Computational Fluid Dynamics

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Module No. # 06

Dealing with complexity of physics of flow

Lecture No. # 32

Topic

Derivation of the energy conservation equation

So, let us start with the derivation of the energy equation. And the derivation of the energy equation will be similar to how we have derived the momentum conservation equation, in the mass conservation equation. We take a control volume and we write down the, we give a statement of the energy equation for it, and then, we evaluate each of the terms, noting that the energy that we are dealing with can be changed, the energy content in the control volume can be changed by the addition of heat into the control volume externally by the work done on the fluid, which is contained in the control volume by external forces. And it can also result from the advection of energy in, because part of the along with the flow, and any other source terms that may be there, that can be come into picture.

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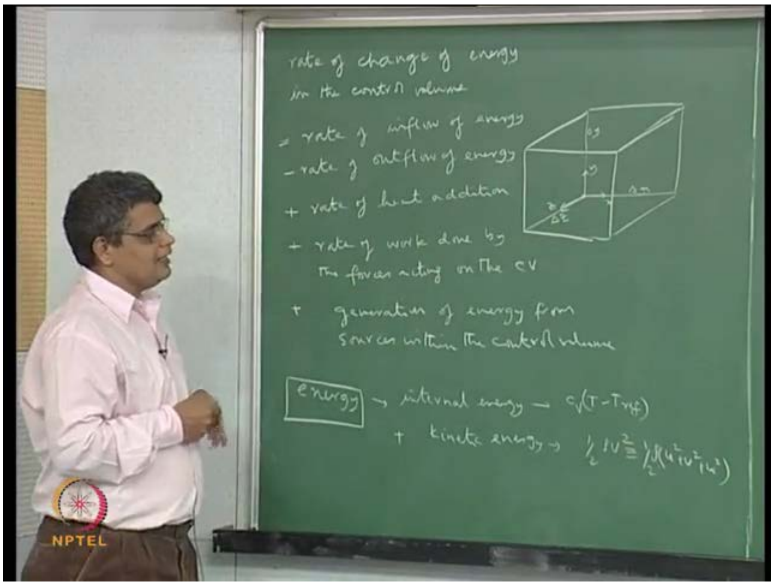

So, what we do is that, we take a control volume like what we have done earlier. And as usual we look at the origin at this point, this is our x direction, y direction and z direction. The control volume has delta x width in this direction, a height of delta y in this direction and depth or the length of delta z in this direction. And this a typical control volume, and we can say that, the rate of change of energy in the control volume is equal to rate of inflow of energy.

So, when there is inflow, then the energy increases minus rate of outflow of energy plus rate of heat addition to the control, to the fluid contained in the control volume. We are distinguishing heat as a specific form of energy plus the rate of work done by the forces acting on the control volume; these are external forces and we have made the point that there are stresses - viscous stresses - and also the pressures forces, which may be acting. And when we have a force acting on a particular control volume, then it can do work and that becomes part of the external work. And if there is external work done on the system on the fluid which may be control volume, then its energy content increases.

So, that is why we have plus rate of work done plus volumetric plus generation of energy from sources within the control volume.

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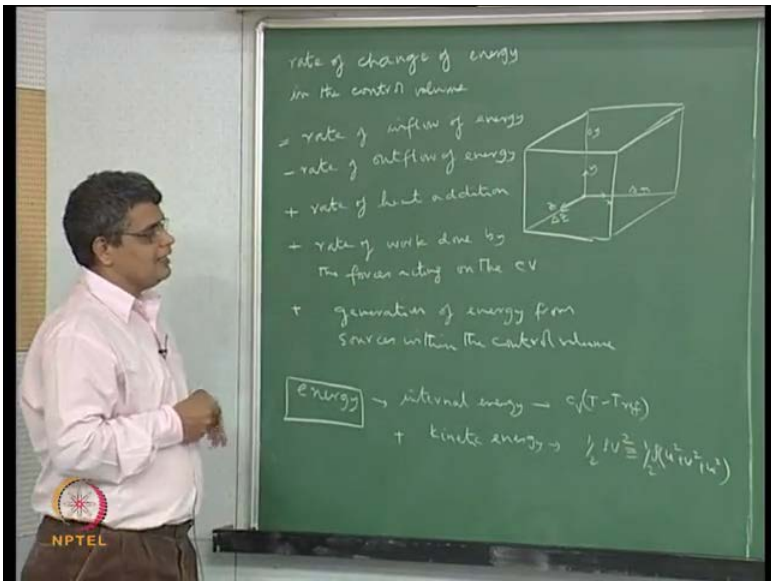

So, this gives us a balance a statement to the balance of the energy, that is contained within the control volume, or energy possessed by the fluid, which is contained within the control volume. Now, first of all, we have to clarify what we mean by energy here, what energy are we talking about. We are talking about energy as that associated with the, what is known as the internal energy, which is c v times T minus T reference.

So, this is the energy associated with the fluid having a temperature T, where c v here is the specific heat at constant volume, plus kinetic energy, and this is associated with the velocity half rho v square; and in the specific case, where we have three velocity components, this becomes half of rho times u square plus v square plus w square, where u, v, w are the three components of velocity.

So, we have potential energy, also is that is considered as a form of energy, but we will see that, when we consider gravitational force as an external force which is acting on this, then the gravitational potential energy associated with the gravitation field will come in through this particular term, rate of work done by the forces acting on the control volume. So, that we do not have to include that in this.

So, we are looking at the rate of change of energy - rate of change of internal energy and kinetic energy - possessed by the fluid within the control volume, which is attributable to rate of inflow of energy, because this control volume is actually a small part of a the fluid expands. And fluid will be flowing through and as it flows through, it brings in water property task, and therefore, the fluid coming in with the property of internal energy, which is associated with the temperature and the velocity; it also brings in energy. And similarly, any fluid that is going out of the flow domain will also carry away with it, the energy - internal energy - associated with this temperature at the outlet point and the velocity at the outlet point.

So, these are the two terms that are coming here; and heat addition from outside is something that we are considering specially. And we already have acknowledged the fact that, there are forces acting on it, and rate of work done by the forces that are coming here. Now, this generation of energy is a special case, and if there is a generation, for example, in a due to a nuclear reaction which may be happening inside or due to a chemical reaction which may be happening within that, we can definitely consider that in the…

So, we would not expand on what specifically the generation, how the generation of energy is taking place, but what we have in mind here are, the generation from factors which are distributed entirely within the control volume. So, we are not looking at a generation at a particular point, at a particular point within the overall flow domain, but something that is distributed throughout. So, the most appropriate thing probably would be a nuclear reaction or a chemical reaction, which may be happening within that.

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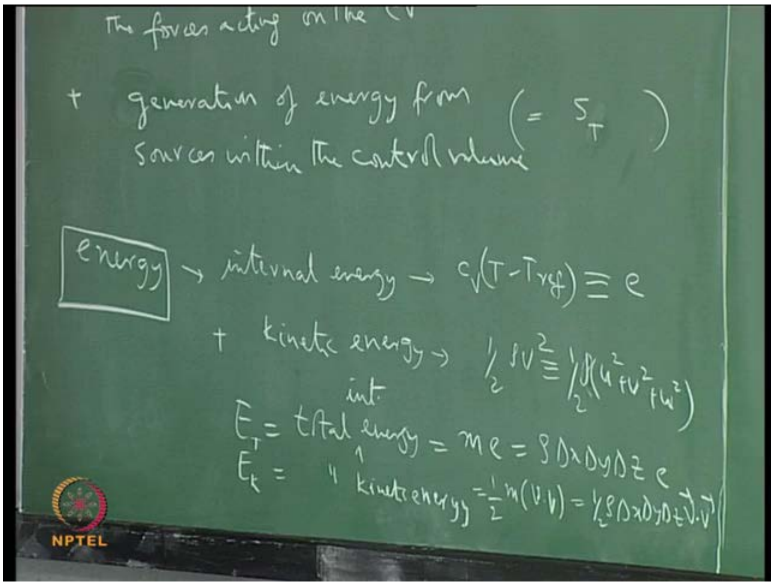

So, we will just put this term as, some s phi s; the s T here with the subscript capital T indicates that, this is the thermal energy source term; we will leave it at that particular thing. And we will try to evaluate each of this in terms of the properties that we know, so that we can write down the overall energy balance equation.

So, let us write an expression for this; we know that we are talking about internal energy and kinetic energy; and we write down the specific energy part of it, that is, the energy per unit mass. So, we can call this, this particular internal energy as equivalent of small e.

So, small e is such that, capital E which is the total internal energy is m times e, where m is the total mass is contain within this. So, that is rho time's delta x delta y delta z times e. So, this gives us the total internal energy. And similarly, the total we will put this as T there, and total kinetic energy I am putting roughly as, this is half m v square; so, that is v dotted with v. So, again this can be written as, half rho delta x delta y delta z times v dotted with v.

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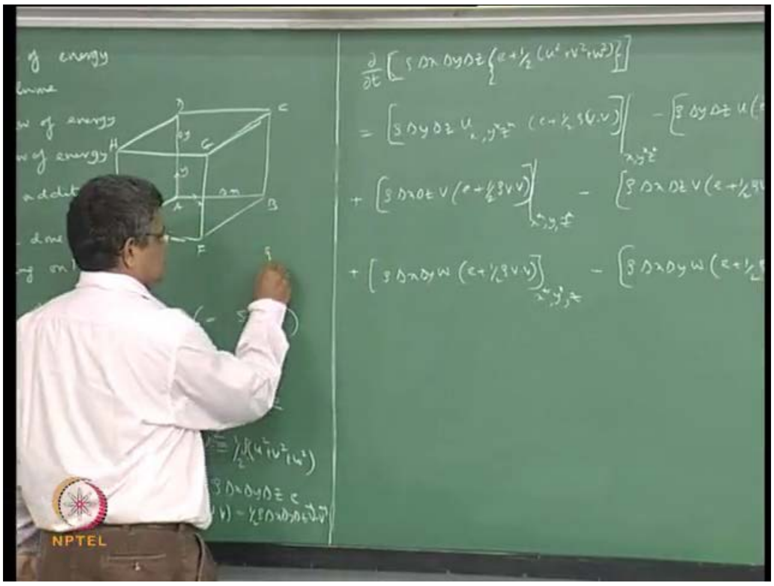

So, we can write therefore, that the rate of change of, time rate of change of energy within the control volume can be written as partial with respective to time, indicating the time rate of change of energy as, rho delta x delta y delta z half e plus half u square plus v square plus w square. So, that is the rate of change of energy, that is in this, is equal to rate of inflow of energy, and inflow is always inflow of something is equal to the mass flow rate times specific quantity there.

So, we can we can write that as, we know that the mass flow rate, if you were to put like A B C D and E F G H as the vertices; usually, we can see that, this is an inflow surface, that is a negative x face, and similar the bottom face is an inflow face and then the back face is an inflow face for this; and the flow rates through each of this is, rho times the cross sectional area, which in this particular case is delta z times delta y times the normal velocity at the centroid.

So, we can write that as, rho delta x delta y delta z times u at x star at x y star z star, where the star quantity indicates the central point of y and z for that particular face. So, this is the flow rate times the specific energy that they have bringing in is e plus half rho v dot v.

So, this is what is coming in through the x face and through the bottom face. So, this is what coming through the left face. So, we can write this whole thing as, at x y star z star. And we can consider immediately also what is leaving through the positive x face, that is, the right face. So, we can write that as minus, since it is leaving, we put minus rho times the cross sectional area delta y delta z times u times e plus half rho v dot v, at x plus delta x y star z star.

So, this is what is going in through the left face and coming out through the right face. Similarly, what is going in through the bottom face? rho times the cross section area, which is delta x delta z times the normal velocity v times the specific quantity. This whole thing evaluated at x star y z star minus, what is leaving through the top face is rho times the cross section area times velocity times of specific quantity; whole thing evaluated at x star y plus delta y z star plus, now we consider the back face. The mass flow rate is rho times the cross section area, which is delta x delta y times the normal velocity w times the specific quantity e plus half rho v dot v. This whole thing evaluated at x star y star z minus, what is leaving through the front face, which is rho times delta x delta y del times w times e plus half rho v dot v, evaluated at x star y star z plus delta z.

So, these six terms represent the rate of inflow minus rate of outflow of energy, through the six faces of a control volume; plus heat of a rate of heat addition, we say that, heat is coming in, for example, by conduction through the faces; and at each face, we have a heat flux q.

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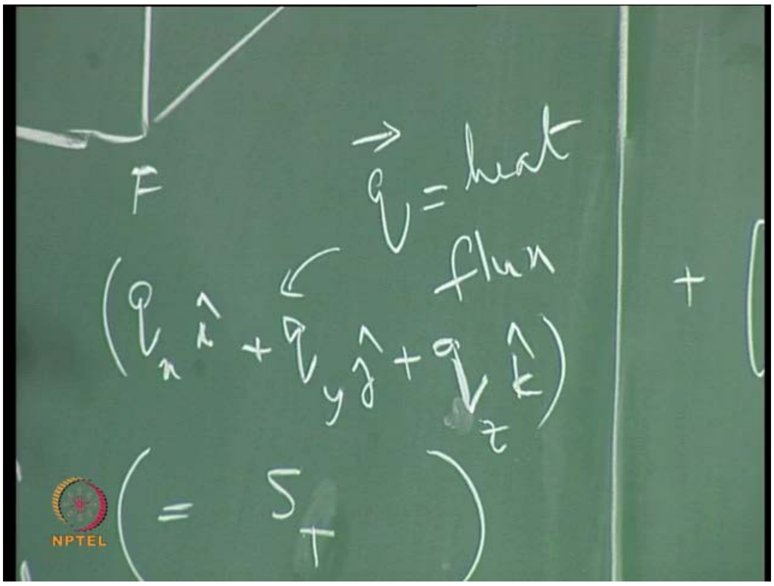

q is the heat flux and this the vector quantity. So, this as three components, q x i plus q y j plus q z k. So, since flux is a vector quantity, there is component in the i direction, j direction and k direction.

So, this heat flux vector as to be dot produced with the area normal vector, in order to find out how much heat is coming in through the particular face. And when we take the dot product, then because we have taken these area faces to coincide with coordinate directions, for an x face, we take the corresponding x face area; these things will be 0. For a y face, that is, bottom face or a top face, take only the q y component of this flux vector - heat flux vector - and multiply by the cross section area delta x delta z. And for a front face, the back face, we take the q z component of this and then multiply with the corresponding area, which is delta x times delta y. So, this because we have taken the faces to coincide with coordinate directions, the evaluation this term will be simplified. So, by definition, q x positive means, that it is going in flux is in the x direction.

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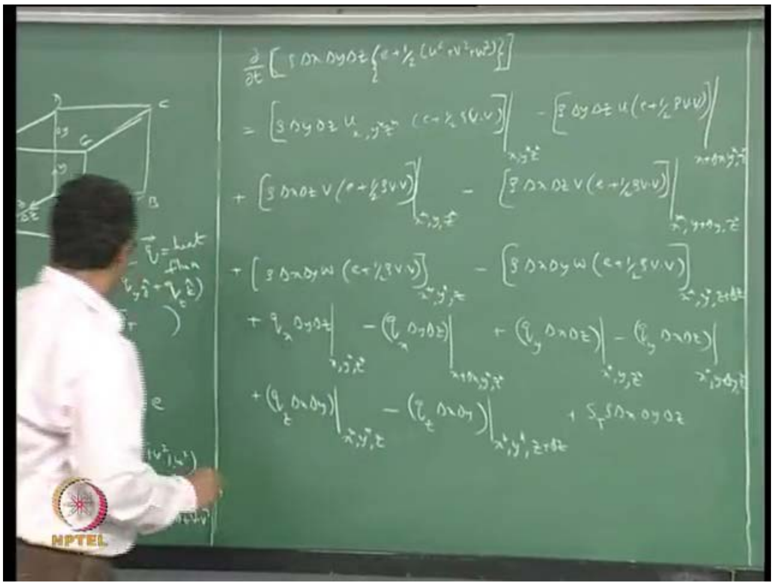

So, we can write, at the left face, at q x is in the positive x direction, so it is coming in. So, q x times the cross sectional area of this particular face, which is delta y delta z, evaluated at x y star z star. And at this face, that is the positive x face, that is the right face, q x is actually leaving; so, that means, loss it is not being added, it is being subtracted the system is losing. So, q x times delta y delta z, at x plus delta x y star z star.

Similarly, when we consider the bottom face, it is a q y component that has to be multiplied by the cross section area, and q y at the bottom face is in the vertically upward direction; therefore, whatever heat flux that is the q y, positive value of q y at the bottom face will be coming into the control volume. So, it is added cross sectional area delta x delta z, evaluated at x star y z star. The heat flux through the top faces is something that is leaving; so, that is q y times delta x delta z with minus sign, evaluated at x star y plus delta y z star.

So, we have now evaluated four of these faces. Similarly, the back face and the front face; at the back face, the non-zero component of the product of a heat flux vector with the surface area vector will be, q z time the cross sectional area, which is delta x delta y; this is the heat that is coming in and this is evaluated at x star y star z minus q z delta x delta y times; and this is the heat flux that is leaving through the front face, evaluated at x star y star z plus delta z.

Now, what is left? We have the rate of work done by the forces acting on a control volume and we have the source term here. So, we can first deal with this. So, we can if we declare this as the volumetric source term, so this times delta x delta y delta z is the volumetric source term. And if we call this as per unit mass, then it will be rho times this; so, we will call it as per unit mass. So, with that, we can deal this.

Now, the evaluation of the rate work done by the forces acting on it, is a tedious process, but it is a relatively straight forward; we have to identify what the different forces acting on the control volume are. And we know that, there are, on each faces, there are pressure forces and there are viscous stresses; and not only that, there is gravitational force, which is a body force that is acting.

So, let us deal with first of all the gravitational force. The gravitational force has forces are there in a each direction. So, you have forces in the x direction, y direction, z direction; and if we have force f dotted with displacement d s, we will give us the work done.

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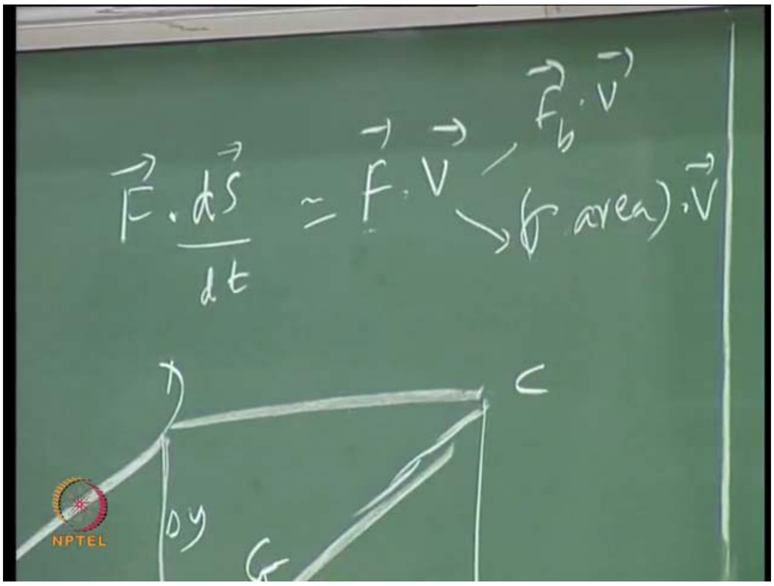

And that divided by time is the rate work done. And we are looking at rate of work done; so, this is can be written as, F dotted with V, where V is the velocity of the particular fluid component. So, this is what will be given to us. In the cases of body forces, then we evaluate the body the force; so, this will be F b dotted with v. And in the case of stresses, we have to look at this stress times the area to convert into a force and that is dotted with v.

So, we have to consider various stress and body forces. And they are three components here, F v will have F b x, f b y, F b z, dotted with v, that is u v w. So, we will get, u times f b x plus v times F b y plus w times Fb z. When we consider F b the body force is to be coming from the gravitational thing, then we can write it down as, rho g x times u plus rho g y times v plus rho g z times w so this and the total volume.

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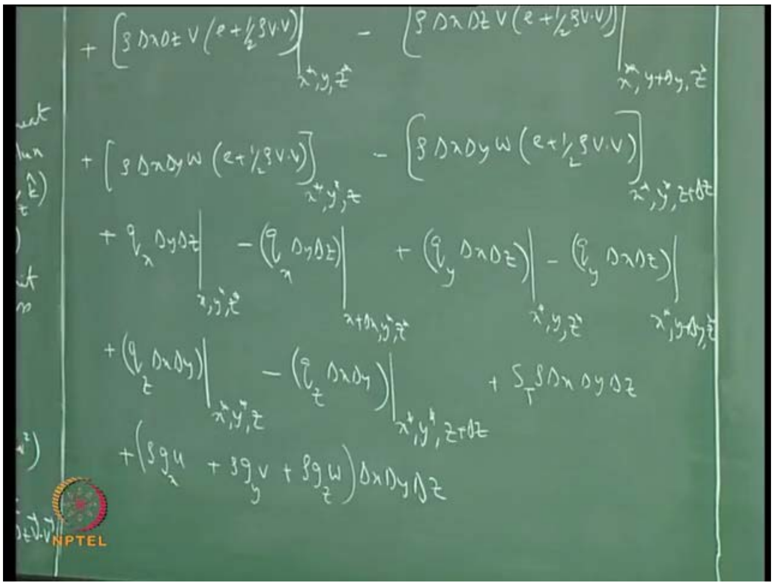

So, this is the contribution of the gravitational force as the only non-zero body force that we considering, and g x, g y, g z are the three components in the x y direction; so, the gravitational vector.

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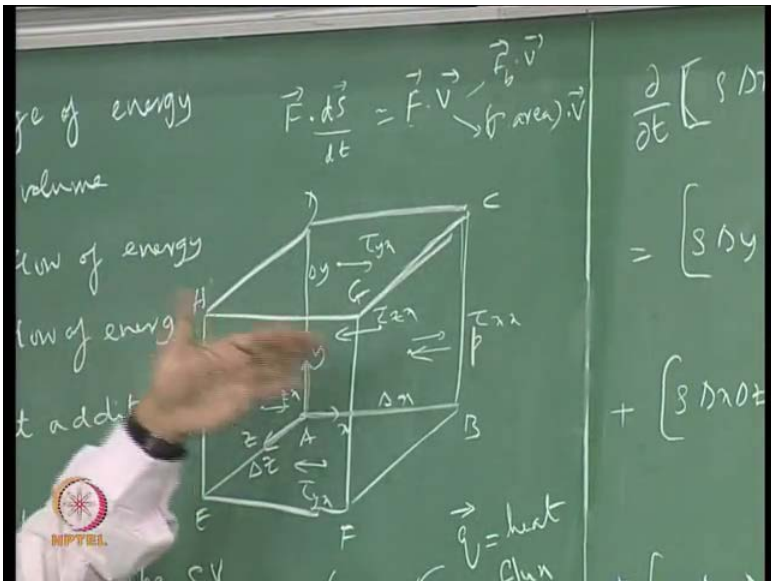

Now, what about the stresses here? We have to look at all the stresses acting in the x direction and y direction and so on. So, let us consider, for example, the x direction, if we take this particular face, at the centroid we have pressure acting as a compressive force; and we also have the viscous stresses; viscous stresses, there is a viscous stress tau xx and we have on, on this particular case, that is only stress that is acting on in the x direction.

At the top here, we have tau yx, and at the bottom, we have tau yx oriented in this direction. And then, we have on this side, we have tau xx on the negative face acting in the negative direction; and the back face, we have tau zx acting in the negative z direction; on the front face, we have tau zx acting in the positive x direction. So, each of this stresses has to be multiplied by the appropriate area. and we have to take the corresponding velocity component. And since all of them are acting in the x direction, the velocity component here will be u.

So, we take, for example, this p here, pressure force acting on this; and similarly, we have pressure force acting on at this face.

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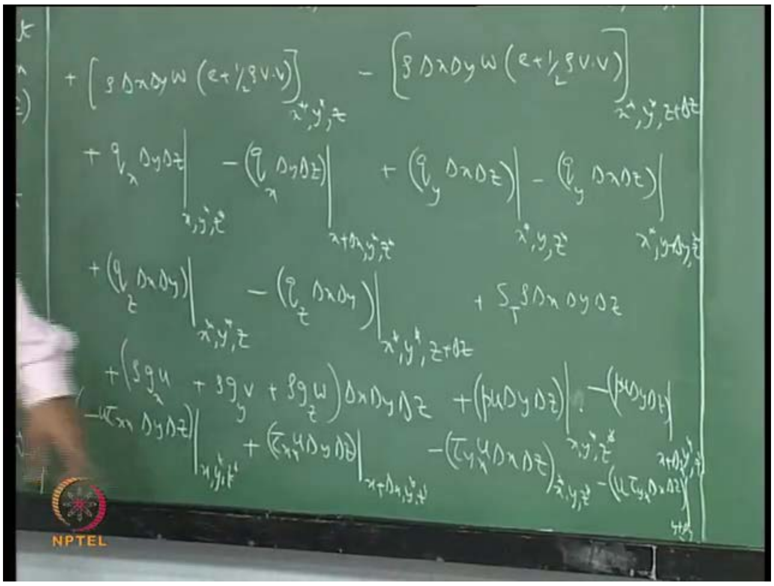

So, if you consider just the x direction forces, and out of that x direction stresses, and out of the pressure as a stress acting, on the x face - left face - the contribution is plus p times the area, which is delta y delta z, evaluated at xy star z star. And on the right face, it is acting in the opposite direction in the negative x direction, so that is minus p delta y delta z, evaluated at x plus delta xy star z star.

Now, consider the stresses tau x x acting on the x face, that is acting in the negative x direction. So, we can write this as, tau minus tau xx delta y delta z, at x y star z star; on the positive x face, then that is tau xx delta y delta z, at x plus delt a xy star z star. The negative y face, that is the bottom face, that is minus tau y x times the area on which is acting is delta x times delta z.

We forget to put the corresponding velocities, because it is just the force, and here this is multiplied by u. We have to take this inside; so, we will put this as p times u here, and p times u tau xx u and tau y xu; there is also another tau xx, yes here, u times tau xx, this is at the bottom face. So, that is x star y z star. And at the top face, again this is u times tau yx delta x delta z, evaluated at y plus delta y.

We notice that, even though this is tau y x here, this is acting on the bottom face and top face. We still multiply with the u component, because this a force stress which is acting in the x direction. So, when we take the dot product like this, then we should take only the u vector - u component. So, finally, we have the left back face and the front face; back face will give us minus tau zx times u times the area, which is delta x delta y. This whole thing evaluated at x star y star z.

And the front face will gives us tau z x u delta x delta y, at x star y star z plus delta z. So, these are all the rate of work done terms coming from the forces stresses acting on in the x direction. We have two pressure components and six stress components, each of which is multiplied by u here, the stress times the corresponding area which is changing; and all of them multiplied by the same velocity component coming from the velocity here, for the rate of vector. And we can similarly write down the six stress components and two pressure components, acting on the y in the y direction. And those six will add the corresponding energy terms - rate of work done terms - and there will be another six stresses and another two pressure forces acting in the z direction.

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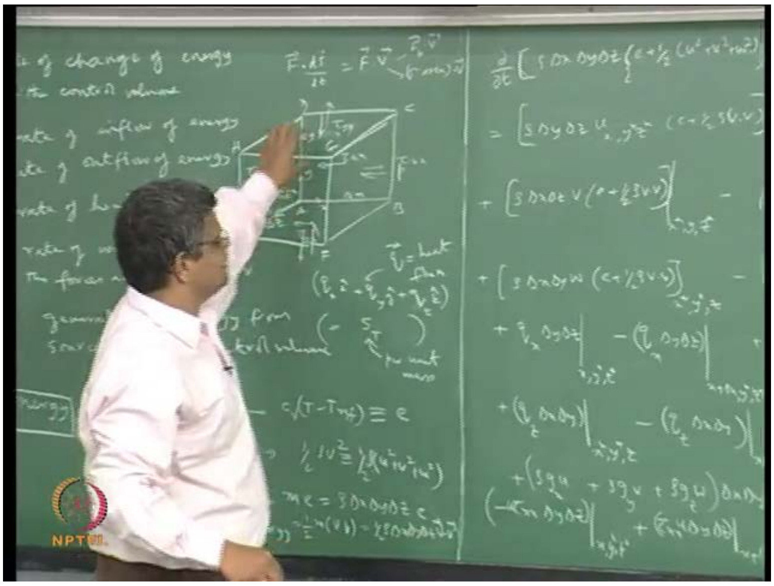

So, if we consider, for example, the pressure here and the pressure here, and there will be a tau yy, which is acting in this direction and tau yy acting in this direction, so we can write them down them. And when we consider all the stresses acting in the y direction, we have to multiply by the v component or the velocity vector.

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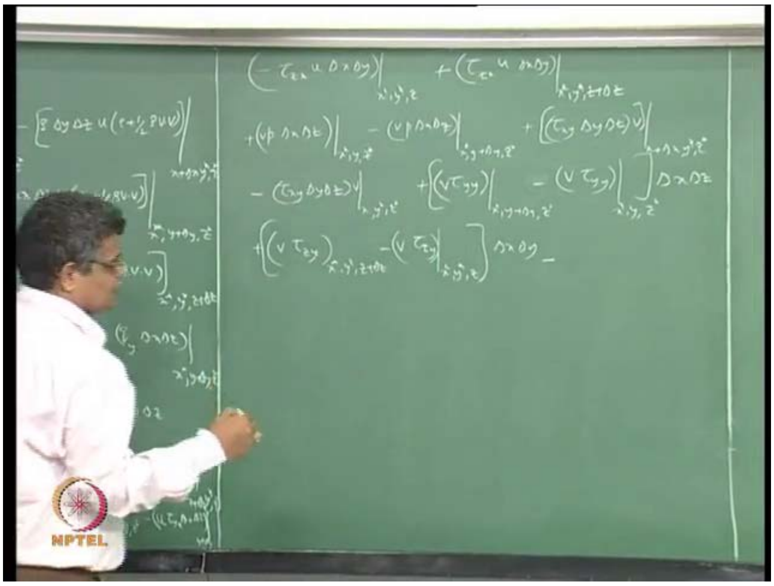

So, let us just do that, minus p times the area which is delta x delta z times v, evaluated this is plus v at the bottom face, x star yz star minus v p delta x delta z at x star y plus delta y z star; so, these are the two pressures. And the stresses if you consider the positive x face, then that is tau xy times that area will be delta y delta z, multiplied by v; and this whole thing is evaluated at x plus delta x y star z star. And the same thing on the negative x face will be acting in the negative y direction, so that is minus tau x y delta y delta z times v, at x y star z star.

Go bit faster Now, tau yy times v at x star y plus delta y z star plus v minus v tau yy at x star y z star. This whole thing is multiplied by the corresponding area, which is delta x delta z, plus v times tau zx at x star y star z plus delta z minus v tau, z y here, z y, evaluated at a x star y star z; this whole thing multiplied by that area which is delta x delta y. So, we have 1 2, 1 2 3 4 5 6, all these things multiplied by the velocity component v; these are the rate of work done by the stresses acting in the y direction.

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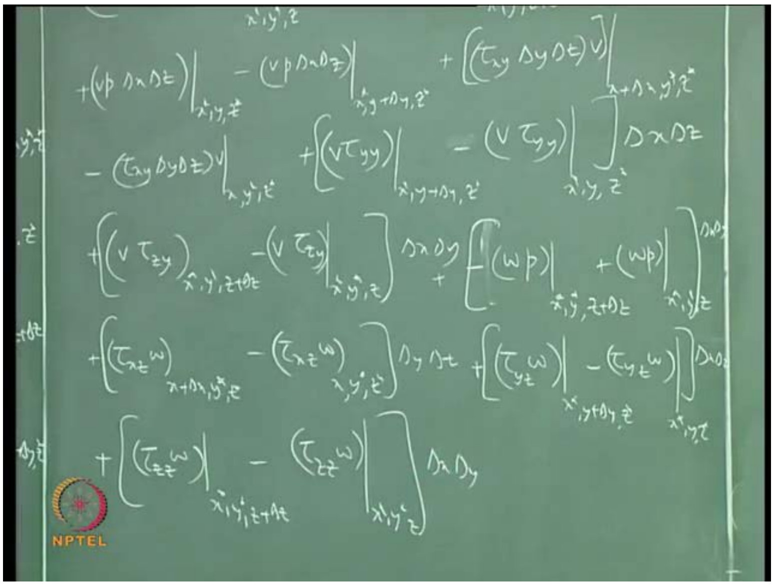

And will have six more terms coming from the z direction; we can write them down here, and all these six stresses and two pressures will be multiplied by the w component. So, that is w times p, at x star y star z plus delta z by, plus w times p, at x star y star z; this whole thing multiplied by the area, which is delta x delta y. And the stresses in the a on the x faces will be plus tau x z times w, at x plus delta x y star z star, minus tau xz w times x y star z star; this whole thing multiplied by the area which is delta y delta z.

On the two y faces plus tau y z times w at x star y plus delta y z star minus tau y z and w, at x star y z star; both these multiplied by the corresponding area, which is delta x delta z. And the stress on the front and back faces, so that is tau zz times w, evaluated at x star y star z plus delta z, minus tau zz w at x star y star z; this whole thing by times the corresponding area is delta x delta y.

So, this is the overall expression for the energy equation. So, we see that, e plus - internal energy - plus kinetic energy being brought in and taken out by the flow, and the heat that is being brought in and taken out by the certain by the particular mechanism. This is the heat flux and then the gravitational work done, source term here, and then the stresses.

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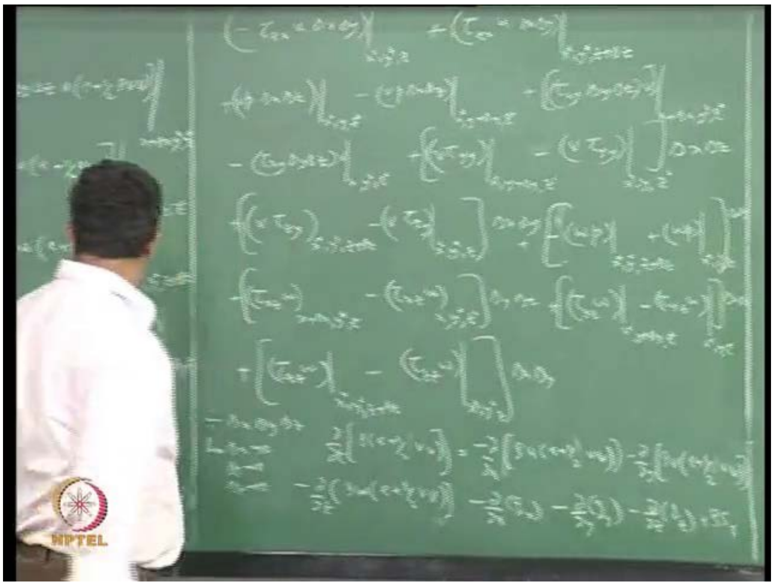

And as usual, we divide the whole equation by delta x delta y delta z, and take the limit as delta x tends to 1, and delta y tends to 0 and delta z tends to 0. And if you do that, then we can notice that from here, this term here will become dou by dou t of rho e plus half rho v dot v equal to…

Now, if we consider these two terms and then divide, we get divided by delta x; and when we take the limit, that becomes dou by dou x; and this is, this becomes minus dou by dou x, because this is coming at x plus t x with a minus sign. So, we can write that as, minus dou by dou x of rho u times e plus half, and these two will give us similarly, minus dou by dou y times rho v times e plus half, there is no. And a these two will gives us rho w minus dou by dou z of rho w tends e plus half v dot v I think; we have put an extra, it is I think we have put an extra rho in this things that should not be there, that is relatively straight forward for us to correct.

So, these are the flux terms, these are the energy fluxes being convicted ah with the flow. And now, we can consider these things; these two divided by delta x delta y delta z, and with the limit as delta x tends to give 0 will give as, minus dou by dou x of q x minus dou by dou y of q y minus dou by dou z of q z, where q x q y q z are fluxes in the x y z directions of heat, that is of the heat flux vector associated with the heat conduction.

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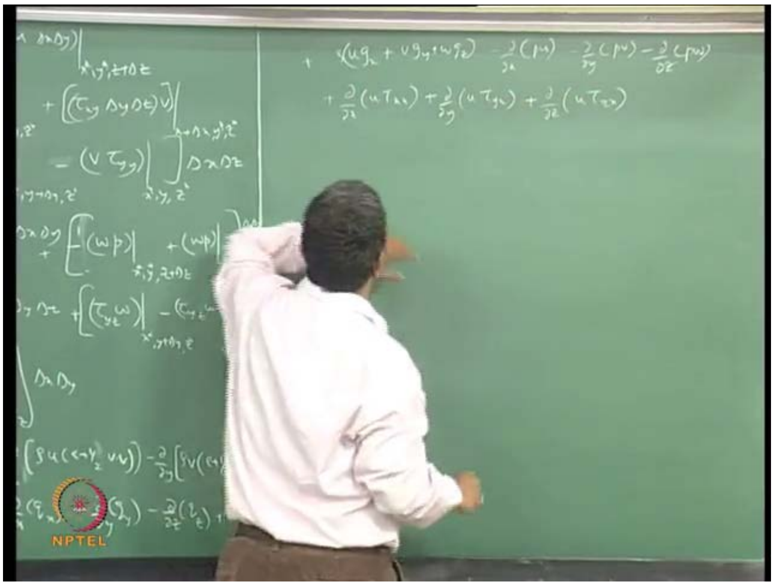

And we have this rho times s t, that is the external source term. And when we consider these things, this rho u gx plus v gy plus w gz as the rate of work done by the external gravity vector, and the pressure here, this is p u at x minus p u at x plus delta x. So, that becomes minus dou by dou x of p u. And similarly, here it is v p at y minus v p at y plus delta y dividing by delta x delta y delta z, and taking a limit will give us minus dou by dou y of p v and minus dou by dou z of p w.

Now, let us consider the stresses; here, it is u tau xx x plus delta x minus u tau xx at x divided by delta x, that is what we will get; so, that becomes plus dou by dou x of u tau xx. And similarly, these two terms here, this is y plus delta y; there should be a plus sign here, that is acting on the positive x direction. So, this is u tau x yx at y plus by delta y and u tau yx at y with minus sign. So, that will give us, plus dou by dou y with u tau y x; and similarly, the z faces will give us plus dou by dou z of u tau z x.

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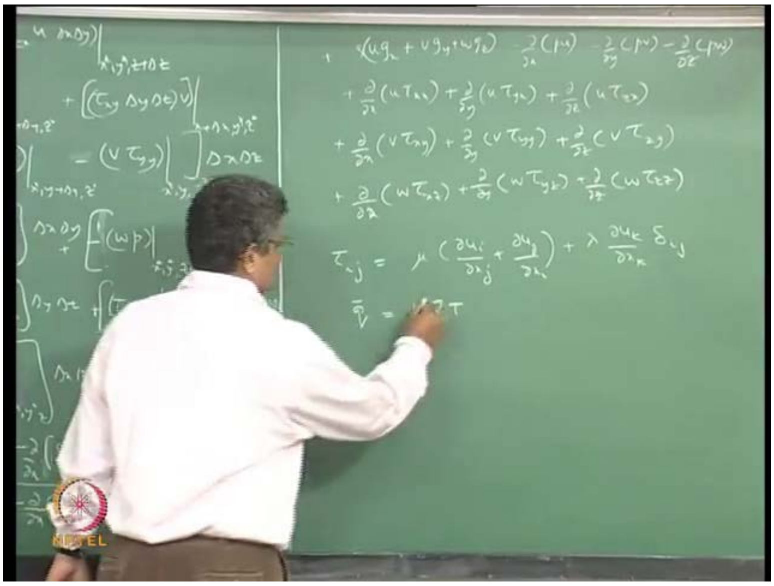

So, contributions coming from the stresses acting in the x direction. And we can similarly write, dou by dou x of v tau x y, that is stress acting on the y face, on the x on the y direction on the x faces plus dou by dou y of v tau yy plus dou by dou z of v tau zy, plus stresses acting in the z direction dou by dou x of w tau xz plus dou by dou y of w tau yz plus dou by dou z of w tau zz.

So, this is the overall equation - the energy equation. And as usual we can bring this to the left hand side and then we can write this. So, as of now, we have not, this statement is not complete, because we have not specified what this heat fluxes are.

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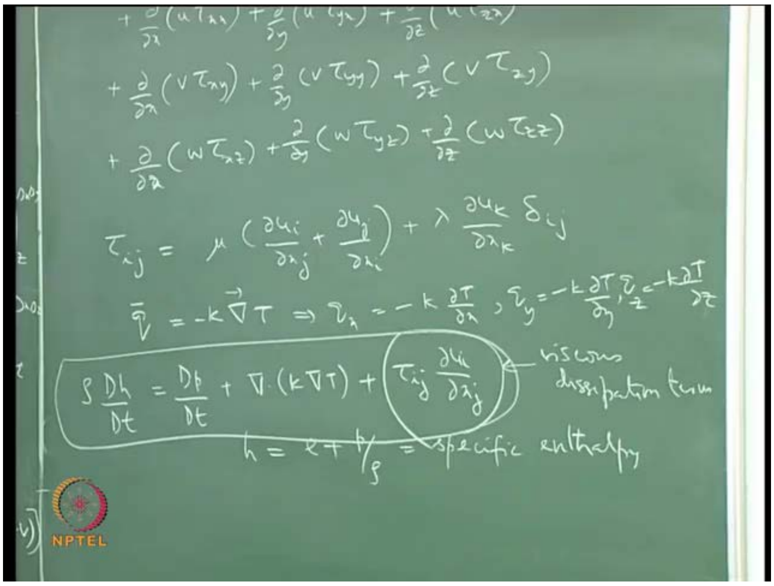

We know what stresses here are; we can say that tau ij for an Newtonian fluid is mu times dou u i by dou xj plus dou u j dou xi plus lambda times dou u k dou x k delta ij. So, what remains for us is to specify q; and for this, we can take the Fourier law of heat conduction. So, this q is equal to k gradient of temperature; therefore, q x with a minus sign, where k is thermal conductivity. So, q x is equal to minus k dou t by dou x, and q y is minus k dou t by dou y and q z is minus k dou t by dou z.

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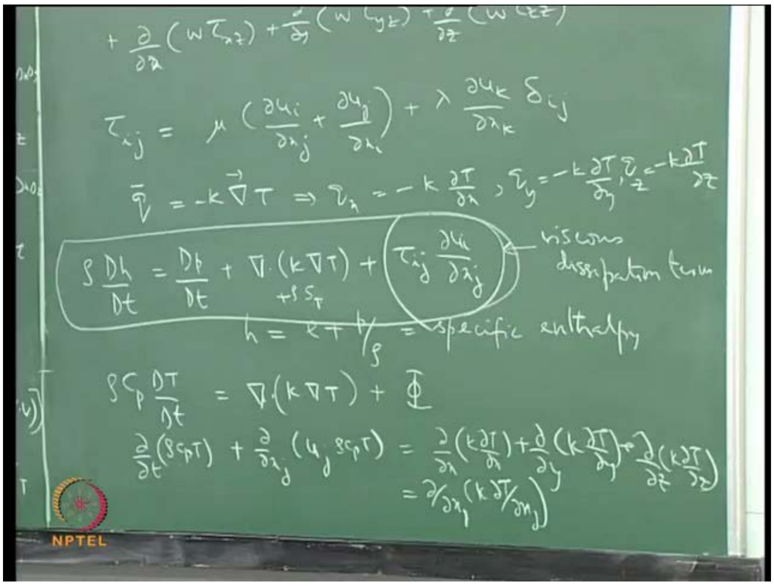

So, we can substitute all these expressions into this and we can come up with overall energy equation can which we can write like this, where h is the enthalpy - this is a specific enthalpy - and tau yj given by this expression, and k is the thermal conductivity and capital D by D t is a substantial derivative. And in this compact form, we can derive this, and the same thing where this particular term is called the viscous dissipation term.

So, all the frictional forces that are arising, all the viscous stresses are actually contributing to this desperation here, through this; we can show that this is a positive quantity, and therefore, that the whole thing is with a negative sign; it comes with a negative sign. This is a source term; so, the viscous desperation is actually a positive quantity. So, that is, if the flow is taking place, then the viscous stress will actually increase the temperature.

In a most heat transfer cases, unless we are taking about highly viscous fluids, this particular term will not be significant and you can come up with much simpler expression. And for cases with cost and density, we can write this as a simpler form, where this is a rho c p times, t will give us the enthalpy essentially, k is the thermal conductivity, and for constant, this is viscous desperation term. If we neglect this, we can write down a simple form of this, dou by dot t of rho c p t plus dou by dou x j of u j rho c p t equal to dou by dou x of k dou t by dou x.

We can make use of the induction notation; let me finish this dou by dou xj of k dou t by dou xj. And of course, we have neglected, we have left out source term; s t is here - rho s t. When there is no s t, then it comes like this, and which we readily identify as a standard scalar transport equation, where the scalar is c p times t.

So, then we have it falls into a scalar transport equation and this k is the effective diffusive as k. So, this an extra equation which is coming into the overall set of equations that we have, which of the standard from that we are familiar with, which brings in only one new variable, which is the temperature or the enthalpy. And of course, we have the properties of the fluid like, specific heat is coming, the thermal conductivity is coming into this.

So, when we want deal with energy equation, when we are want to deal with non isothermal flows, then we have to solve an equation of this form, which is a standard scalar transport equation form, which introduces one extra variable and which also brings in an extra equation; therefore, we are not disturbing the overall balance of equations verses variables. So, in this where, we can take care of ah all non-isothermal forces.