Computational Fluid Dynamics

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Module No. # 05

Solution of linear algebraic equations

Lecture No. # 27

Application to the Laplace equation

So far, some textbook cases, one can see what it is and we will take one such textbook case.

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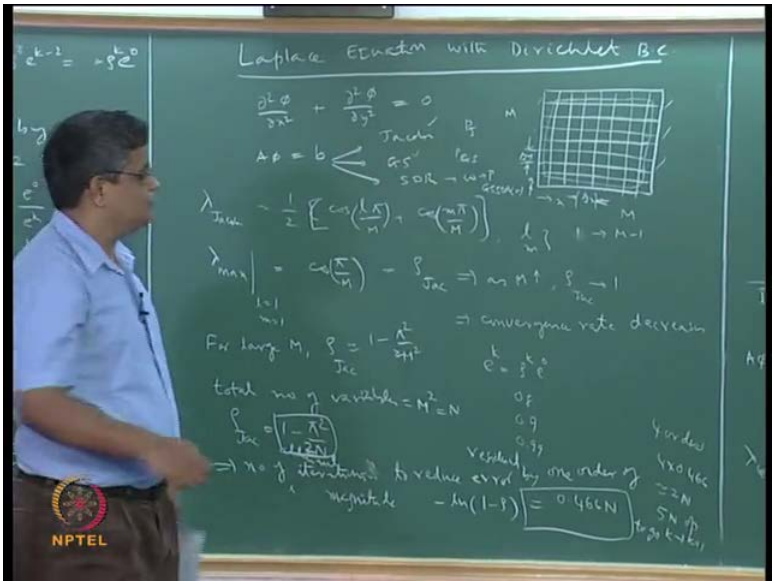

We will take the case of a Laplace equation with Dirichlet boundary conditions. So essentially, we are looking at and the two-dimensional case dou square phi by dou x square plus dou square phi by dou y square equal to 0; and we have a domain, the boundary conditions are given, and we can divide this into grids like this, and as many as we wish the finer the grid, the more accurate will be the solution; and for this particular case we are assuming that we have constant spacing delta x here, and a constant spacing delta y, because these are boundary conditions; that means, all these boundary points are known, and it is only the interior points here are unknown.

So, we can convert this using your central differencing into some A phi equal to b; and we can decompose this A in to d minus e minus f, and from that we can attempt to solve this using either Jacobi method or Gauss Seidel method or the SOR method by specifying the value of the over relaxation factor.

So, from for this particular case, we can construct the iteration matrix; and we can for each case, we can have a different iteration matrix: P Jacobi, P Gauss Seidel and P Gauss Seidel SOR, in which the iteration matrix depends on the value of omega. So, each of them have their eigen values.

So, for this particular case, for the Laplace equation with constant spacing and Dirichlet boundary conditions, we can show that the eigen values of for the Jacobi method are given by 1 half of cosine l pi by M plus cosine m pi by M, where l and m are the indices in the x and y directions. So, they take values from 0 to M minus 1, where M is the number of divisions in this direction, and this direction. So, we are looking at a square thing, and we are looking at equal number of divisions on both the sides.

So, these are, they are going from from 1 to M minus 1 number of iterations eigen values; so you have so many eigen values; and this we can… If you what is of interest is, not all the set of eigen values, but only the set the maximum eigen value, and the maximum eigen value occurs, when l is equal to 1 and m is equal to 1, because in that case, you can see that maximum value of l or m is M minus 1. So, this is only going to go between 0 and pi. So, within that, the maximum value will occur, when this is equal close to 0. So, the lowest value of l and m will give you the maximum value of this. So, when l is equal to m equal to 1, then we get the maximum value and that is equal to cosine pi by M. So, this is 1 plus 1 2. So, this cancels out with this.

So, this is the spectral radius of the iteration matrix obtained by applying the Jacobi method to the Laplace equation with uniform spacing, and here M is the number of divisions in the x direction and y direction, and what we see is that as we can see from here, as m increases, then spectral radius for the Jacobi method approaches 1; and as spectral radius approaches 1, then the convergence rate decreases; why? Because we have said that epsilon k is equal to rho k times epsilon 0 for long values of k, roughly speaking. So, if rho is of the order of 0.8, then per iteration, the error is reducing by a factor of 20 percent.

If rho is 0.9, it is only reducing by 10 percent; if it is 0.99, per iteration it is reducing only by 1 percent. So, as the number of divisions increases, then cos pi is approaching closer and closer to 1, and as this approaches closer and closer to 1, the error reduction factor per iteration, it is going to decrease.

So, this is something that we we have to pay the penalty, when we go to the larger and larger grids of not only the number of multiplications that are required… The number of multiplications in this remains the same, but the rate of convergence will be will be decreasing in this.

So, that is one one of the factors that we have to consider, and we also have to do for more and more number of these rhos here. So, with with this, we can make an estimate; for large M, we can write rho Jacobi as roughly equal to 1 minus pi square by M square; and if we divide this into M number here, and M number, the total number of variables is

M square. So, that is equal to N.

So, that means that rho Jacobi is is 1 minus pi square by N. So for this, the number of iterations that is the number of additional iterations let us put this as additional iterations to reduce error by one order of magnitude.

We should point out here that although we are talking about the error here, essentially we are talking about the residual is probably the better way of describing this, because error is not known, but residual is that can be something that can be computed.

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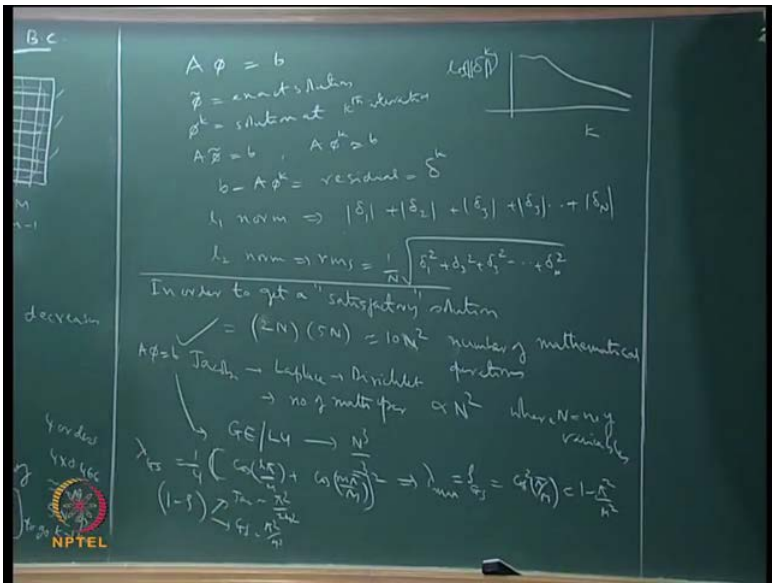

And if you have A phi equal to b, then and if phi tilda is the exact solution, and phi k is the solution at k th iteration, then we can say that A phi tilda is equal to b, because that is the exact solution and A phi k is only approximately equal to b, and we can say that b minus A phi k is the residual. So, which we can call as delta; and this is by how much this equation is not satisfied; and this residual obviously, b is a constant, A is a constant, but phi k is not a constant. So, this residual depends on the iteration on the current value of phi k.

So, it is this residual that can be computed because at the end of each iteration, we have an an estimate of the solution. So, we can substitute in this, and we can evaluate the residual. So, that residual when plotted against k is something that can be plotted at the end of each iteration, and that residual would show a variation, which for a convergent scheme would be decreasing like, at a constant rate when you plot it on a a log scale. So, this residual here is not a simple quantity, it is a vector quantity, because for each equation, there are n numbers of equations. So, there are n values in this.

So, this is a column vector n, we can take, we need to plot the norm of this vector, and the norm of that vector can be either the l 1 norm, which means that it is the sum of the absolute values or it can be the l 2 norm, which is the r m s - root mean square value. So, that is 1by N of delta 1 square plus delta 2 square plus delta 3 squares like this.

So, any of either l 1 norm or l 2 norm of the given equation A phi equal to b with an estimated solution can be computed. So this norm, the variation of this norm was with k is an indication of the rate of convergence. So, this is, it is this norm that we trying to… So, the lower this value it is, the lesser is the degree of dissatisfaction of the current solution to the exact equation.

So, we would like to reduce this residual to a very low value, and that residual norm variation here, follows the same way as the eigen values. So, the spectral radius and the number of iterations that are required to reduce the error or the residual here is given essentially by norm of is given by minus log 1 minus rho, and because our rho is given by this, then it is given for the case of the Jacobi method as 0.466N, where N is equal to M square.

So, for the Jacobi method, for the case of the Laplace equation, the with equal division in both x and y, and the number of divisions being M with dirichlet boundary conditions, the number of iterations to reduce the residual by one order of magnitude is 0.466N, where N is now the total number of variables.

So, if we want to reduce by the error by 4 orders of magnitude that is we are looking at the fourth decimal accuracy that is a kind of situation; then the number of iterations required will be 4 times 0.466. So, that is about 2N number of iterations are required; now each iteration, so that is to go from k to k plus 1 requires about 5N number of operations to go from k to k plus 1.

So, that means that in order to get a satisfactory solution, we are putting the satisfactory in quotation marks, by which we mean that the residual norm has been decreased by 4 orders of magnitude. So, then we need 5N, 2N number of iterations and each iteration costing 5N number of mathematical operations that is 10N square number of mathematical operations. So this is this is how we can arrive at the number of mathematical operations that are required for a given iterative scheme, this is not an exact estimate.

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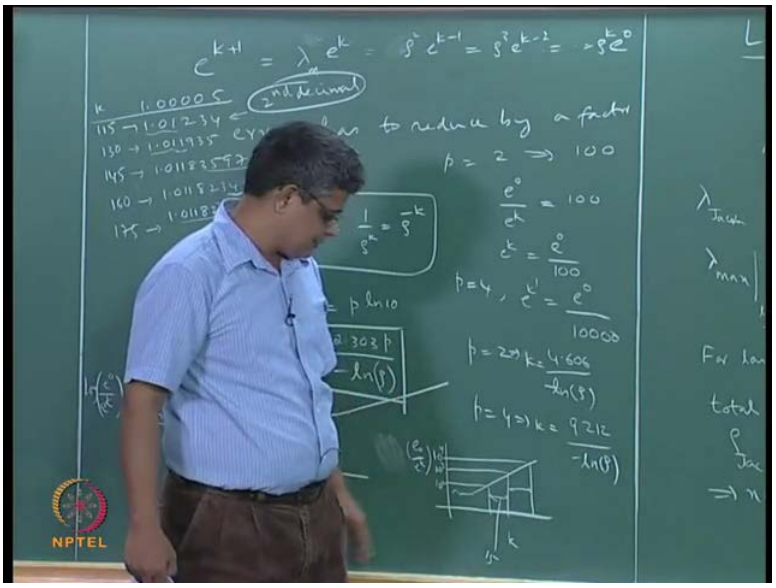

Firstly, because we are looking at the asymptotic rate of convergence that is when this error reduction factor gets into a logarithmic range, after some number of initial initial iterations; this sum number depends on the spectrum of eigen values and we have seen that the spectrum of eigen values for Laplace equations varies like this; this and it is only the maximum value that is going to determine the asymptotic convergence rate. So, it is only that the value that we are estimating here, is the number here, in this region not this number.

So, if we reach this this as asymptotic rate of convergence only after say 100 iterations like what we have put here, then we are not counting the initial 100 iterations that are required to get into the second decimal accuracy. So, this is only a rough order, but typically this is considered to be an estimate a reasonable estimate of reaching a satisfactory solution, a reasonable estimate of the number of mathematical operations that are required for getting a satisfactory solution and the point is that this value this factor 10 here, maybe 20, maybe 30, maybe something like that, but it is not dependent on the number of it is not directly dependent on the number of variables that we have.

One can argue as we are pointing out here that as m increases, the number of eigen values obviously increases, and there are more and more number of eigen values having larger and larger things. So that, it the falling in to the asymptotic convergence rate becomes slower and slower, and we need to have more and more number of operations

that are required, before we get in to this asymptotic convergence rate. So, we can make all those kind of arguments, and because of all those arguments, this is only a reasonable estimate. So, this is not an exact estimate.

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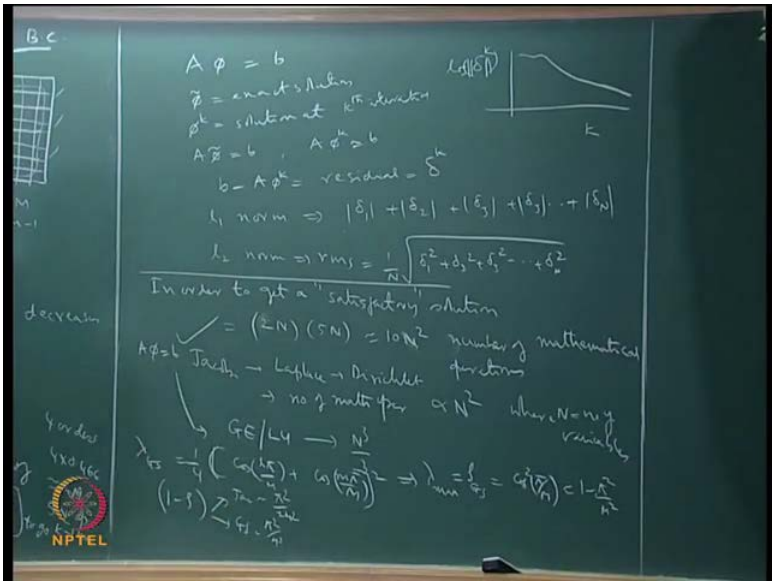

But one can say that for the Jacobi method for a Laplace equation with Dirichlet boundary conditions, the number of mathematical operations is proportional to N square, where N is the number of variables; and we want to compare this with the case of the same A phi equal to b, then using Gaussian equation or L u decomposition method, which requires N cube by 3 number of operations.

So, when we talk about large number of variables, in such a case, this factor of 10 or 20 or 50 does not make so much difference, and we can look at N cube and N square thing, and we can declare that the Jacobi method is superior to the Gaussian elimination method.

Now, we have looked at the Jacobi method and we can also consider the Gauss Seidel method, because even for the Gauss Seidel method for this particular thing, we have the expression for the full spectrum of eigen values, and in in this particular case, the eigen values for the Gauss Seidel method are given by 1 by 4 of whole square, and from this we can write lambda max that is equal to the spectral areas for the Gauss Seidel method is again when l is equal to 1 and m is equal to 1, and this is given by cosine square phi by M, and in the case where M is large, this is roughly equal to 1 minus pi square by N

square, I think here it is pi square by 2N, in the case of Jacobi methods pi square by 2N here this is 2M square here and in this case this is…

So, we can see that 1 minus rho for the Jacobi method is pi square by 2M square, for the Gauss Seidel method, it is phi square by M square; that means that the factor of reduction of the residual per thing, per iteration is less in the case of Jacobi method than in the case of Gauss Seidel method, and as a result of this, the number of iterations that are required to reduce the residual by a factor is exactly for this particular problem is half as much as what is required for the Jacobi method; for this particular case that is for the Laplace equation with the Dirichlet boundary conditions with constant division like this; in this particular case, the number of iterations required to reduce the residual by a factor of 10 is half as much as what is required for the Gauss Seidel method.

So, if it takes 2 N numbers of iterations for 4 orders of reduction residual for the case of Jacobi method; in the case of Gauss Seidel method it requires only N; and each iteration takes 5 N number of operations. So, instead of 10 N square we will have 5 N square. So, for the case of Jacobi method, for the case of Gauss Seidel method, the number of operations required is still proportional to N square with a slightly with a lesser constant.

So, as compared to the Gaussian elimination or L u decomposition method, for the same problem of Laplace equation, we have N square number of operations that are required for the basic iterative methods. And we can show through a similar kind of analysis for the case of SOR method with an omega here, then again the spectral radius will now be a function of the omega parameter; and for this particular case, we can try to find that value of omega for which the spectral radius is minimum, because we want the smallest spectral radius. So, that the residual reduction factor is more per iteration.

So under those conditions, we can show that for the Gauss Seidel SOR method, the number of operations required to reduce the error is not N, but it is square root of N; for the case of gauss Seidel SOR method in order to get the satisfactory condition, we need only square root of N number of iterations, each iteration again costing about 5N. So, the number of total number of mathematical operations that is required is N to the power of 1.5.So, this is for the case of optimal SOR method.

So for the case of optimal SOR method where the parameter omega has been optimized, so that the spectral radius is as low as possible for the case of Laplace equation; in such a

case the number of iterations that is that are required for one order of reduction of the residual varies as square root of N that is whereas, for Gauss Seidel method and Jacobi method it varies as N.

So, in such a case, the total number of mathematical methods required for the convergence, varies as N raised to the power 1.5 instead of 2, and that as compared to this for a direct method, this makes the SOR method, optimal SOR method as a very attractive scheme, but this result here mathematical result is true only for symmetric positive definite matrix matrices, which is what we get if we take a Laplace equation or a puissant equation, and then use some central differencing kind of methods to do the dirichlet boundary conditions.

For the general case, this is not necessarily true and when you have nonsymmetrical matrices, then this result may not be true; but even then experience shows that the iterative methods will converge, the total number of mathematical operations required will be probably significantly less than what is required for the direct methods, and as a general principle, we make use of these basic methods for the solution of large system of equations like A phi equal to b.

In the next lecture, we will look at some methods, which take advantage of the iterative methods in terms of getting to a faster convergence, but we also put in wherever possible the advantage of a direct method in order to derive combined methods, which will give an overall reduction in terms of the total number of mathematical operations. So, those combined methods are what we will discuss in the next lecture.