Computational Fluid Dynamics
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Solution of Navier - stokes equation

Lecture No. # 19

Topic

Artificial compressibility method and the stream function - vorticity method for the solution of NS equations and their limitation

Let us, now take a look at artificial compressibility method, this is one of the early methods for dealing with evaluation of the pressure specifically for incompressible flow. We will try to list out the method, so that we have a clear grip of how the method is done. We will take incompressible flow, unsteady flow, and then we will write down, how we solve the corresponding equations, that is a set of the continuity, and the three momentum equations for the four variables u, v, w and p.

As we have discussed earlier, we will still retain the x momentum equation for u, y momentum equation for v, w momentum z momentum equation for w, and we will introduce an artificial compressibility and artificial relation between density and pressure in order to bring out pressure from the continuity equation.

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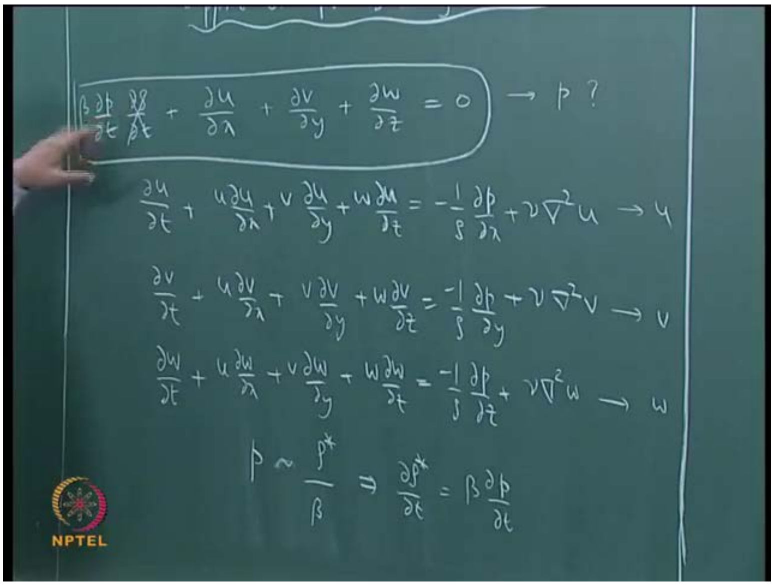

So, the method artificial compressibility approache for solving incompressible NS equation, Navier - Stokes equations. So, we are writing down the continuity equation in this particular case, using rectangular coordinates as, dou u by dou x plus dou v by dou y plus dou w by dou z equal to 0.

The x momentum equation is dou u by dou t plus u dou u by dou x plus v dou u dou y plus w dou u by dou z equal to minus 1 by rho dou p by dou x plus nu del square u; here rho is the density, which is constant, nu is the kinematic viscosity and del square is obviously the Laplace, which will have three terms del square u by del del x square, dou square u by dou x square plus dou square u by dou y square plus dou square u by dou z square.

And we similarly, have the y momentum equation, it is always nice to able to write down this things, w dou u by dou z equal to minus 1 by rho dou p by dou y plus nu del square v; and finally, the z momentum equation is dou w by dou t plus u dou w by dou x plus v dou w by dou y plus w dou w by dou z equal to minus 1 by rho dou p by dou z plus nu del square w.

And we have said that this equation will be evaluated for u, and this equation for v, and this equation for w; and therefore, the remaining variable that is the pressure must be evaluated from this, and the question comes how, because pressure is not appearing in this.

So, this is where we introduced an artificial relation between density and pressure; artificial, because we have assumed incompressibility; that means that there is no relation between pressure and density, but we introduced the p varies as rho by beta, where beta is a factor. So that it is linear propositional to density and from this, we can write example, duo rho by d t partial of rho, with respect to t as beta dou p by dou t.

So, we must and this is the term; that we introduce here in a compressible flow, we have d rho by d t term appearing here, plus rho term appearing in each of these equation; and we solve this for rho for a compressible flow, and since in an incompressible flow, this term will not appear, but we now, introduced this term here and we replace this with beta dou p by dou t. So, we can write this as beta dou p by dou t; in place of this.

So, we rewrite only the continuity equation like this, and to distinguish between the rho the real density of the fluid and this fictitious density, we can put rho star now, so this density that is appearing in the artificial compressibility has no bearing on the momentum equation expect through by acting through the pressure, which is a same as what appears in the momentum equitation. So, we add one extra term in the continuity equation, which is now explicitly written in terms of p.

So, now we can write for example, of forward differencing approximation for p, and then we can solve this for p, now in this form, which in the which the continuity equation is written as beta dou p dou t plus dou u by dou x plus dou v by dou y plus dou w by dou z equal to 0; is a form with which we are familiar. And we can discretized this equation using explicit for example, explicit forward differencing for the time derivative, and we can use central differencing or upwind differencing as appropriate for these things.

And we can write an appropriate an equation, where p i j k n plus 1 is given in terms of p i j k n; and u i n u i j k n; I am putting the underscore here to say that is a vector. So, that; we have u, v, w components coming in this. So, if you choose an explicit method, then we can use this equation to get p i j k n plus 1; and that p i j n plus 1; can now be used to evaluate the the spatial derivatives of each kind that appear in the three momentum equations; and we can march forward.

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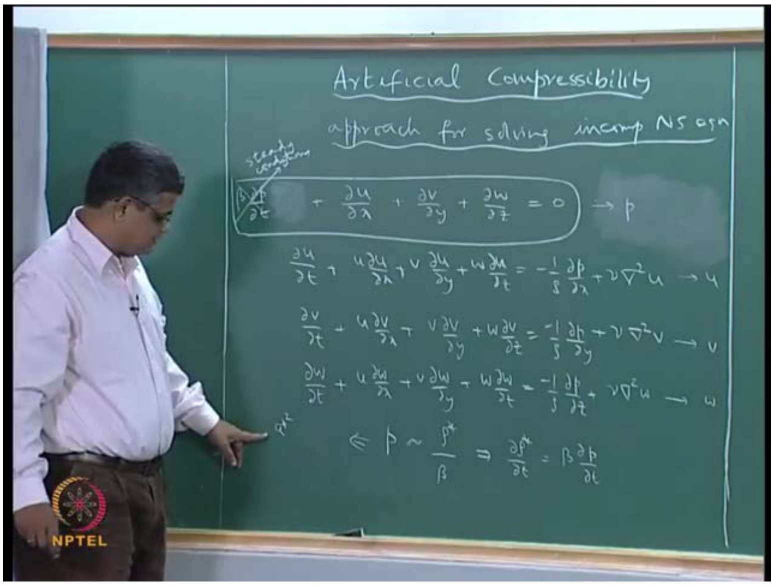

So, in this set of extended equation, let us just erase this, so that it is clearer. So, this extended set of equations is what we solve using a conventional method, which is applicable for compressible flows. For example, we have described the McCormack method, and we can use the McCormack method for the solution of these equations; and so, this will give us p and that p will go into into this, and then we will able to solve this, and then this, and then this, and then we can go to next times and so on.

So, from here onwards the method will be very similar to the McCormack method or any other method; that we have; that we can use for for the solution of these equations. So, by introducing this term, we have created linkage between the momentum equations and the continuity equation via the pressure, which now appears specifically, in the continuity equation, is it a justified method - is it possible for us to introduce any term, it is ok as long as we know that it is wrong, we know that these equations are not the correct equations, but there will be the correct equations, when this term disappears and when will it disappear, under steady condition. So, this goes to 0, under steady conditions.

So, under steady conditions, we are solving exactly the same equations; that we wish to solve; that is the incompressible Navier - stokes equation subject to the boundary conditions, which we would have included in this unsteady calculation. So, this the final steady state solution is not affected by the introduction of this extra term, but this extra term enables us to create that linkage between the continuity equation and the momentum equations. So, that we can solve them together using methods that we are familiar with for developed for compressible flow.

Now, how to fix this value of beta, beta if if we put it like this, then the speed of sound a star square, I put a is usually given as speed of sound, I put a star, because this is a fictitious speed of sound and it is given by d by d p; and that is nothing but 1 by beta. So, the speed of sound of this particular medium, the fictitious compressible medium is 1 by square root of B beta. And so, we have to choose beta, in such a way that the mach number of this fictitious compressible flow is not too high.

So, we can choose a mach number of the order of 0.5 or so; and once, we choose, we can use that as the criterion for choosing the value of beta. Now, the particular method that we use for solving this for example, if you use an explicit McCormack method, then that has a delta t restriction on how we can go through the transient solution, if you use some other method, it may have it is own delta t, the restriction the time step limitation based on the delta x and delta y; and other parameters that appear in this, which will restrict the value of… So, it is sufficient to specify, what the value of beta here is; and sufficient to specify the method for the solution of this; and the corresponding time step limitation on a chosen grid.

So, this method is completely specified by by beta of the order of 0.5 or something like that; so, that it is not highly compressible and it is definitely not supersonic mach number of greater than 1; and of course, because it is treated as an unsteady equation, we have to specify also the initial conditions. So, we have the velocity initial conditions, and the pressure initial condition coming either from the rho star or once we specify p, then we have the corresponding rho star coming from this equation.

So, this method can be used to get the steady state solution for three - dimensional flows, because under steady conditions, we are solving the exact equations, we are solving these equations; and these are exactly the same as the incompressible Navier - stokes equations, but the way that we get to the steady solution is through this artificial compressibility, which provides us the bridge between the momentum equations and the continuity equation via that pressure, which is link to the artificial density.

So, we must make sure that this density here is not the same as rho star, because then that would complicate matters, this is the true density of the incompressible medium, and this is the true kinematic viscosity the incompressible medium, so that is why, this is put explicitly in the form of pressure here, not in the form of density; and we also see that the density term does not appear here.

So, in this way, we can make use of the artificial compressibility method to solve the incompressible Navier - stokes equations for a steady three - dimensional flow and obviously, this can be also be used for two - dimensional flow, but this has a restriction that this will not be useful for time accurate methods.

So, if you want to find out, how the velocity and pressure change as a function of time from a given initial condition, then this method is not correct, because we are spoiling the transient evolution of the flow by introducing an incorrect term, which does not belong there. So, this is an extra term, which should not be there, it is used only first to get a solution to the coupled equations.

If you want to get a time accurate solution, then we have to use other methods, otherwise this method is quite popular, and it has been used it has been used successfully for a number of cases; now, what if you want to do at time accurate solution. So, in such a case, we can use another approach called the stream function - vorticity approach, which again looks at, how to evaluate pressure by bypassing it completely

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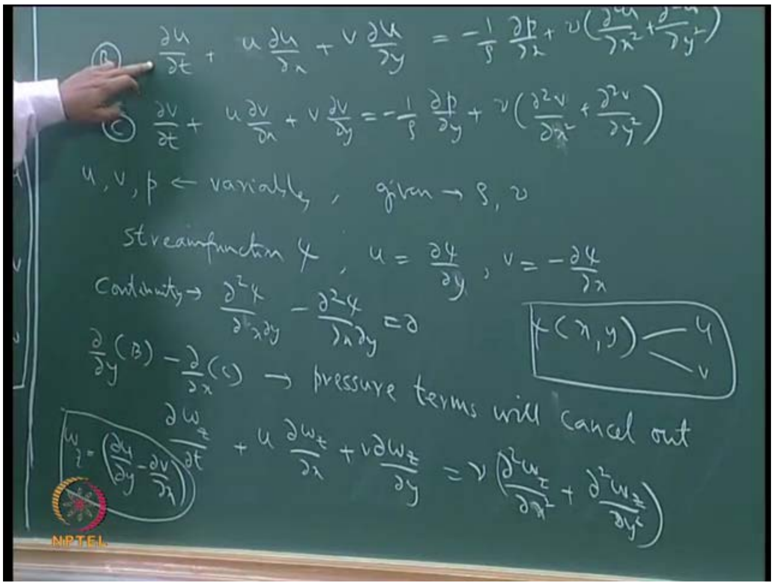

So, let us see, how it is done, so we can look at the stream function.

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Our basic knowledge of fluid mechanics, tells us that stream function is a is a function, which is defined only for two - dimensional flows. So, this method is obviously applicable only for two - dimensional flows. So, let us first of fall write down the two - dimensional incompressible Navier - stokes equations, dou u by dou x plus dou v by dou y equal to 0; and dou u by dou t.

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And we have the y momentum equation, which is dou u by dou t plus u dou v by dou x plus v dou v by dou y equal to minus 1 by rho dou p by dou v y plus nu del v by dou x square, we have three equations A, B and C; and we have three variables u, v and p. So, these are the variables, and given are rho and nu and of course, the corresponding initial conditions and boundary conditions. So, we are given constant values of rho and nu here, which correspond to the incompressible fluid; and we need to find u, v, p as a function of x and y; x, y and t subject to these equations; and the initial and boundary conditions.

Again, this is the variable, this is the equation to solve for u; and this is the equation to solve for v; and we have the same old problem of evaluating pressure from an equation, which does not contain pressure as a as a variable. So, what we do is that instead of trying to use this to get pressure, we try to eliminate pressure completely.

Stream function psi is defined, in such a way that u is equal to dou psi by dou y; and v is equal to minus dou psi by dou x. So, the stream function is a mathematical function, which is defined in such a way that u is equal to dou psi by dou y; and v is equal to minus dou psi by dou x, this is for a rectangular coordinate system.

If we have a radial coordinate system, cylindrical polar coordinate system or a spherical coordinate system for each type of two - dimensional flow, we can find a stream function with a variable definition, but we can find a stream function, which will satisfy the continuity equation, if you want to substitute this into this, then it is straight away easy to show that the continuity equation will reduce to…

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So, psi is a function which satisfies the continuity equation by definition; and it is possible to define the two velocity components, in terms of the psi first derivatives of the psi, in such a way that the continuity equation is satisfied. So, this definition is only for Cartesian coordinates within the (x, y) plane like this.

In a radial coordinate system with r and theta components as the non zero velocity components ,you have a definition for v, r and v, theta; and if you have r and z as the non zero velocity components, you have a different definition for the velocity components like that; and in spherical coordinate system, you can have another set of definitions for this, in such a way that the continuity equation is satisfied by the stream function; that is if you know the stream function, if we know psi as a function of x and y throughout the domain from this, we can get u and v by differentiation.

So, and this is what we would like to do, we would like to solve for psi here, and we can also eliminate pressure from these equations, because pressure is the irritating variable here, is the difficult variable, which is not enabling us must to solve the three equations simultaneously; and how can we eliminate pressure.

If we take the derivative with respect to y of all the terms in the equation B and subtract from it the derivative with respective x of all the terms in equation C, then this term here becomes dou square p by dou dou y dou x; and this square, this term here becomes dou square p by dou x dou y, and because we are looking at a smooth function, smooth variation of p, we can interchange the two orders of the two derivatives, and then these two will cancel out.

So, if you were to do dou by dou y of B minus dou by dou x of C, then pressure will drop out, will be cancelled, will cancel out. So, we can we can derive the whole thing, but we leave it as an as an exercise; and we can rewrite this combined equation in the form of a vorticity transport equation, which will be like

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And here, omega z is the vorticity, which is defined as…

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So, this is the definition for example, we can find out the origin of these, if we take dou by dou y of this B.

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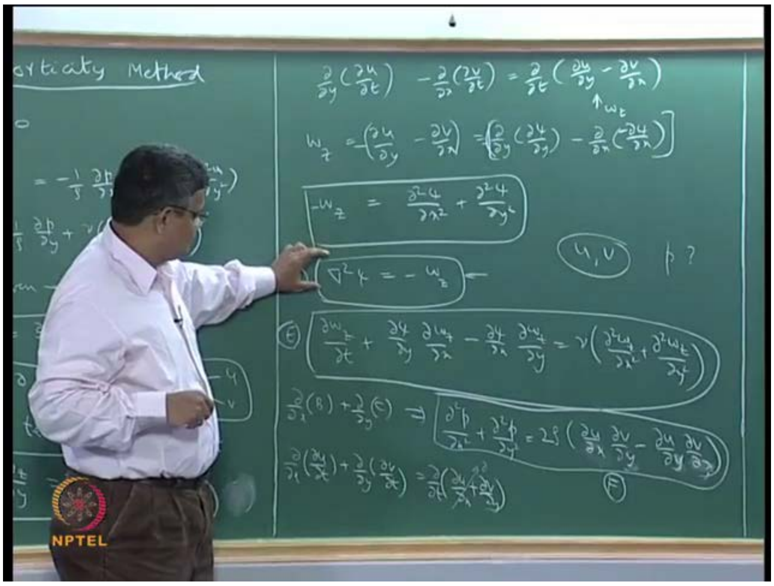

The first term will give as dou by dou y of dou u by dou t; and the first term of this will be minus dou by dou x of dou v by dou t. So, we are looking at only the first term of this operation; that is appearing that will give us the time dependent term; and we can write this as dou by dou t of dou u by dou y minus dou v by dou x; and this by definition is the vorticity in the z plane.

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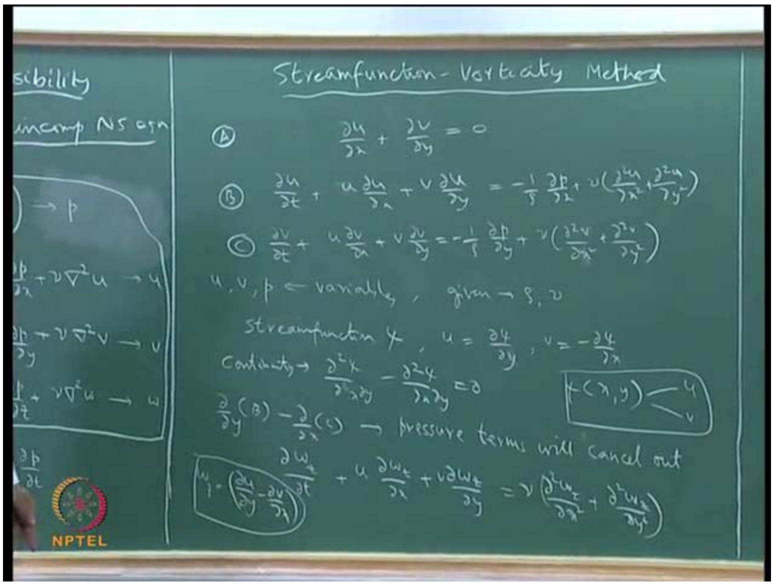

So, in the in the (x, y) plane is the vorticity. So, what we have been able to achieve is an equation here, which is the combination of the two momentum equations, which does not have pressure in it, but of course, which has the kinematic viscosity and a new variable omega z, which is expressed only in terms of the velocity gradients and not in terms of pressure. So, we are able to combine these two equations in such a way that pressure is is cancelled out.

Now, what is the advantage, we have gained in this; we have not gained much, if we were to leave it like this, but if we were to couple it in order to solve this this equation, we still need to know u and v, this is we can consider, this as a generic scalar transport equation, where omega z is the scalar; and these are the convention term and this is the diffusion term ,this can be solved readily using for example, the (( )), upwind scheme and central schemes for these things.

So that can be done provided, we know u and v and how can we get u and v, we already have this stream function here, if you know the stream function, we could get u and v from this, and then once, we have the u and v, we can solve this. Now, how do we get the stream function? So, we know that stream, the stream function such that this is the definition and we also know that vorticity is defined like this, can we completely eliminate u and v by expressing the velocity components in terms of this.

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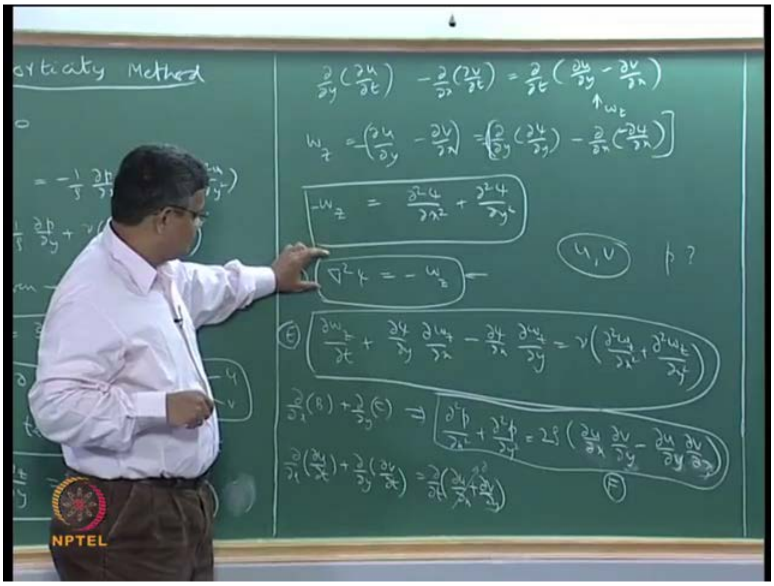

So, let us just see, so omega z is dou u by dou y minus dou v by dou x; and we know that u is equal to dou square psi by dou psi by dou y. so that is dou by dou by of dou psi by dou y; and v is minus dou psi by dou x; so, minus dou by dou x of; so that is equal to dou square psi by dou x square plus dou square psi by dou y square. So, depending on how we define this u here, we can put a minus sign here and a plus sign here or either way it is possible. We can have different equations for this.

I think, this omega z will be dou u by dou x minus dou dou u by dou y. So, we can put a minus here; and then we can put a minus all the way through here; so that we get minus omega z equal to this or we can say that del square psi is equal to minus omega z, and then we have these equations.

So, now we can see assumption approximate of a of a sequential solution, if you were to start out with, we have two equations. So, we have this equation, let us say, equation D and equation E; and this equation E can be written entirely in terms of stream function, because we know that u is given by this; for example, we can write dou psi by dou y dou psi by dou x dou omega z by dou y equal to nu. So, this is our equation E.

So, when you look at these equations, they get two variables that are appearing here. The psi, which is the function of x, y and t; and omega z, which is also a function of u and v so therefore, it is also a function of x, y. So, in this way, we have taken these equations here involving three variables u, v and p.

And we have written in terms of two new variables psi and omega z, in which neither the velocity components nor the pressure appear. So, if you are able to solve these things together, then we can get psi and omega z from this; and from psi we can get u and v. So, the idea of the stream function - vorticity method is to start with some initial; for example, initial velocity field as an as an initial guess from the from the initial conditions. Once, we have the initial conditions and the initial vorticity, we can solve for this and once, we get psi here from that we can get the u and v; and once you get u and v here, we can solve this equation to get omega z.

So, from a known vorticity as to begin with we can solve this; and then, we can solve this equation to get omega z here; and then we can move on in in that way; so and if it is a steady flow that we are interested, then this will disappear; and we can solve these two in an iterative way, to get the overall solution.

So, we can have either a time evolving method or a steady flow also is possible in in these things. So, using these two together, we can get u and v. So, we are not directly solving for u and v by solving the momentum equation, we are solving directly that is using partial differential equations and finite difference approximations; we are solving only for the stream function and vorticity; from the stream function we deduce u and v, what about pressure - how can get pressures, because the solution our solution requires not only for u, v, but also p; only then we can say that it is a complete solution.

So, in this particular case, we can, we have combined B and C; in such a way that we eliminated pressure; so that we are able to get the vorticity transport equation. Now, we can combine B and C; in such a way that we get an explicit expression for pressure in terms of the velocities. So, let us we can do that by in the following way, we can do dou by dou x of B plus dou by dou y of C.

So, essentially we are taking a divergence of the momentum equation; whereas, this is taking a curl of the momentum equation. So, if we do this, then we can see that this becomes dou square p by dou x square; and this becomes dou square p by dou a square and then we have dou by dou t like this; so and we can go through a derivation, and then we can simplify the equation.

Finally, we can show that the pressure is given by this.

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Is there a factor of 2 here, this is dou u by dou x dou v by dou y; this requires some simplification. Let us just see, how the time dependent term gets eliminated, we can take these two terms; and see what this divergence operated just to these two terms. For example, so we are taking dou by dou x of dou u by dou t plus dou by dou y of dou v by dou t.

So, this is we can write this as dou by dou t of dou u by dou x plus dou v by dou y; and this the volume in the brackets is equal to 0 from the continuity equation. So, this thing goes to 0. So, we can see that when we take the divergence of these two, then the time dependant terms will go to 0. We have to do further more algebra in order to to get rid of these things, but it can be done, it has been done and we can derive an equation for pressure like this, we just put this clearly.

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Now, what is the advantage of this, let us call this equation as equation F. Now, what this means is that; if you are in a position to get u and v from solving this D and E together, then we can substitute those things here, we can evaluate all the derivatives here; and so, the right hand side is known; and we can solve this for pressure and that is the approach that we have in the stream function - vorticity method.

So, the evaluation of the Navier - stokes equations, the incompressible form of Navier - stokes equations is not done in the u, v, p mode. So, this is not done in the so called primitive variables mode, primitive variables being u, v and p; it is it is done in the form of newly defined variables stream function – vorticity; and these three equations coupled equations are are reformulated into two coupled equations involving psi and the stream function and the vorticity.

For example, we can solve this, only if you know omega z; and omega z is known, only if you solve psi. So, you have to solve these things simultaneously, but these are forms that are known to us, because this is like a scalar transport equation; and this is like Poisson equation.

So, we can solve this by discretization and linearization is obviously necessary in some not linearization, but when we solve this, we when we solve this we do not know what this is; so we have to make some assumptions here; and we have to solve these things iteratively, solve this within assumed value of psi, and then solve this with the calculated value of omega; and then, we get a new values of psi, you go back to this; and then reevaluate. So, you have to solve this two simultaneously and iteratively, numerically can be done using finite differences at the end of which we get psi and omega.

And once we get psi, we can take a the differentiation with respect to y and x to get the u and v components; once we get the u and v components, we can evaluate the right hand side of the Poisson equation for pressure; and once this is known at every grid point, then we can do a discretization; for example, central differences for this will give us a discretized equation, which we can solve for pressure at t, x, y.

So, this is the overall approach, so when we are considering steady state conditions, when this does not appear; the method requires us to evaluate to write down numerical approximations for psi for a given value of omega z; and once we evaluate that; we can put that in this and then solve for omega z as a function of x and y.

And we can go back to this and at each grid point, you have omega z i j; we have this and then we reevaluate psi at i j and using the psi i j that is evaluated from this, we evaluate the derivatives, and then we discretize the rest of the equation with omega z as the as the variable here; and then we solve this again for omega z. So, when we solve this equation, the Poisson equation for stream function, this is given or estimated or calculated; and this equation is written in the form of…

So, this equation becomes A psi equal to b; with psi the stream function being the variable here, now when we solve this equation, this is evaluated or given or estimated and this equation in the discretized form becomes A prime omega z equal to b prime. So, we solve this for psi i j; and we solve this this equation for omega z i j. So, we have to do this iteratively, so at the end of that we have a psi i j and omega i j i j omega z i j which satisfy both the equations. So, at that point, we can say that we have got a solution for the stream function at x and y with those stream functions.

Now, we can evaluate u at i j and v at i j; once we have u at i j and v at i j; we use those values to evaluate these derivatives; and we evaluate the right hand side term. Now, this becomes a third equation, A double prime p equal to b double prime, where b is known and we solve this directly for p i j.

So, here there is iteration for the solution of this matrix equation, if we use for example, the Gauss Siedel method; if we use a direct method, and then there is no iteration required. So, the pressure is decoupled from the momentum equations in this particular way. So, there is in the solution of the reformulated momentum and continuity equations, we are not solving directly for pressure.

So, we are getting the velocities u and v directly and with the known u and v, we solve a Poisson equation for pressure without any further iteration. All this is done, under steady conditions or at a single time step. So, if you do all this for a single time step at this point, you can go to the next time step and then revaluate. So, at each time step, we have to do all this kind of things; and so, it is a bit more complicated for unsteady flows.

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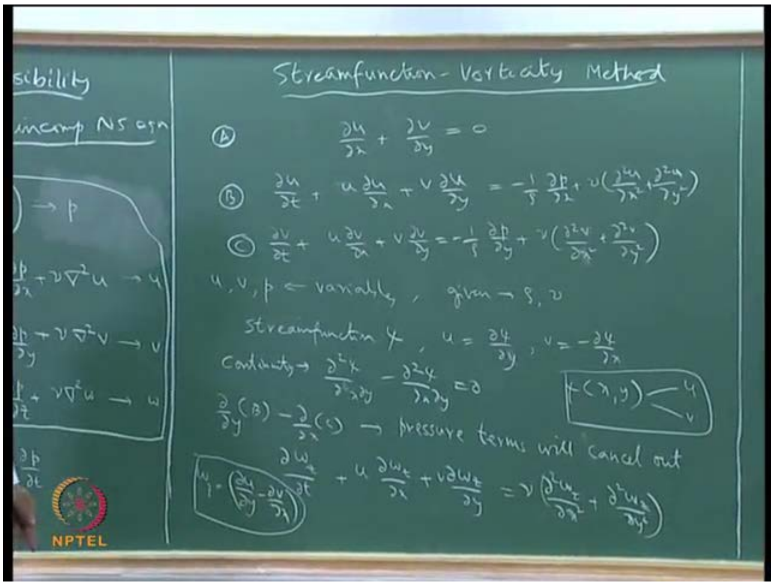

But it can be used and this provide us a means of solving the Navier - stokes equations together for u, v and p as a function of x, y and t; without making any compromises, this time evolution as predicted by the stream function - vorticity method is accurate. So, in that point, this method is superior to artificial compressibility method, but it has a disadvantage.

Whereas, this equation can be used for a three - dimensional flow; so that we get u, v and w; this equation can be used only for two dimensions, because only for two dimensions can we find a stream function, which satisfies the continuity equation. So, only in such cases, can we use this method. So, this method is useful only for two - dimensional flows; and it can be used for steady or unsteady, this approach is useful for steady three - dimensional flows, this method draws upon the compressible flow methods.

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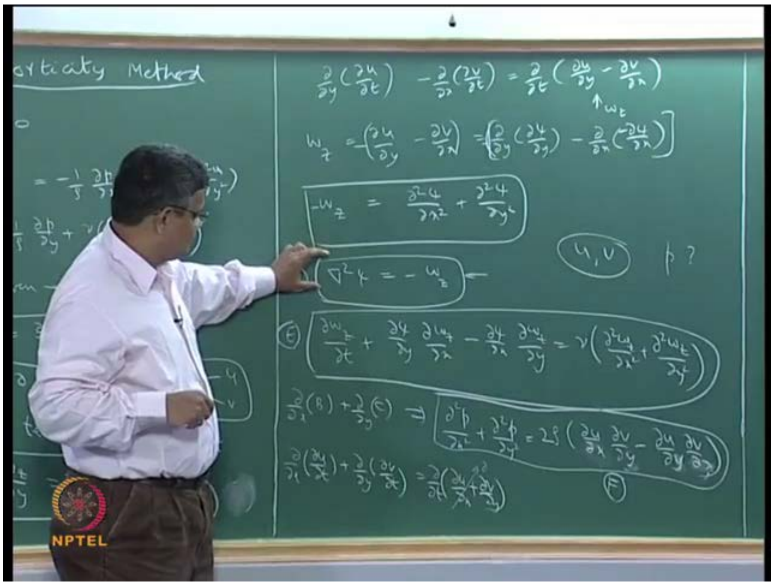

Whereas, here we are not making any statement about compressibility at all, we are just looking at the reformulated equation as a Poisson equation for stream function; and Poisson equation for pressure; in this particular case, it is given in terms of explicit quantities on the B side, on the right hand side.

Whereas, here the poisson equation the value coming here is an unknown, it is linked to the transport equation for vorticity. So, in that sense here, we are not invoking compressibility at all, both the methods have have been applied and they are applied for a number of cases, but they both have limitations, in terms of what kind of flows we can apply them to.

So, in the next lecture, we look at a more generic method, which does not have either the time step, the time limitation, unsteady limitation associated to the artificial compressibility method or the two - dimensional flow limitation only associated with the stream function - vorticity method. So, those in which we again make use of an equation for pressure; and then we solve in primitive variable form, not in terms of psi and omega z, but directly for u, v and p. So, that is it for now.