Computational Fluid Dynamics

Prof. SreenivasJayanti

Department of Chemical Engineering

Indian Institute of Technology, Madras

Module No.# 05

Solution of Linear Algebraic Equations

Lecture No. #23

Direct Methods for Linear Algebraic Equations

Gaussian Elimination Method

Let's look at some direct methods for the solution of A phi equal to b.

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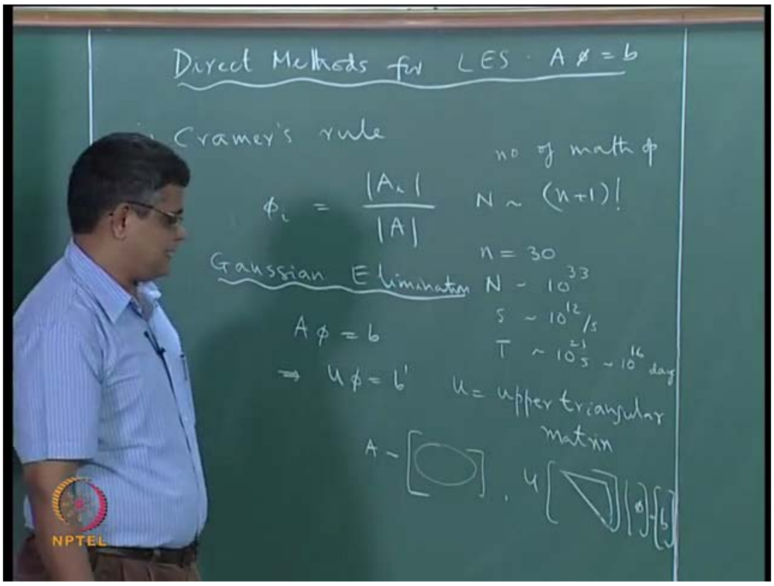

So, we will call this as direct methods for linear equations. So, these are linear equation solvers LES, A phi equal to b and as we have said earlier, all the coefficients A are known and all the coefficients b are known therefore, this constitutes a linear equation a set of linear equations. And we also mentioned that a direct method is a is a method in which after a finite number ofsteps, you get the overall solution.

We mentioned several methods, the first method that we need to look at, we need to mention for the sake of completion is the Cramers rule in which phi i is given bya i by determinant of A i by A, where A i is the coefficient matrix A here in which the ith column, where i represents thevariable that we are looking for, the ith column is replaced by the column vector b. This way you have to replace the ithcolumn by column vector b and find the determinant of the resultant matrix and divide that by the determinant of the coefficient matrix. So, this method isis taught in allhigh school syllabuscourses and the advantage of this method, that this will work for any nonsingular matrixA is outweighed by the difficulty of this for large matrices.

The number of mathematical manipulations mathematical operations for this method is N plus1 factorial, where N is the number of variables that we have. So, if N is 30, the number of mathematical operations N is of the order of 10 to the power of 33. So, if you have a speed of mathematical operations of 10 to the power of 12 per second, this would takethe time taken for the solution of 30 equations using this method would be 10 to the power 21 seconds.

So, that is a very huge number that is 10 to the power, let us say 15,16 days. So, this why this method we mention it for the sake ofcompletion and especially in cases, where n is very small, may be this method would can still be used; if you have n equal to 4,may be this method ismuch easier and n equal to 3, we can use these kind of methods.

The the most general method that onecan recommend is a Gaussian elimination method;this is the probably the mostuseful and efficient methods method for the solution of A phi equal to b, in case where a does nothave any special features like Bandedness orsymmetry or sparseness. So, in such cases when A is almost fully populated without any specificstructure, diagonal structure orother structure symmetry in such a case this can be taken to be the most efficient method and how do we do this? So, if you have A phi equal to b, the objective is to convert this into a matrix like u phi equal tod prime, where u is an upper triangular matrix.

And upper triangular matrix is of,if Ais like this, then with all thesecoefficients here, u will have coefficients only in this. So, if you now put this u times phi equal to b, you have n equations here and withas you go down down in this way, you will have fewer and fewer variables right at the bottom, you have a single variable here. So, that equation can be solved directly to give you phin.Now, you come to the upperlevel, there you have n minus1 and n as thetwovariables and n is already known. So, you can solve this directly forphi n minus1 and then, if you come to the next row here, next higher row you have which will be n minus 2 n minus1 n and by now you have computed n minus1 n, you can directly getn minus 2.

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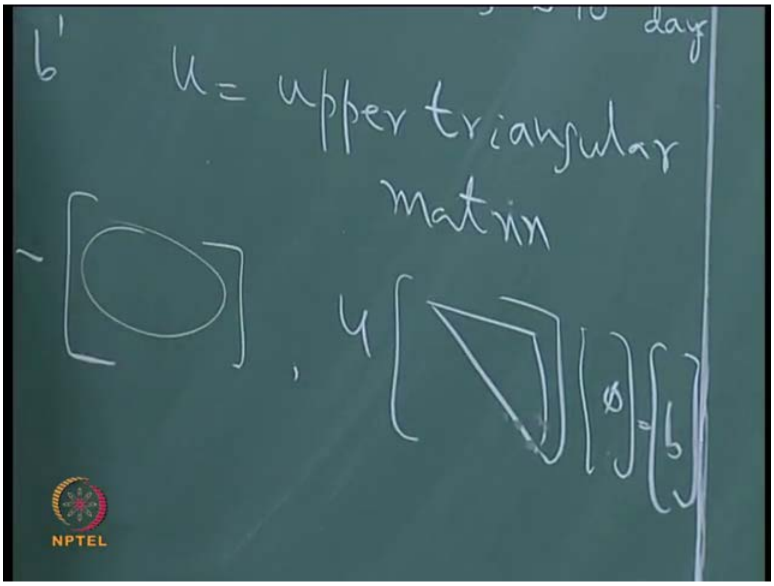

So, the solution of u phi equal to b, where u is an upper triangular matrix is something that can be done easily by back substitution. So, the idea of the Gaussian elimination is to successively eliminate these variables; you have phi1 phi 2 phi 3 phi 4 like that. So, you eliminate phi1 from all these equations, phi 2 from all the lower equations, phi 3 from all the lower equations and soon, until you get a single variable left in thelast equation and thereby in an in the first stage of elimination, you convert A froma matrixwhich has coefficients all over the place intoone which has coefficients only nonzero coefficients only in the upper triangular part of the original matrix A and the solution of this can be done efficiently.

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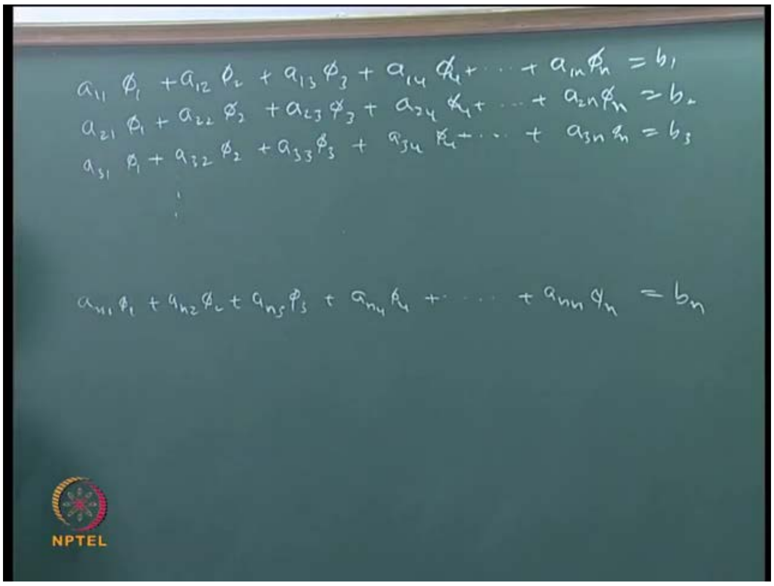

Now, how do we do thiselimination? So, let us take a specific case a1 1 phi 1 a1 2 phi 2 plus a1 3 phi 3 plus a1 4 phi 4 plus a1 n phi n equal to b1.The next equation will be a 21phi1 plus a 2 2 phi 2 plus a 2 3 phi 3 plus a 2 4 phi 4 plus a 2 n phi n equal to b 2 and we can write onemore, a 31phi1 plus a 3 2 phi 2 plus a 3 3 phi 3 plus a 3 4 phi 4 plus a 3 n phi n equal to b 3 and we have n such equations and the last onewill be a n1 phi1 plus a n 2 phi 2 plus a n 3 phi 3 plus a n 4 phi 4 plus a n n phi n equal to b n.

All these coefficients b are in this side and all the coefficients a1 1 and1 2 and all these things, they form the matrix the coefficient matrix a and phi that is phi1 phi 2 phi 3 phi 4 phi 5, all those things all the way up to phi n are the variables.

So, in this, the idea of the elimination process is to make use of this equation which is known as the pivot equation and eliminate phi1 from all these equations and how do we do that? If you say that, if you multiply this equation by a 2by a1 1 and subtract from this equation, then you get this term will be a 2 1by a1 1 times a1 1. So, this a1 1 will cancel out and what you have is a 2 1phi1,when you subtract from this, phi1 getscancelled out and this term here will become…

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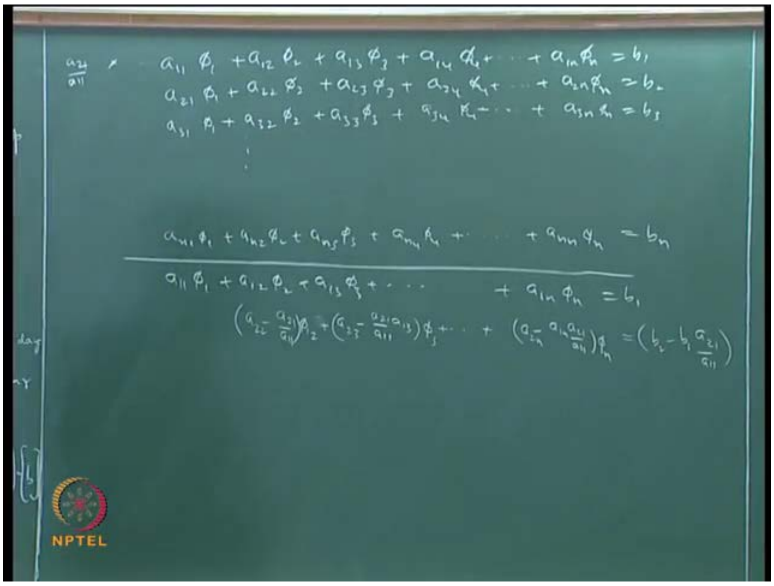

So, you have the first equation as it is,a1 1 phi1 a1 2 phi 2 plus a1 n phi n equal to b b1 and here this becomes 0, because we have cancelled this out and this equation here willbecome… your subtracting this from this; this multiplied by a 2 1 by a1 1. So, this will now become a 2 2 minus a 21 by a1 1 times a1 2 phi 2 and similarly,this oneherethis is what we have this, a 2 3 will become plus a 2 3 minus a 2 1 by a1 1 times a1 3 phi 3 plus soon plus a 2 n minus a1 n a 2by a1 1 times phi n equal to b 2 minus b1 times a 2 1by a1 1.

So, we have modified the second equation by multiplying the first equation - the pivot equation - by this value a 2 1by a1 1 and then, subtracting that resulting equation from this, so that the secondequation gets modified like this and it does not have phi 1.

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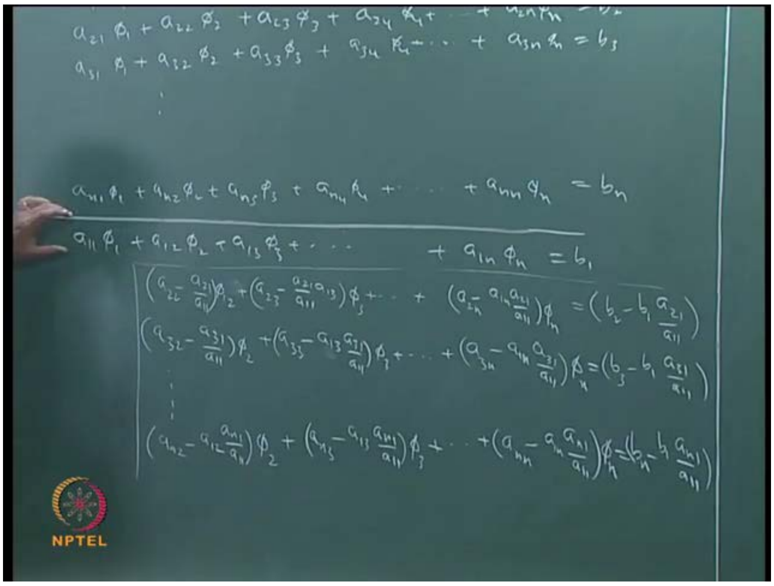

Similarly, we take the third equation and then, we multiply this by a 3 1 by a 11. So, a 3 1 by a11 times the pivot equation. So, this much is subtracted from this; sowhen we do that, a1 1this and this will cancel out; you get a 3 1phi1,when you subtract from this, that cancels out and this term here becomes a 3 2 minus a 3 1by a1 1 times phi 2 plus a 3 3 minus,where a1 3 bya 3 1by a1 1 times phi 3 plus a 3 n minus a1 n times a 3 1by a1 1 times phi n equal to b 3 minus b1 times a 3 1by a1 1.

So, we have eliminated from the third equation also phi1 by multiplying the pivot equation by this product, the product of the coefficient of the first equationthe first term divided by the coefficient of the first term of the pivot equation. So, by doing that, we are able to eliminate phi1 from the third equation also and then, we go on and then do it like this for all the n minus1 equationand then finally, we have… So, this is zero plus zero plus zero plus you will have a n 2 minus a1 1a1 2 times a n1 by a1 1 phi 2 plus a n 3 minus a1 3 a n1 by a1 1 phi 3 plus a n n minus a1 n a n1 by a1 1 phi n equal to b n minus b1 by a n1 by a1 1.

So, what we have now is is a set of equations in whicha1 1 appears only in the first equation and in all the rest of the equations we do not have a1 1 and this becomes a smaller set with n minus1 variables.

So, now we keep this like this; we take this this set of equations as something like a phi equal to b and then,this is written aswe eliminate there is something here yeah we eliminate, we retain this and then, we eliminate phi 2 from this, phi 2 from the next equation, phi 2 from the next equation and soon and then, we eliminate phi 2 from all thesucceeding equations all the n minus 2 succeeding equations; at the end of that,we will havea1 1 only in the first equation,phi1 only in the first equation, phi 2 only in the second equation and then, you will have a subset of n minus 3 equations in whichphi 3,phi 4 and all those things appear and then,you can eliminateyou can retain the third oneand eliminate phi 3 from all the rest of the equations and then, we can go on doing this until we eliminate all, but phi n in the last equation. So, this is if this is original one.

(Refer Slide Time: 16:40) Step1 will give us this.

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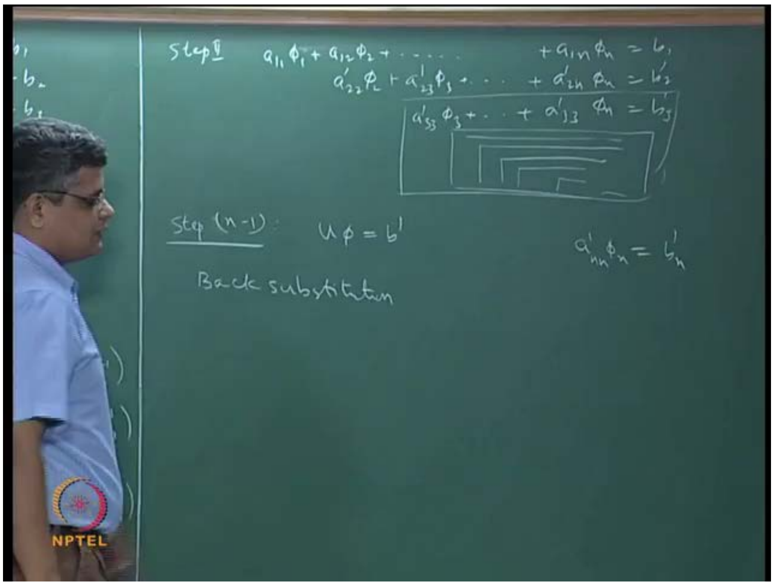

And step 2 will give us this, a1 1 phi1 plus a1 2 phi 2 plus a1 n phi n equal to b n and then,a 2 2 prime phi 2;I am puttingprime here, becausethat is modified; a 2 3 phi phi 3 plus a 2 n phi phi n equal to b 2 prime and so on.

And. So, now, if we eliminate phi 2 from all these things,we will have a 3 3 prime phi 3 plus a 3 3 soon. So, we will have a subset of these equations n minus 2 equations in which you have this. So, we we keep this and then from this subset, we eliminate phi 3. So, that we havethis like this and after this, we retain the top oneand from this subset we eliminate phi 4 and again we eliminate phi 5 like this and then, we go on like this until we are left withan upper triangular matrix which will have finally, like this and thenat the end of step n minus1,we will have all the first equations and then we will have a n n prime phi n equal to b n prime, where this prime of course has been multiplied several times. So, this is the final form and we have an upper triangular matrix and this is the form we have as u prime equal tou phi equal to b prime. So, and thiswe solve by back substitution.

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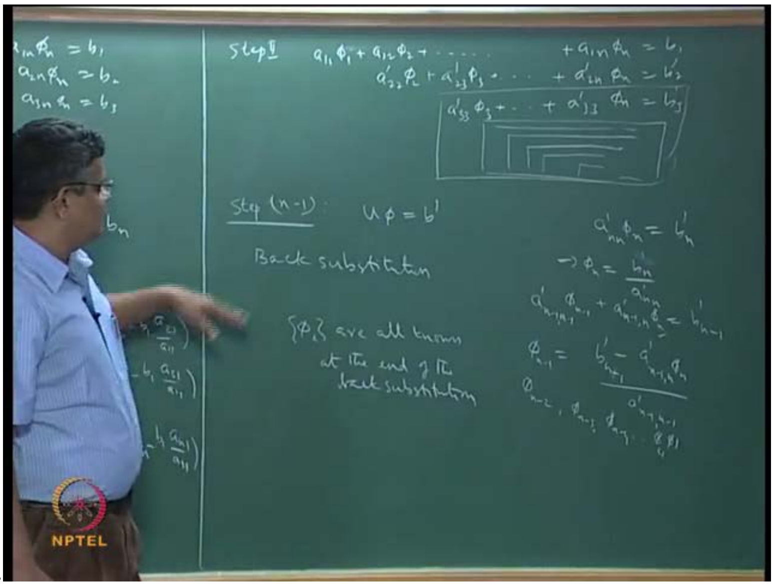

We will solve this for phi n equal to b n prime by a n n prime and then above this we have a n minus1 primen minus1 phi n minus1 plus an minus1 n prime phi n equal to b n minus1 prime. So, this is theresulting equation here and in this we already know this value from here and then we canevaluate this. So, this is phi n minus1 will be equal tob n minus1 prime minus a prime n minus1 n phi n divided by a prime n minus1 n minus 1. So, like this then we go to phi n minus 2 and then phi n minus 3 phi n minus 4 up to1 up to phi 2 and then phi 1.

So, we havein the Gaussian elimination, we havea forward elimination stage in which we operate on n minus1 equations to eliminate phi1 and then, we operate on n minus 2 equations to eliminate phi 2 and then n minus 3 equations to eliminatephi 3 like that and then we ultimately end up withan upper triangular matrixand that triangularupper triangular matrix is solved by back substitution from the bottom row up to the first row in a series of substitutions ofvalues or variables which are already determined. So, at the end of the back substitution, we get all the phi i valuesare all known at the end of the back substitution.

So, the Gaussian elimination is a good illustration of a direct method. You have a series of orchestrated steps by which you successively eliminate phi1,phi 2,phi 3,phi 4 in a time consuming tedious calculation procedure to convert a intoan upper triangular a phi equal to b into an u phi equal to b prime and this is then solved efficiently by a back substitution process and what is the number of mathematical operations? We have said that for theCramers rule it is n plus1 factorial, this looks like a very tedious thing you have to do somany multiplications and soon. We can make an estimate ofof this, for example, to do the first step we have this;we have to multiplywe have to determine this coefficient here with which we are multiplying each term here and then subtracting it. So, we need onemultiplication here and then, we want towe know that this is going to be 0,because we have done this. So, from all these n minus1 n minus1terms that are here plus the right hand side term, we have to multiply by this. So, that is n minus1 n minus1 multiplications to do this multiplication here up to this and then nth multiplication. So,that gives youa total of n minus1 of this plus1 plus 1. So, that is n plus1 multiplications are requiredto eliminate phi1 from1 equation and we have to do there are n minus1 equationsfrom that are from which we have to eliminate phi minus 1.

(Refer Slide Time: 24:23) So, the first step here takes n by nn timesn plus1 number ofoperationssorry n minus1 number of equationare there in which to eliminate phi1 and for each elimination, we requiren plus1 terms. So, this is for the first step and for the second step of course, we have a smaller set. So, we haveless number of forequations to deal with. So, the second step will require n minus 2 eliminationrow elimination each requiring n multiplications and then the third onewill require n minus 3 times n minus1 plus soon all the way up towhen we have only onething here and then we havetwomultiplications for the last one.

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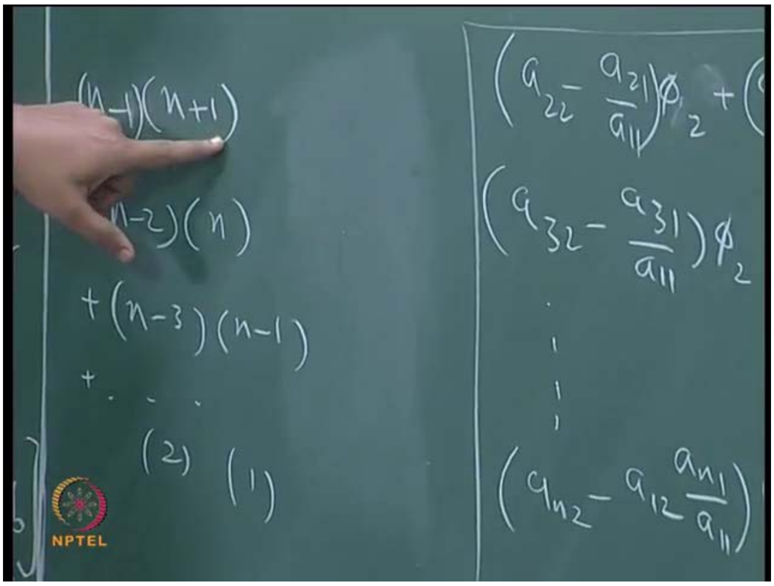

So, the total number ofthe total number of multiplications and of course, divisions that arerequired to do; step1 elimination is n minus1 times n plus1;step 2 is n minus 2 times n; step 3 is n minus 3 times n minus1 and soon like this.

So, the total number required is the sum of all these things. So, we can write this as sum of i times i plus 2 this where i goes from1 to n minus1.If you want to get exactly this, thenthis this thing here is probably wrong, because this is 1and 3. So, this must be 3 and1 in which case this will be correct, because the difference between thesetwois 2. So,this will be1 and 3 obviously, because in the last equation, we have this term and then thephi ncoefficient to be determined and the b coefficient to be determined. So, it is 3 times onehere. So, this formulaworks. So, the total number of mathematical multiplications and divisions required for the elimination is this.

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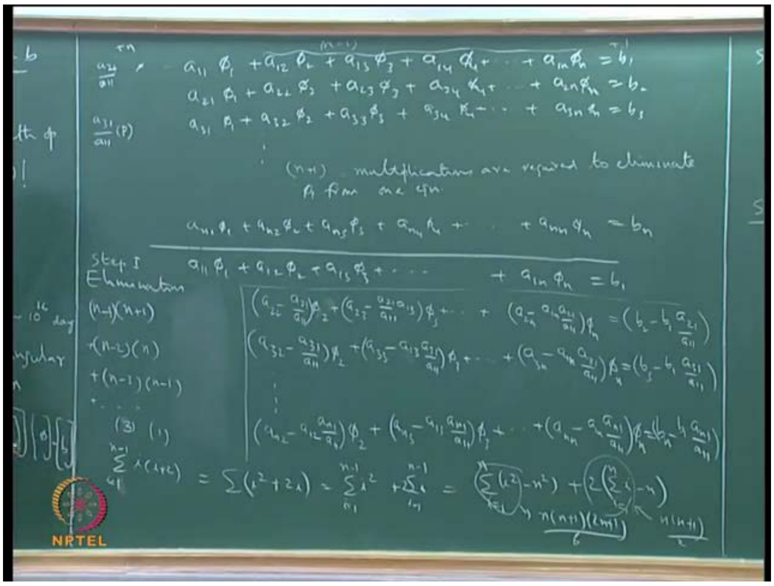

So, this is i square plus 2 i. So, that is sum of i square plus sum of 2 sum of in, where i is from1 to n minus1 and this also i equal to n to n minus1 and we have a formula forthis. So, we can write this as i square minus n square; now i going from1 to n.

So, the last term will be n square minus n square, sowe are cancelling out plus 2 times i equal to1 to n minus n. So, we have readymade formulas for this; when i goes from1 to n, then this is this particular thing is n times n plus1 times 2 n plus1 times 6 and thisthis particular thing isn times n plus1 by two. So, we can substitute these things and then finally, show thatthe number of operations required in the elimination process is something like n cube by 3 plus 2 n square or1 n square some something like this, some 3 n or something like that.

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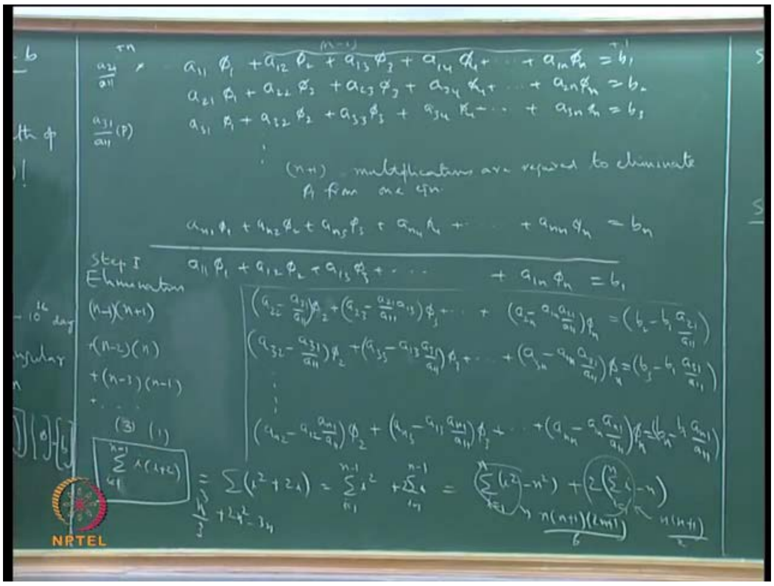

What is important is that, when n tends to large numbers like 1000, then this will be 10 to the power 9 and this will be 10 square. So, that is onlythis isthis is a billion and this will be a million. So, billion plus 2 millionor billion plus1 million isnothing much. So, this tends to n cube by 3 for large n.

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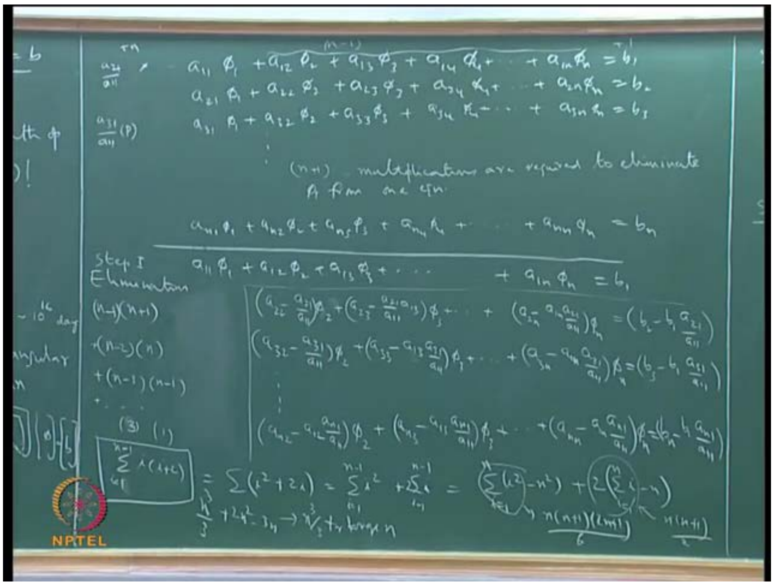

So, the idea is that in the Gaussian elimination process method, the elimination stage requires about n cube by 3 number ofsteps and whatever back substitution. So, back substitution onecan similarly make an operation count,for example, for the first for the bottom most row, we have takenone calculation, for the second bottom most row we require onecalculation here and onemore division here. So, that is 2 and for third onewe need 3.

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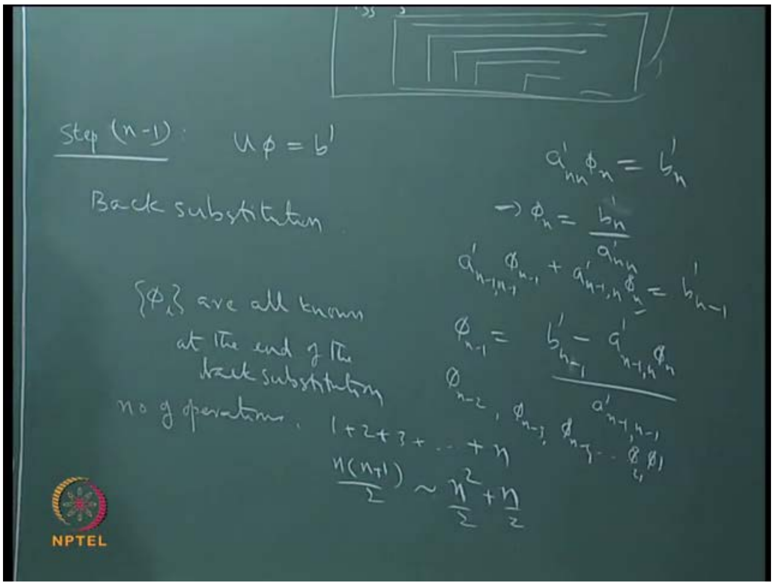

So, it becomes the number of operations is1 plus 2 plus 3 up to n. So, this isSo, this isn square by 2 plus n by 2 like this.and So, for large value of n, the back substitution takes only n square and the forward substitution takes n cubed. So, the point that we are trying to make is that for the entire solution for the Gaussian elimination solution, the number of multiplications or divisions required is of the order of n cube by 3 and what is also important to note in this is, that the solution phi i is obtained only after we do the elimination process and then the back substitution. And the back substation constitutes only a small fraction of theoverall computation required, because the elimination process requires n cube number of operations and back substitution requires only n square number of operations. So, a small fraction if n is 1000,then only 0.1percent of the time is requiredfor back substitution and 99. 9 percent effort is required for the forward elimination process and when is even larger, that fraction correspondingly decreases.

So, what this means is that which is characteristic of a direct method and especially when the number of operations goes as n cubed or n square and all that is, that the solution from a direct method is known only after we do a large number of computation without even getting a hint of what the solution is. Only after we do the 99.9 percent of the overall solution effort do we getany idea of what the solution is?There is no intermediate guessestimate of what the solution is and that is the disadvantage of a direct method when we want to incorporate this in an overall iterative procedure.

Typically, we have seen that inthe method that we had been using forthe simultaneous solution of the coupled equations of the Navier stokes equation, we make upthe a phi equal to b by making some estimates, forfor example, v and w and p when we want to solve the u momentum equation.So, that means, that this solution this a phi equal to b has coefficients which are determined, but which are not fully correct. So, even though we know that this coefficients are not fully correct, these are approximate we cannot get an approximate answer from the direct method, until we do 99.9 percent of the overall computation effort then it is too late. So, that is the disadvantage that a direct method typically has. Even though that is a case the direct method has the assurance that no matter what the structure of a is as long as there is a unique solution that is as long as the determinant of a is not zero, we can be confident that this method will give us a solution. So, we have the assurance of the widest possible applicability of the Gaussian elimination for the solution of a phi equal to b; you do not have to worry whether diagonal dominance is there or if it is not there. So, that kind of condition is not…Another feature of the Gaussian elimination method is that, we can see that in step1 when you want to eliminate phi1,all the coefficients of the lower things are getting multipliedbya1,a 21and then divided by a1 1 like this and then you are doing subtractions and then once you come to the thirdnext step, even this a 3 3, which is originally modified to eliminate phi1 is now again modified by division and multiplication by some other numbers to eliminate phi 2 and when you go to the next step, the phi 4 coefficient here which is already modified forphi 2 and phi1 is again modified by further multiplications. So, as you go further and further down, each coefficient is modified several times by multiplication, division like that. So, unless you are careful,you canthe round of error in representing thesenumberswill becomeintolerable when you have finite precision of arithmeticand when you have large number of equations.

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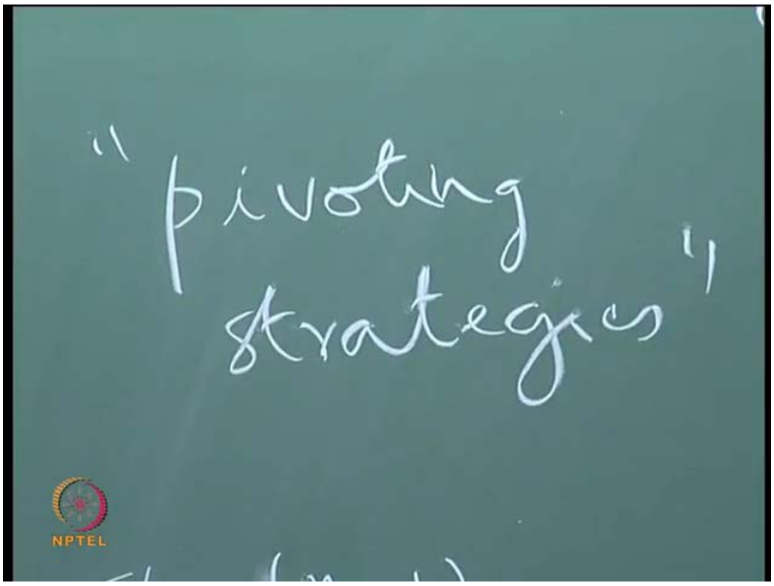

So, that is why in order to reduce this error, strategies like what is known as the pivoting various pivoting strategiesare incorporatedin trying to find out the best pivot equation in order todo the elimination process.

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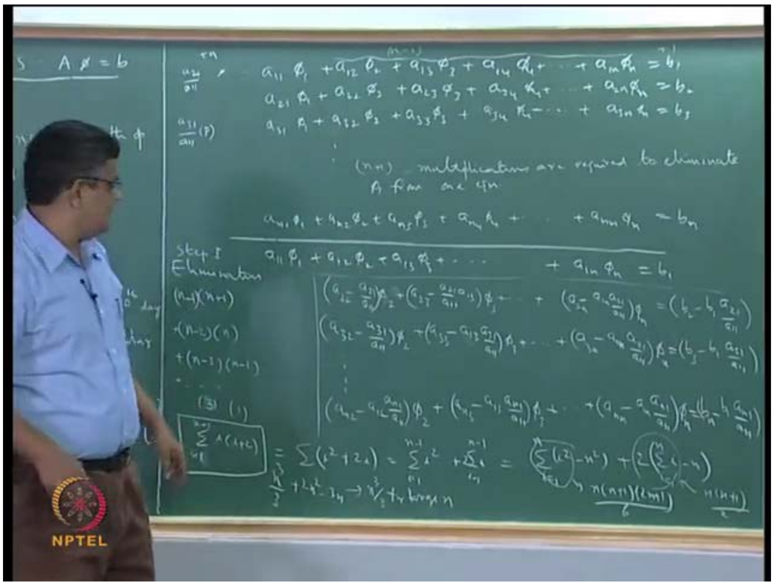

So, at each step,for example, you take that coefficient which has the highest value, because then if since you are dividing by this a1 1,all these things if a1 1 happens to be very small, then this coefficient here gets unnecessarily magnifiedand then, when you are trying tosubtract from this, then because this is magnified here,the this subtraction may lose some significant digits.

So, you would like to you'd like to tominimize the those kind of errors by choosing that coefficient among these things which has the highest value as the pivot. So, you havea row wise exchange of exchange of rows or exchange of columns is usually done to identify the right kind ofpivot to do the elimination process.

So, usually when you are looking at a phi equal to b type of matrix solution, where a is full that is more or less with many coefficients which are nonzero, in such a case you would have to have some sort of pivoting strategy that has to be…andyou have to do more much more than doing writing a program here, you have todo more logic type ofprogramming steps to introduce a pivoting strategy. So, that successive eliminationand modification of the coefficient does not lead torun off type of situations.

So, including all those pivoting strategies, Gaussian elimination is considered as the most efficient method for the solution of A phi equal to b, when A is full and does not have any structure. So, from that point of view this isa very good method.

When we come to c f d type of problems, where a is mostly sparse, we know thateven if we have thousand equations, that is thousand variables the value of onevariable is influenced only by only the 4 neighboring points or six neighboring points or 2 neighboring points. So, that means, that each row will have onlya few nonzero coefficients. So, in such a case the buildup ofarithmetic error because of successive multiplications and subtractions and all that is going to be less when you forfor a matrix obtained from a c f d solution.

Obtained for a c f d solution, because the discretization will make sure that only a few of the entire row have nonzero coefficient. So, in such a case the round of error and pivoting strategy is not. So, important for a c f d type of a phi equal to b, but for a general case it is very muchtrue.

So, the point that we want to take from this is that, this is a general method which can be used for any nonsingular A phi equal to b type of equation and it requires n cube number ofmathematical operations, which is a great improvement on n plus1 factorial,for example, when you say n equal to 30, then this only 30 cubed. So, that is aboutthirty thousandoperations, where as this is giving to 10 to the power of 3 3 number of operations. So, this isvery good for typicallyand less than thousand, we can safely use this this kind of thingandthere are certain applications in which they would prefer to use these type ofelimination type of methods, becausefor a very course grid, may be you can you can use this type ofsituation.