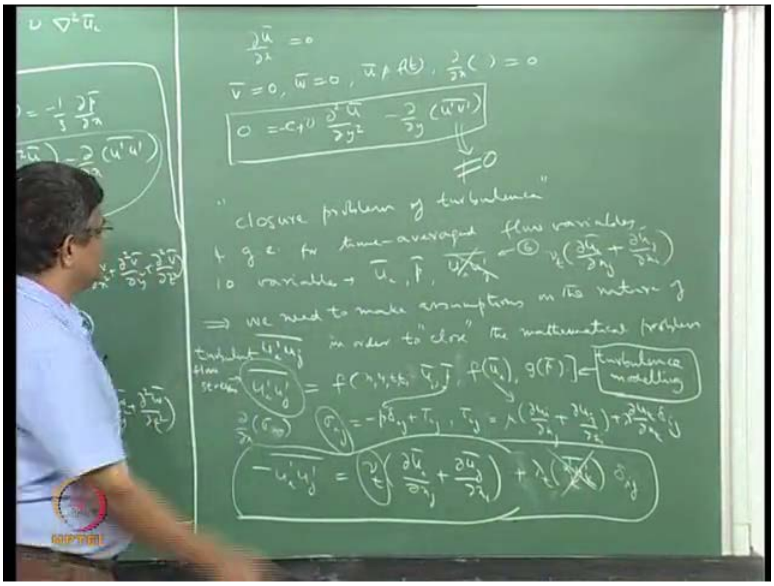

So, in such a case, this is the continuity equation; we will give as this. we are looking at the case where v bar is 0, w bar is 0 and the flow is steady. Therefore, u bar is not a function of time and the flow is fully developed. That means, that dou by dou x of anything is equal to 0 except pressure and pressure gradient is constant and it is a given constant from the boundary conditions.

So that, the idea is that when we have the time average form of Navier Stokes equations and when we are in the time average behavior, that is when we are interested in how u bar, v bar, w bar and p bar vary with x y z t, in such a case we would be, we would like to be solving a simpler form of the equations and we have said that even though that the flow is turbulent and therefore, 3 dimensional and transient for the case where they… one can assume steadiness and fully developed flow condition in the turbulent flow sense. That is, variation of a time averaged quantity in the x direction of any quantity is equal to 0 is the fully developed condition for a in the turbulent flow sense of this and similarly, not variation, not the time averaged quantity of a particular variable not being a functions of time; here is the concept of steadiness under turbulent flow. So, under those conditions, we expect these equations to exhibit the same kind of 1 dimensionality or 2 dimensionality or the sense of fully developedness or the sense of steadiness that we actually see in experiments and in, as in the case of lamina flow. So, we are saying that for the case of steady fully developed turbulent flow between 2 parallel plates, we have no cross velocity components. The flow is fully developed variation of any quantity is equal to 0 with respective to the spatial x derivative and any quantity is all quantities are independent of time.

So, under those conditions the x momentum equation will give this equal to 0. This quantity is equal to 0 because it is dou by dou x of any quantity. Any time errors quantity is 0; this is 0 because v bar is 0; this is 0 because w bar is 0; here this is constant and in this case, this is 0 because it is variation with respect to x this is 0. So, only remaining term is this one and here, this will be 0 because, this quantity is variation with respect to x. So, this is a time average quantity. So, this should be 0 and this is also 0. So, what we are left with in this particular case will be 0 equal to nu del square u bar by dou y square dou y square and then we have this one minus dou by dou y of u prime v prime bar. So, this is the 1 dimensional, steady fully developed flow form of the x momentum equation and we immediately see that in this case, this cannot be 0 because, if it is 0 we just have the same old Poisson form of the equation for lamina flow. So, that means, that if that were to be 0, if the fluctuating quantities were to be 0, then we should have identical velocity profile for lamina flow and turbulent flow between 2 infinitely long parallel plates driven where the flow is driven by a constant pressure gradient ok.

We have a constant here so, and we know that is not true. We know that the turbulent flow velocity profile is very different from lamina flow. So, that means, that this quantity is not 0. So, that is why we cannot say that among all these quantities, the extra quantities that are appearing here we have said that this cannot be not 0; this cannot be not 0; this cannot be not 0 and even this cannot be not 0. This cannot be 0 because, if that is the case then, we would have the lamina profile for turbulent flow. So, in the general case, we cannot, we cannot cross out any of these things and we are left with the problem of 4 equations and 10 variables. So, this is the closure problem; the closure problem of turbulence where you have 4 governing equations for time averaged flow variables and we have 10 equation 10 variables which are the time average quantities u i there are 3 of these p bar and then the time average the time average of fluctuating velocity components u i u i prime u j prime and there are 6 of these.

So, we are faced with the problem of having too many variables and too few numbers of equations. So, what is the way around it? One possibility, one argument that we can make is that the difference between the lamina flow part and the turbulent flow part is fundamentally nothing because we have just done some mathematical operations here.

We have just integrated with respect to time both, the left hand side and right hand side and in the process, we have got some extra quantities and we have got expressions for the time averaged quantities of the velocities. So, one may argue that we can therefore, in a similar way by doing further mathematical manipulations, we can derive equations for these quantities also and we will see that it is possible straight away and without much of difficulty, but in the process of generating equations for these we will find that we will be introducing more fluctuations. For example, when we in the process of generating equation for u bar that is, the time average quantity of u velocity we have introduced, we have had to introduce the time average. The time average of fluctuating quantities u prime v prime if you were to write an equation for u prime v prime bar, then we will find that we may get u prime v prime w prime which again becomes unknown. So, this problem of closure cannot be overcome by doing further manipulations of these equations and then trying to develop more equations for these time average quantities.

So, the problem closure cannot be done through more averaging and manipulations of the basic equations. In other words, we cannot overcome the problem of closure in turbulence flow if we resort only to the fundamental equations; that is, the fundamental Navier Stokes equations. So, we have to bring in some empiricism. We have to bring in the kind of empiricism that we have brought in making the assumption of a Newtonian fluid and isotropic medium in order to bring about Navier Stokes equations from the momentum conservation equations. So, in deriving the Navier Stokes equations, we have made some assumptions and similarly, if you wanted to close this problem and come up with equal number of variables as the number of equations available, we have to make further assumptions and only under those assumptions we will have these equations being valid. So, we need to make assumptions on the nature of these quantities u i prime u j prime in order to close in inverted brackets. So, there is nothing figuratively, no door being closed close the mathematical problem.

So, what kind of assumptions we can make? What kind of guess work we can? What kind of additional knowledge we can bring about these things? What we should keep in mind is that, it is not these quantities that are coming directly into this. It is the spatial gradients of these quantities that are coming here. If you look at the time average Navier Stokes equations, it is dou by dou y of this dou by dou z of this dou by dou x of this is, it is the gradients of these things that are coming. So, we need to know; we need to have an expression for. So, what we are looking for is u i prime v j prime as a function of something. So, that we can then as a function of x y z t and any other quantities that we need to have for example, we can have it as a function of u bar u i prime bar and p prime and we can have a function of u i prime bar a function of p bar and so on, ok?

So, if we were, if we had an expression like this then wherever we had this u i prime v j prime, we could substitute this and then substitute these in this equations. With the result that all this fluctuating quantities these things will be taken out of the things and we will have only a system of 4 equations and 4 unknowns. So, at this point it is useful to recollect what we have done in the case of momentum equations. In the case of momentum equations, just as we had these derivatives of these things, we also had spatial derivatives of for example, sigma i j, sigma x y, like that and sigma y z, sigma x x, all those things were coming. So, at that point, in order to get around this we decompose this sigma i j in to minus p delta i j plus tau i j where we said that tau i j is the viscous stress tensor and using the linearity assumption between the viscous stress and the deformation rate tensor we finally, express tau i j as nu dou u i by dou x j plus dou u j by dou x i plus dou u k by dou x k lambda times this times delta i j ok.

So, and therefore, these expressions will give us an expression for sigma i j in terms of known quantities; that is, p here which is being solved. This is the new variable that was introduced at that time and gradients of the velocities. So, we would like to have a similar kind of expression for the stresses for these fluctuating quantities in terms of x y z and the known quantities. For example, if you were to compare this or this thing once we substitute here, this with this then this p and this p will be coming and this function of u i is what we are essentially saying here ok.

So, in that sense just as we have derived expressions for the stress tensor in terms of the velocity gradients of the strained tensors, we would like to derive a similar expression for this u i prime u j prime bar that is the time average quantity. So, these things in terms of variables that we are anyway following for which are any way appearing as variables of interest for us and variables of interest are the time averaged velocity components and pressure from which we can get time averaged drag coefficient, time averaged pressure gradient and so on, for a given flow rate. So, this is what we need to do and this is what turbulence modeling is about. Turbulence modeling is about expressing this u i prime u j prime over bar; that is the time average of these quantities which are appearing as a result of the time averaging process. Let us express these things in terms of the variables that we are solving for, is the essence of turbulence modeling and it is also necessary because, as we have shown here, these quantities cannot be neglected. Now, if you want to do this turbulence modeling and if you want to make this, we need to make some we need to have some idea of what this is and what kind of modeling we can make. So, the easiest way of looking at these things is to compare it with the case that we had earlier when we were deriving the Navier Stokes equations and say that these stresses, these quantities here are very similar to these viscous stresses. So, we can call this as turbulent stresses ok.

So, turbulent flow stresses which are arising because of the fluctuations there is a lot of imagination which goes into the concept of equating this with the stresses, but mathematically one can see that we can rewrite these 2 parts here. In this you can write this in an equivalent way as dou by dou x of nu dou u by dou y minus u prime v prime bar; will give us a combination of this term and this term plus dou by dou y of nu dou u bar by dou y. We have the bar quantity this is, sorry, this is dou by dou x of u prime v prime bar is this and this and these 2 is u dou square u bar by dou y square dou by dou y. So, we can say dou by dou y of nu d u by dou d by d y minus u prime v prime bar and these 2 can put… can you put together to write as dou by dou z of nu dou u by dou z minus u prime w prime bar? So, we are saying that these terms which are coming in the x momentum equation can be written in this form and therefore, these turbulent fluctuations the time averaged quantities are directly adding to the corresponding viscous stresses which are appearing here. And, what are these viscous stresses? These are, these can be considered as momentum flux terms associated with momentum transfer arising out of viscosity. So, this is a momentum flux which is arising out of viscosity and velocity gradients in the x direction and this is the velocity gradient in the y direction which is causing a momentum flux in the y momentum flux of the in the x direction and similarly like this.

So, it is because of this the first modeling turbulence modeling which is proposed for this was by Boussinesq and that is in 1875 are s1875. So, that is he expressed u i prime u j prime minus of this as nu t some turbulent viscosity times dou u bar by dou x j plus dou u j bar by dou x i. So, this is we have an extended expression which we can see; but this particular expression here is very similar to the expression that we have. We can add that extra things plus let us call this as lambda turbulent times dou by dou x u i prime u i prime. So, this can be the expression; this is delta u k, so as not to confuse this. So, we can write it in an expression which is analogist to this and thereby, we can see that these things are considered as momentum additional momentum fluxes which are arising out of a turbulent viscosity and in this sense these can be called as turbulent stresses and we note that it is a minus of this particular quantity which is equated to the turbulent viscosity and this also is consistent with the general observation that in turbulent flow we have additional moment flux which is causing additional friction factor and an additional mechanism for heat transfer and mass transfer. That is why we have a higher height transfer coefficients and transport coefficients in turbulent flow, ok.

So, that is, that observation is supportive of this kind of description; that the additional momentum flux which is coming in the form of these quantities, the fluctuating quantities which are coming up as additional terms in the time averaged form can be equated to turbulent viscous stresses with a turbulent viscosity and the gradients of the time average quantities of the velocities of the appropriate velocities like this and if we were to do that then we can directly substitute this in these things and get rid of all these turbulent fluctuations. So, if we neglect this for the time being and then we look at this then, the problem of turbulent closure can be completely overcome if you were able to specify this nu t. So, that we can substitute this here and we can say that this whole thing is replaced nu t dou i dou u bar by dou u j bar by dou x i. So, if you do that then we have u bar which are also appearing here and the only variable is nu t.

So, the problem of turbulent closure can be at once brought down to specification of turbulent viscosity. We will see how this particular idea of Boussinesq has been explored and then brought up in to fusion in the form of turbulence closure which is used even to this day in some form of modeling.