Computational Fluid Dynamics

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Module No. # 3

Template for the numerical solution of the generic scalar transport equation

Lecture No. # 15

Interpretation of the stability condition

Stability analysis of the generic scalar equation and the concept of upwinding

Diffusive and dissipative errors in numerical solution

Introduction to the concept of TVD scheme

We have seen the Von Neumann way of analyzing the discretized equation for stability and we have seen that, it is very general method applicable for a linear equation with periodic boundary conditions. And within these restrictions, we can apply this to the generic scalar transport equation and come up with the scheme, which is, from a stability point of view, which is acceptable.

But before we do that, before we going to the next step, let us try to see why there should be stability concern, why there should be, for example, a courant number limitation on the delta t - the time step - that we can take for a given grid. So, the there probably many interpretations, but we can go back in interpreting the stability to the idea of the nature, of the mathematical equations that we are trying to solve. We have seen that typically in a scalar transport equation, we have a hyperbolic nature, in which a solution is like… a wave like a solution, which goes forward in a particular direction at a specific speed like that or we can have a diffusive type of equation, which progresses in all directions in space. So, we can, that means, that are the solution that we are looking at, should also exhibit these kinds of properties.

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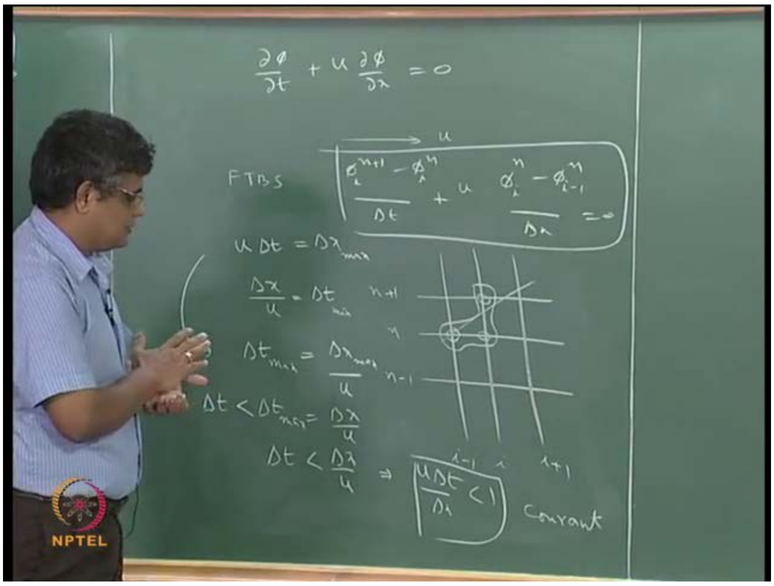

So, that I think will impose certain conditions on, what delta t that we can use for a given delta x. If you consider the simple case of the one-dimensional wave equation, this represents the case of a wave, which is moving in the forward x direction at a particular speed given by u. Now, we are trying to solve this using, for example, FTBS scheme as phi i n plus 1 minus phi i n by delta t plus u phi i n by delta x equal to 0.

Let us look at the computational molecular for this; we have different constant n lines and constant i lines, we say that, this is I, i minus 1, i plus 1, n, n minus 1 and n plus 1. We are looking at a solution for phi i n plus 1; so, that is for this point. And this is being described in terms of phi i n and phi i minus 1 n in this. So, the computation molecular is this and we are seeking a solution for this point, in terms of the preceding points at this. And we know from the well posedness of a problem, that for every problem, there is a zone of influence and there is a zone of dependence.

So, that means, that if our computation is to be correct, then the value of this must depend; if it were to depend on these things, then these two points must lie within the zone of dependence for this particular point. And also, if this is going to influence this value, then this must lie this point must lie within the zone of influence of this particular point; only then we can have well posedness. And what determines the zone of dependence or influence in this particular case? We have a hyperbolic equation, we have information going in the positive x direction at a particular velocity and it is moving at a speed of u - the information is moving at a speed of u. So, in a given time delta t, it would traverse a distance - maximum distance - of delta x or for a given time of delta, for a given spatial distance of delta x moving at a constant speed of del u, it will, it will take a time of delta t minimum, in order to traverse a distance of delta x. So, these things will determine the zone of influence and the zone of dependence.

So, this delta x is the maximum distance that information from, for example, this point u phi i plus i minus 1 n can travels in the x direction over a given time t. And we must make sure that, the delta x here is less than this. And similarly, when you look at it in this direction, for a given delta x u here, it can traverse, it will take a minimum of delta t in order to have an influence on this.

So, now, if you were to say that, if you were to delta t maximum from here, is therefore delta x max divided by u; and if you were to apply this to this particular case with a delta x of with delta x of delta... So, for a given grid of delta x and for a given velocity of u, delta t max that is permissible is this. And therefore, the used delta t, the delta t that we must use, must be less than the maximum permissible value; therefore, delta t must be less than delta x by u for a given grid spacing and for a given propagating velocity. So, this means that, u delta t by delta x must be less than 1, and this is the courant content, that we have derived as a stability condition for this.

So, if this condition is not satisfied, we are either attributing too higher velocity for the information from the wave to be passing; or for a given delta x and delta t, the information that is coming here will not be able to reach this; so, that means that, this point here will not lie in the zone of dependence; and therefore, to seek a solution for this in terms of this, will be incorrect and that is the kind of problem that surfaces if you have too higher value of delta t. So, we can interpret this stability limit in the, in the terms of zone of dependence and zone of influence. And the way that we seek the solution and the computational molecule that comes into this into picture, as arising out of this will impose certain restrictions on the delta t and delta x for a hyperbolic problem, and therefore, one can readily interpret the limitations that we have on this for a simple case.

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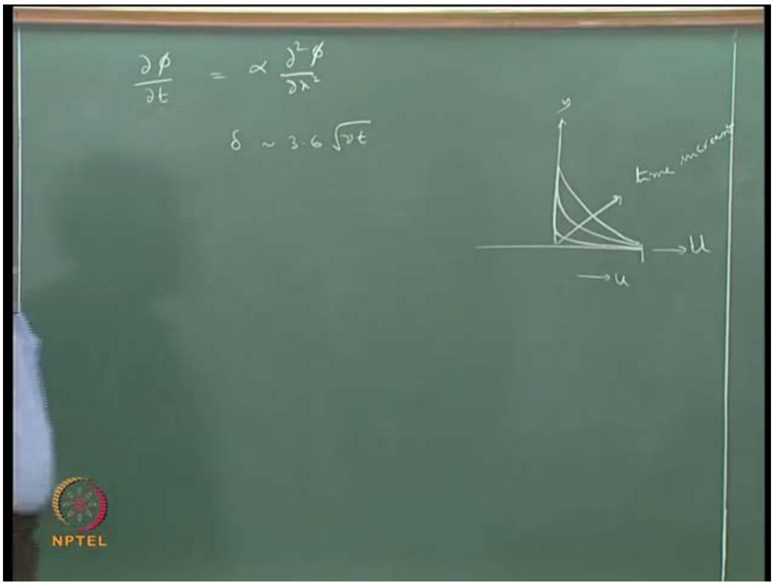

As you go on making more and more complicated approximations here, then it may be more difficult, but we can visualize the need for a time step limitation for a case like this, from the physical argument, that half zone of dependence and zone of influence. Now, we have seen the other extreme, where advection is negligible and the variation of phi with respect to time is given purely in terms of diffusion; think, we might have used d for diffusivity, but let us just consider this. So, this is the case which is a diffusive transient diffusion and this is transient convection. The difference between the convection and diffusion problem is that, convection problem has a hyperbolic nature; so, that means, that it moves at a velocity given by u in a particular direction, whereas diffusion information is propagating in all directions.

So, it is not… with cost and diffusivity, it is an isotropic thing. So, it does not have a constant direction of propagation. So, the question is, to what, why we should have a limitation on this arises? So, here, we can look at what this diffusion is actually implying. And here, we are looking at a case of transient diffusion; so, that is something is diffusing in a particular direction.

And one of the famous examples of this is strokes first problem, where you have a plate in an infinite medium and this is suddenly moved at a particular velocity u, and it is set into constant motion. And this is the y direction, its infinite expands; so, this is flow in this direction and flow in this direction. Because the plate is moved in the horizontal direction, they will be a no slip; because of no slip condition, the flow will be moving here; and as time progresses, the fluid away from the wall also will be acquiring velocity.

So, for small times, the velocity profile will be like this, that is, we are looking at u in this direction and y in this direction. So, only over a short time, we will have non zero velocity; otherwise, the velocity will be like this.

Now, as time progresses, the velocity profile will keep on increasing, will keep changing as time increases. So, we can see that, there is some the information of a moving wall here is propagating in this direction and there is a certain directional thing. And we also know roughly, at least for small wise, we can say that the information is propagating, this boundary layer is shifting, the thickness of the boundary layer is typically given in terms of some 3.6 times square root of nu t.

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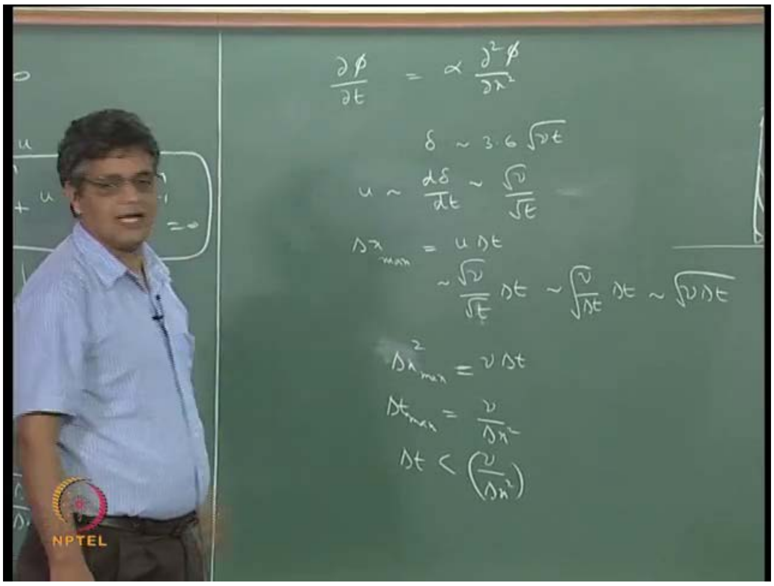

So, from this, we can since the boundary layer is growing at this, we can find out the velocity of the movement of the boundary layer and this will be this will vary as square root of nu divided by square root of t.

So, this we can say is the velocity with which information is propagating in the x direction; and for small delta t, for small time, this will be… So, now, once we have this u, we can therefore say that delta x max, the maximum distance that this information is propagating is u times delta t and that is… And for small time steps, we can say that this is roughly nu by delta t square root time delta t or nu delta t square root.

So, we know therefore, that the delta t we can now put this as delta x square max is nu delta t. And we can say that, delta x max is nu by delta x square for a given grid, and therefore, delta t must be less than nu delta x square.

Now, the factor of half and all those things may come from more proper evaluation of this, but one can see that why there is a maximum delta t coming even from a diffusive case, because there is a certain velocity with which information is propagating, from one grid point to another grid point even in this particular case.

So, one can make these kind of arguments to understand, why there will be time step constants which would limit the maximum time step, that we can take for a given grid; for a purely convective case, in which case, it is much easier to understand; and for a more diffusive case, in which we have to use only hand waving arguments like this. So, it boils down to how we are evaluating the i plus 1 value, in terms of the neighboring values and the computational molecules associated with this.

So, with the these things, with this kind of understanding, that there are certain combinations of delta t, delta x, and u diffusivity and all those things, which limit the kind of approximations that we can take for a given partial differential equation, and also the limit of consistency that we must adopt. Now, let us go to the scalar transport equation and see what kind of approximation we can take, and let us see we can come up with a template, which will satisfy both the consistency condition and the stability condition for a linearized scalar transport equation.

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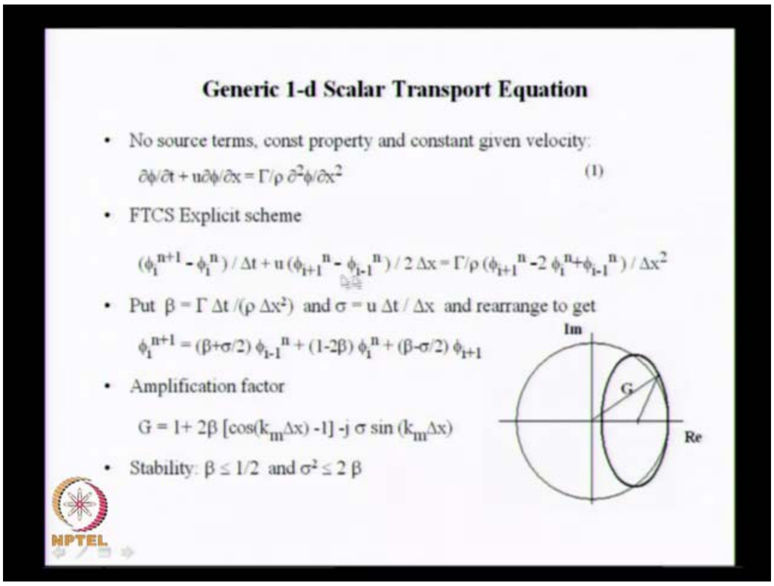

So, let us consider the generic one dimensional scalar transport equation. For simplicity, we are considering a case with no source terms, constant properties of the fluid and a constant given velocity. So, in such a case, the scalar transport equation can be written like this in one dimensional form; we have the accumulation term, constant property is now we have divided everything by rho - all the terms by rho, which is the density - and u is the velocity of advection with which the scalar phi is being transported in positive x direction; gamma here is the diffusivity and gamma by diffusivity...

The diffusion for the example capital T that we have used divided by rho here; in the case momentum transfer, this gamma will be the dynamic viscosity and gamma by rho will be the kinematic viscosity.

So, gamma by rho here will have something like meter square per second as a unit. So, this is the diffusion term in one dimension. So, this whole equation is the scalar transport equation, and this can represent different momentum equations without the pressure gradient, because we have taken that the source term is 0 in this. So, we can now consider an FTCS explicit scheme for this, and so, we can write dou phi by dou t as phi i n plus 1 minus phi i n by delta t. We can write u d phi by d x as u phi i plus 1 n minus phi i 1 n dived by 2 delta x; so, we are looking it as a central in space. And we have, this is also defined with a central in space; so, gamma by rho which is assumed to be constant, phi i plus 1 n minus 2 phi n plus phi i minus 1 n divided by delta x square. So, the overall scheme is first order accurate in time and second order accurate in space, and it is explicit.

We know that, if we had just the terms on the left hand side, FTCS scheme would not do us a stable thing. But with the combination of a diffusion property, diffusive term on the right hand side for which we know that, FTCS scheme would give a conditional stability, that is, gamma by rho times delta t by delta x square should be less than half, is one particular thing. Now, because we have expressed it in a in simple forward in time and central in space without any further approximations, it is readily seen that this discretized scheme will satisfy the consistency property, but will it satisfy that the stability condition. If so, then we have a possible scheme template for the generic scalar transport equation.

So, for the sake of simplicity and ease of a notation, we will put beta equal to gamma delta t by rho delta x square. So, we are taking this delta t here and then like this, and sigma is our courant number u delta t by delta x. So, we can see that, that term associated - the combinational terms associated - with the diffusive term and courant number associated with the advective term, both will appear in this; and by rearranging this, we can write phi i n plus 1, which is the value that we are seeking, in terms of beta and gamma by 2 like this, beta plus sigma by 2 times phi i minus 1 n plus 1 minus 2 beta phi i n plus beta minus sigma by 2 phi i plus 1 n.

So, we have phi i minus 1 is coming here, i plus 1 coming here, and we can see that, in both cases, we have beta term appearing from this. And we also have the courant number term, which is coming from this. And here, we have no 2 in the denominator, and here, we have 2 in the denominator; so, we have sigma by 2 which is coming here and beta which is coming from this.

And we also note that, when we take this phi i plus 1 to the right hand side, so that we can express in terms of phi i phi i plus n plus 1 in terms of these. Then, this becomes minus sigma by 2; that is why we have a coefficient here. And phi i minus 1 in the convective term appears with a minus sign; so, when we take it to the right hand side, this becomes plus sigma by 2, we have beta plus sigma by 2. And here, it we have only this 1, that is minus; when we take it to the right hand side, we have plus 1 here. And phi i n is appearing with a 2 beta, minus 2 beta here; so, we have 1 minus 2 beta terms phi i.

So, we are expressing in the FTCS scheme of this scalar transport equation, phi i n plus 1 in terms of phi i minus 1, phi i and phi i plus 1; so, we have a discretized equation. We can show that this is consistent; if you want to get stability, we can apply the Neumann boundary - Neumann analysis - assuming periodic boundary condition.

And we can derive an amplification factor G for this scheme to be given by, G equal to 1 plus 2 beta cosine Km delta x minus 1 minus j times sigma sine Km delta x, where the Km delta x is the phi that we have been using in this, and j is square root of minus 1, and beta and sigma are the parameters associated with the discretization scheme. And we can see that, there is a real part associated with this and there is a there is an imaginary part associated with this. And we can we can plot the real and imaginary values of G on these axis here; and for different values of beta and sigma, once we fix these parameters values, we can plot G as a function of Km delta x, and Km delta x various between 0 and phi. So, for these things, we will we will get a solution like this. And we can see that, there are two conditions that need to be satisfied here, and that beta here is less than half and sigma square must be less than 2.

So, the dual conditions are, that beta is less than half, if we neglect the diffusion term, we have sigma to be less than 1; and given and if we neglect this, we have beta to be less than half. So, here, we have the dual condition that beta must be less than half; so, that means, that sigma square must be less than 2 beta, so that this G is always less than 1. And since beta has a maximum value of half, then 2 beta will mean that sigma has a maximum value of 1.

So, both the courant number condition and the transient diffusive case condition of beta less than half, both are in a way satisfied, but beta can take values of this and this. So, if you take a beta of, say 1 4th for a particular delta x and for a particular diffusivity, we choose beta, we choose a value of delta t such that beta is 1 4th, then sigma square here must be less than 2 beta; so, sigma square must be less than half. Now, what is sigma square? So, sigma square is sigma is u delta t by delta x. We have already fix this delta t and delta x; so, u must be such that sigma square will be less than 1 half. So, that means, that the choice of delta t is now determined not only by delta x and u but also by beta. So, you have to choose such a value of delta t, which satisfies both this condition and this condition.

So, you cannot say that I will take beta equal to half. For a given delta x, I will choose delta t such that beta equal to half; if you take that and if you fix the value of delta t, and since you fix value of delta x here, u must be such that the courant number condition is satisfied. If that is not satisfied, then you have to choose a different delta t, so that you satisfy this condition and this condition. So, the choice of delta t is a bit more tricky in this particular case, but it is possible to have a solution scheme of an FTCS explicit scheme which is conditionally stable, for a certain choice of delta t and delta x, for given diffusivity here and for a given velocity here.

So, in that senses, we know that this is a consistent scheme. So, as long as we choose beta and sigma properly, then we have a resulting discretization scheme - FTCS explicit scheme - which is consistent and stable; and therefore, we can serve as a satisfactory template for the scalar transport equation. This gives us one possibility, which is first order in time and second order in space, and it is easy to get a solution using this. Now, even though it is stable, this particular discretization has a well-known limitation and the limitation is that the oscillations.

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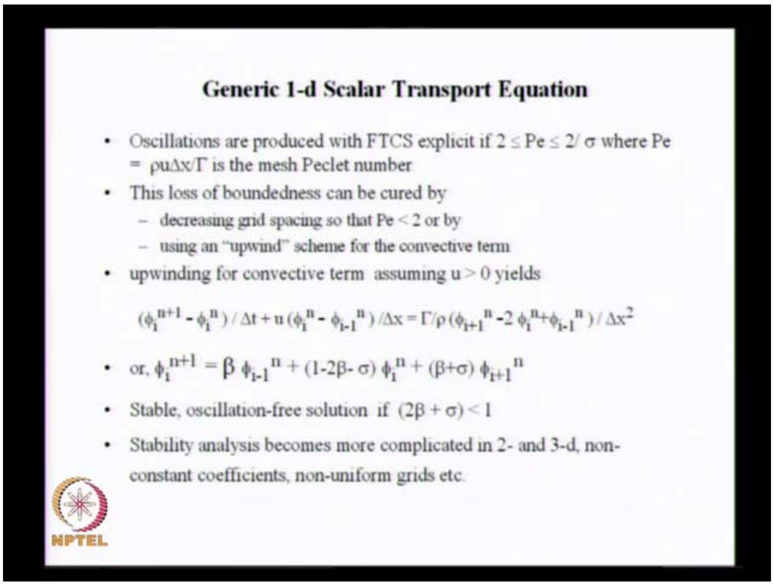

So, the value of phi that is computed using this particular thing may produce a spatially oscillating solution, which is not there in the original equation. So, because of that, this and that is produced in a certain range of parameters; it is produced, when peclet number which is defined; it is more like a mesh peclet number. Peclet number is usually, for example, Reynolds number times Prandtl number, where Reynolds number characterize the momentum transfer, and Prandtl number is a property of the thermal diffusivity and momentum diffusivity.

So, that we can define, since we have advection and diffusion, we can define a mesh Peclet number as rho u delta x by gamma here. So, the gamma is the associated to the diffusion, and u is associated with the convection term. So, when the Peclet number defined like this, which is a function of the parameters appearing in the equation plus the mesh size that we have chosen or the delta spacing delta x that we have chosen; if the Peclet number lies between 2 and 2 by sigma, where sigma is a courant number, then we can have spatial oscillations produced artificially, that is, in the numerical solution, but which are also stable; the oscillations will not go with time, but they are there.

And what this means is, that a solution a value of phi, for example, phi may be a concentration or mass fraction of a particular species, and it may have to lie between 0 and 1. Or it may be a particular quantity which has to be only positive; for example, we will see that, something like kinetic - turbulent kinetic energy - which always as to be 0 or positive. So, they have a certain range in which the values can change. Now, the computed solution may exhibit oscillation, because of which the solution may go out of bounds, it may become less than 0 or greater than the maximum value, that it can go physically from a physical point of view.

So, for example, you can have mass fraction which is negative or greater than 1, because the computed the exact solution may be 0. But because of the oscillations that are produced in numerical solution, it becomes 0 minus something at some spatial location and 0 plus something at some spatial location; it is not able to exactly give you a value of 0 at a particular point. So, because of that, you have a loss of boundedness; and that loss of boundedness is an undesirable property, because if a solution has to be cannot be negative, and if at some point of the time, you are taking the square root of that, then the computed solution will give a problem, because we expect it to be non-negative; and therefore, we are taking a square root. But the computed solution has turned out to be negative, because of the spatial oscillation that is introduced in this; and therefore, it when we try to take square root, we get an error in the computation.

So, how to this loss of boundedness is undesirable in a numerical scheme. So, even though we have a stable and consistent solution scheme using the FTCS approximation for the generic scalar transport equation, it is still not entirely satisfactory, because for certain range of parameters, it is possible to get an unbounded solution or a loss of boundedness is possible in this. Now, one way of reducing, getting out of this problem is to make sure that peclet number is always between these limits.

So, if and we can, for example, reduce the peclet number, we have sigma as always less than or equal to 1 effectively, or less than or equal to sigma square is m; so, that means, that there is a small region in which this can happen. So, by reducing the grid spacing effectively, we can get around this particular problem; there is a small mistake here. So, if the peclet number is less than 2, then we do not have this loss of boundedness. And another way of getting down this, is to use an upwind scheme for a convection term; by doing an analysis, we can show that the loss of boundedness is arising specifically from the use of central differencing approximation for the convention term. And when we use a central differencing term, then oscillations - spurious oscillations - are produced and that is what is actually causing this.

So, instead of using an FTCS scheme throughout, we use a combination of first order and second order discretization here. So, we use a first order scheme for this; so, as to not have those oscillations and we have second order scheme for this. And we know that, when we are looking at an advective transport equation like this, if you are going for an explicit scheme, then only FTBSs scheme; so, that is backward in space will give us a proper solution; therefore, we use forward in time here, backward in space here and central differencing here.

So, this backward in space is a solution that is satisfactory, when u is positive; and when u is negative, we can also show that, for this particular thing, we must be using forward in space. So, the appropriate conditionally stable equation discretization for this part is, backward in space if u is positive, and forward in space if u is negative. So, in both cases, what we are actually saying is that, the scheme must be an upwind scheme; so, that means, that we should always take the value of… if u is positive, then we must be using, we must be taking it in, we must be taking this derivative - this difference - in the direction of u.

So, if u is positive, we take i minus i minus 1; if u use negative, then we take i plus 1 minus i. So, it is coming, we are going in the negative x direction, which is going in the same direction as u. So, that is why it is called upwind scheme. We can look at in that way, that the differencing approximation that we use for the spatial derivative, dou phi by dou x should be chosen in such a way, that it always follows the direction of u. If u is positive, then it is going from… in the positive x direction, so we take it as i minus i 1; and if u is negative, we take it to be coming from i plus 1 to i. So, that is in the negative x direction, so it will be i plus 1 minus i by delta x.

So, that particular scheme is called an upwind scheme. So, we make use of upwinding scheme for the convective term, assuming that u is greater than 0. So, that it is backward differencing and we do like this. And this means that, your phi i n plus 1 is expressed in terms of beta i minus 1 times beta i minus 1 n and 1 minus 2 beta minus sigma times phi i n and beta plus i plus 1 n in the FTBSCS scheme for the entire thing.

So, since we have changed the discretization scheme, the stability characteristics will also change; and since we have again not made any approximations here other than to use straight forward derivatives for the first order derivatives, for the time derivative and the space derivative here, and second order central scheme here, we can assume that the resulting scheme is going to be consistent; the stability of this scheme can be investigated, and you can show that, it is stable, provided that 2 beta plus sigma is less than 1 and so we can get as a stable oscillation free solution, for as long as we satisfy this stability condition and it is also a consistent scheme.

So, for the generic one dimensional scalar transport equation, we can choose an upwind scheme for the convection term, and a central differencing scheme here for the diffusion term and a forward in time for the time dependent term. So, using this, we can come up with conditionally stable consistent scheme which if you apply, will guarantee convergence for this linear scalar transport equation; and if it is non-linear, we can linearize it around a particular point, therefore in such a way that u is constant and the diffusivity is constant, and then we can express it in this way. So, without worrying about the source terms, we can therefore derive a particular scheme, which is not only consistent but also stable and it also produces an oscillation free bounded solution for this particular case.

So, in that sense, this is a scheme which can be recommended. Now, will it give us a satisfactory solution? Will it gives us the exact solution? That is not very certain, because we are making approximations here; and typically when we make these approximations, we have two kinds of errors that are introduced. So, we will discuss these errors and then we will see what kind of errors we may expect in this, and how these errors will pose limitation on the scalar transport equation that we need we want to solve. In our generic equation, we have a time dependent term, advection term, diffusion term and a source term; if the source term is such that, it does not depend on the phi.

The scalar that we are that we are conserving in the scalar transport equation, then the source term does not have much role to play in determining the stability or the consistency; we do not have to worry about that. But if it is, then we also have to include that in overall analysis, and then, see, whether it introduces stability problems and so on.

So, let us leave aside the source term and if we consider the diffusion term, diffusion term usually does not pose any problems we can have, and it is very easy to write a second order accurate approximation for the diffusion term. So, as long as we are using straight forward second order approximations, then we do not usually have consistency problems, and we do not have too much of problem with the term, because it is second order accurate. The most problem that we have is with the convective time; so, that is u dou phi by dou x and this particular thing we have seen, if we if you were to use for the simple wave equation forward differencing with the positive view, it will give us a unsatisfactory solution, it will give an unstable solution and even a central scheme will give us unsatisfactory solution; and backward scheme will give us a conditionally stable scheme.

But backward scheme, when you see is only the first order accurate, now if you want to make it higher order accurate, then we have the difficulty that we, for example, we cannot use the central differencing a straight away. Ok

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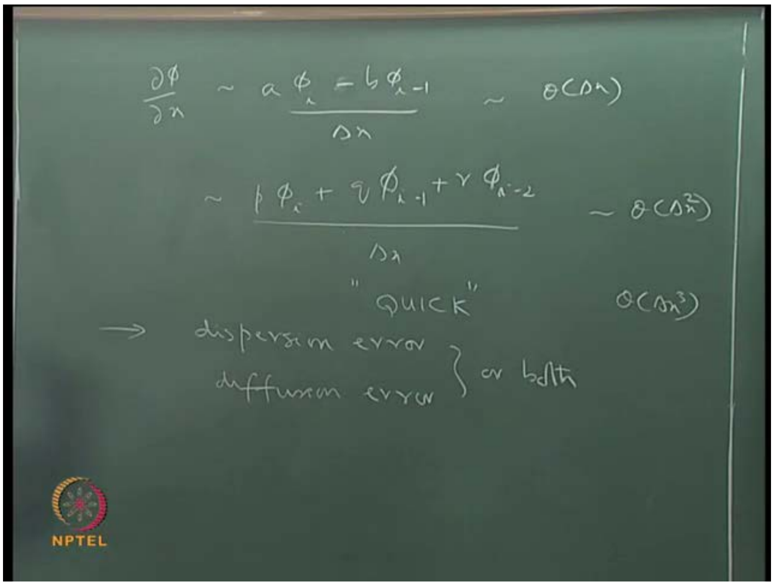

So, but because we have, because we know how to derive one sided approximations for any derivative, we can go back to that. And then, for example, we have dou phi by dou x and this can be represented as a phi i plus b phi minus b phi i minus 1, and this is first order accurate. We can also write it as p phi i plus q phi i minus 0 plus r phi i minus 2 by delta x, which will be second order accurate. So, both this and this are backward schemes, and we can get at least a conditionally stable solutions with these things; we can also make it third order accurate.

So, we can increase the accuracy of this, and for the third order accurate, there is a method called quick scheme, which became popular. And this a third order accurate approximation for the convection term and that gives us good accuracy, and it is been used. But whenever we go from a first order accurate to second order accurate or third order accurate or any higher order accurate solutions, then it is possible to get dispersion error.

So, that means, that we must understand, what we mean by dispersion error or numerical error. So, when we talk about an error in the approximation, then one can get, typically you can get either a dispersion error or a diffusion error or both. What you mean by diffusion error is that, if you consider hyperbolic solution, a wave which is propagating in a particular direction with a particular amplitude at a particular speed, then a pure diffusion error will mean that, the wave will propagating in the same direction at the same speed, but with an amplitude which is reducing with time.

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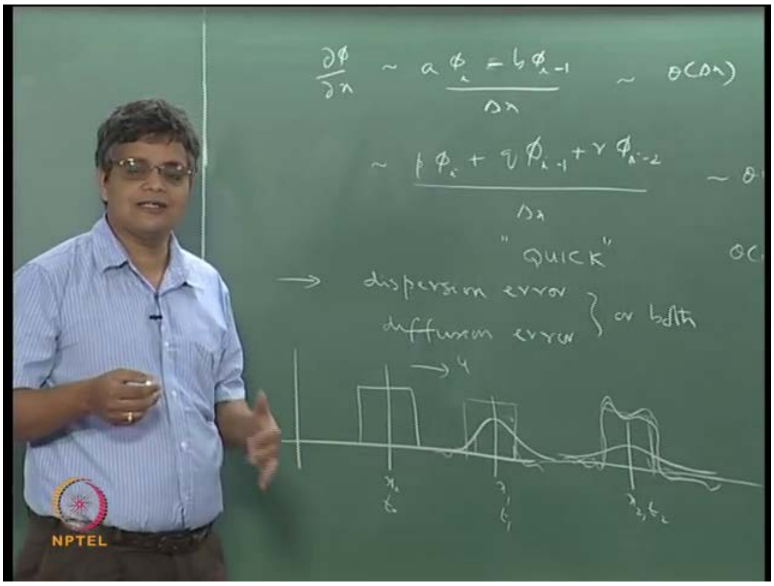

So, if you have a solution, for example, like what we had seen here, this is a square pulse, and after some time, it is propagating at the same speed; it would not be like, this it will be like this; and after some more time, it will be spread out like this. So, it is propagating in the positive at x direction at a constant u.

So, this is at x naught, at particular t naught and it has moved to x 1 at t 1 and x 2 at t 2. And the location of x 1 and x 2 at t 1 and t 2 are as per constant u, which is given in the equation - in the governing equation, and this is what we get with a purely diffusion error. If you have a dispersion error, then typically what happens is that, the wave speed will not be the will not be the same, and a pure dispersion error will mean that the amplitude will remain more or less the same, but the shape of this… In addition to this, typically we may have some oscillatory solution like this, and oscillatory solution like this, like that. And the precise shape of this thing will depend on what kind of initial function that we have.

So, in this particular case, we will have an amplitude, which is preserved in a pure dispersion error, but you will have oscillation, so that it will be more than the maximum value and less than the minimum value like this; and these oscillation are stable oscillations. These are spatial oscillations, in the sense, that they are varying with respect to the spatial position, but it is not an unstable method; we can also have an unstable scheme in which the oscillation is growing with time, but even under stable conditions, we may be getting a spatially varying thing, where there is no spatial variation other than the f of x.

So, this dispersion error effectively means that, we have a wave speed which is a function of wave length. And in order to understand, we can go back to the Fourier series expansion of a given function; and we have said that, a Fourier series expansion for a periodic function will have a finite number of wave component; so, different wave lengths. The smallest wave length being 2 delta x and the maximum wave length being 2 l.

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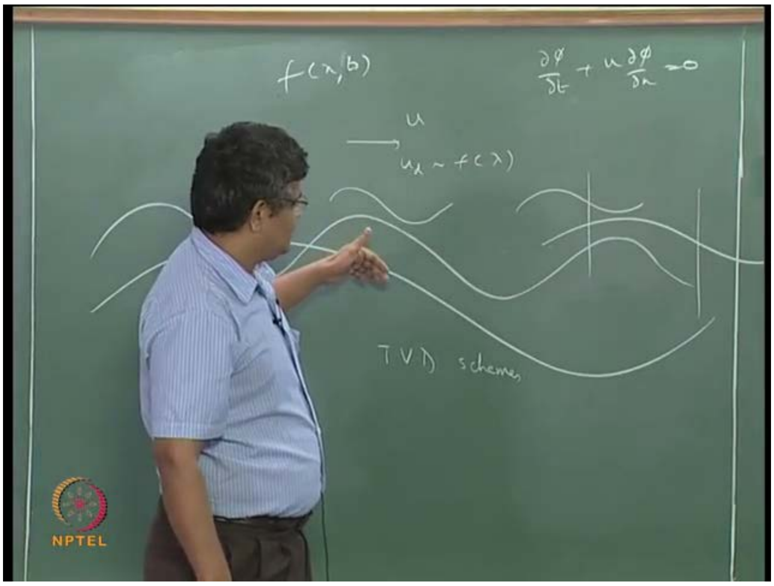

So, we can have a wave small as this and we can have a large wave and we can have something in between also. And a particular functional variation f of x at a particular time is given by a contribution from this, a contribution from this and a contribution from this.

So, now, if we have a dispersion dispersive discretization of this, then the velocity of each wave component will be a function of the wave length. So, in a purely diffusive scheme, all these wave components will have the same velocity u, which coincides with the true solution. But in this particular case, we will have u d dispersive wave velocity, which is a function of lambda, where lambda is wave length of this, and given that the initial function f of x will contain contribution from each of this; you will find that, these wave components will be moving at different speeds. So, that at time equal to t 1, not all of them will have reached the point x 1.

And in the linear wave equation, the initial function f of x at time equal to 0 is such that, the super position of all these things will give us the initial shape. And in order to get the shape at any time, we need to have all these wave components being present at the same point. So, now, if this wave component is moving at some speed, so that it has reached here, and this wave component has moved further, so that it has reached for the same thing, centered around this point and this has moved even here.

So, at this particular point of time, the summation the contribution of all the wave components is not the same as what we started with; therefore, the pulse that we actually get after certain time t 1 does not have the same shape as the original shape; the shape is going to be distorted. And it is some of these wave components, which have not quite reached here maybe something like this, and the contribution of this may also come here and this may have progressed even more like this. So, it is a superposition of a different combination of the same function, if what we are going to get at time equal to t 1 and time equal to t 2.

So, a dispersive error means that, the wave components that initially form the functional variation f of x is going to be of a different component; therefore, we see that, although initially we have a square shape like this, after some time we see some wave components falling out of this is initial box, some which are lagging behind, some which are moving forward. So, this kind spurious variances which are not there at a time equal to t 0 are introduced, because a dispersive error introduces a wave speed, which is a function of the lambda here - the wave length of the particular wave component. And it can be shown that, for a first order derivative, the convective term like this; any scheme which is more than first order will introduce a dispersive error.

So, if you in the interest of increased accuracy, you want to go to a second order scheme or a third order scheme; then, you are introducing dispersive error which may not be so good. And if you are in the interest of having an oscillation free solution, if you want to have only first order scheme, either the backward differencing for positive view or the forward differencing for a negative view, essentially the upwind scheme, then you are stuck with only first order accuracy. So, you have to choose between first order accuracy with an oscillation free solution or higher order accuracy with a spatial oscillations, which may not be so desirable; so, neither of which is really satisfactory.

So, that is why we, even if you have a stable consistent scheme, because of the dispersive and diffusive errors that arise from both the, especially from the advection scheme; we will not get, we may not get an entirely satisfactory solution; we may not get an exact solution for a given scheme. And we have to use more advanced methods, which contain to some extent the dispersive errors that are introduced by going for higher order accuracy.

So, essentially what there try to do is, that when these dispersive errors have their presence felt, when you have strong gradients, for example, you have a strong gradient here, as you go from this value to this value, and again when you come down from this value to this value, you have a strong gradient. So, in the regions of the strong gradients, the disperse the effect of the dispersive error becomes most convincing.

So, for example, there is TVD schemes are one such family of schemes methods, by which the dispersive error arising from these higher order schemes like, the quick schemes or these things, is suppressed in the presences of large gradients; and only the presence of large gradients, to such an extent that the oscillation is reduced.

So, using those kinds of TVD schemes, which try to detect where an oscillation is possible and put enough damping in the oscillation, so that you get a proper solution. So, that kind of approach has resulted in a making much better discretization of the advection schemes; using which, we can increase the accuracy of discretization and still get a stable solution. Let us, let me just plot the different kind of solutions that we may get with different schemes; let us say that we have, this is the wave which is propagating in the x direction at a particular u.

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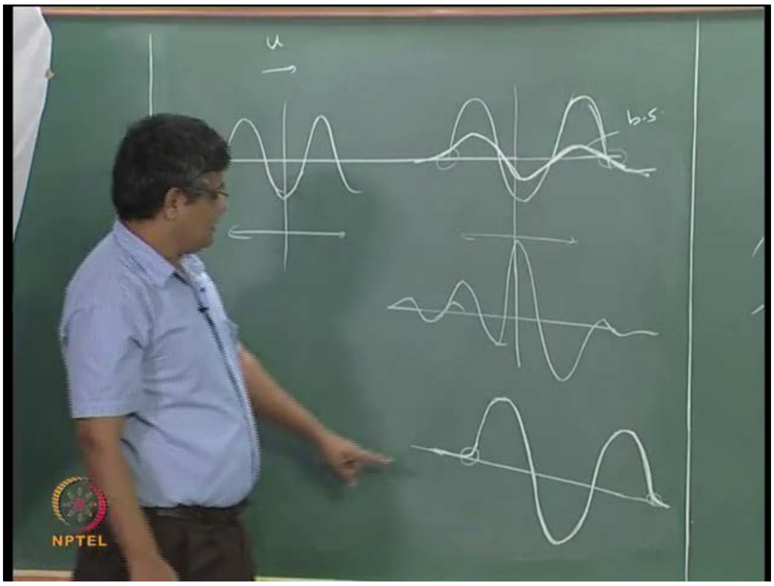

So, if we use a first order scheme which is, for example, in this particular case, first order upwind schemes, so that is backward differencing scheme for this; we introduce heavy diffusion error, so that after sometime, this particular wave will should have, let us say that this is the midpoint, and this is spread of the wave and this is the ideal position, and the spread of the wave is like this. So, ideally it should be, it should be like this, but we may find it. This is what we may get with a backward differencing, which is first order accurate.

So, if we make use of a second order accurate scheme, then this is, we may get, we may get more oscillations than what we what we have here, and especially, around this point and around this point. And if we use TVD schemes, then we recover almost the exact shape. So, we are expecting a solution like this. If this is the exact solution, we may get a solution which follows almost like this.

So, around here and around here, we expect some smoothening, not like this, but not as much smoothening as what we have with backward differencing; and the effect will be much more pronounced, if you have sharp corners like this. So, with a backward differencing, we may get a solution like this; with a TVD scheme, we may get a solution which is like this.

So, that means, that it may also be it may go much closer. So, we are retaining the shape much more faithfully using a TVD scheme and we are reducing the amount of diffusion that is taking place here, and we are also not introducing any oscillating solution. So, we are preserving the boundedness property with more faithful reproduction of the original wave pulse.

So, using these kind of schemes, we can get more accurate solution, but this is not the appropriate time to look at the theory behind tvd schemes, and why they function and how they enable a solution, but these kind of things are available in the literature and interested person can go in detail. So, what we can therefore say is that, we for a generic scalar transport equation, we have forward in time upwinds for the advection term; and a second order central scheme for the diffusion term will give us an oscillation free solution with some amount of smearing, that is, that is a coming forward more than what is inherently there, because of the advection term. And if necessary, if that smearing is found to be too much, then we have go for more advanced schemes like TVD scheme and so on.

So, with this, we can say that, we know how to solve the generic scalar transport equation. We will now see how we can go and look at the solution of all the equations together.