Computational Fluid Dynamics

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Module No. # 03

Template for the numerical solution of the generic scalar transport equation

Lecture No. # 12

Topics

Need for analysis of a discretization scheme

Concepts of consistency, stability and convergence and the equivalence theorem of lax

Analysis for consistency

The example of the linear wave equation has shown that straightforward application of finite difference approximation to a given differential equation, does not necessarily give us proper results. We have seen, in fact, three different ways of discretizing the equation, and we have seen that in all cases, the solution has been not so exemplary. The solution that we got from the by applying the final differences approximations, and then doing the corresponding computation has not given us the correct solution or anything resembling the correct solution, except for one particular combinational parameters, where, they, the courant number was equal to 1. So, this should give us, this should tell us that they something more that we need to do in terms of what kind of approximations that we make before we can start solving them.

So, what it a, what it amounts to saying is that all though we can write down approximations, final difference approximation of given accuracy for a given derivative, not all approximations will give as proper solutions. There some which are better than some others. How can we say which is better and which not good, because in the general case, when we try to solve Navier-Stokes equations using this particular way, then we do not know the solution, and if we knew the solutions, we would not be doing the CFD solution at all.

So, if we do not know the solution and if your numerical method is likely to give rise to errors, then how can we accept such a solution method? Because how can we distinguish the errors from the real solution when we applied t to a practical case, where we do not know the solution and where we depend on the accuracy of the method to provide the correct solution? So, we need to have some method, some way of saying a formal way of saying that if we do this kind of approximations and if we follow this kind of procedure, then we will get the satisfactory solution. So, that kind of assurance necessary and that kind of assurance is what we going to discuss in this class.

So, we are looking at the general case of the scalar transport equation with a time dependent term, advection terms, diffusion term, source term, and we are looking at a specific idealization of that in form of the wave equation. And when we look at that, there seems to no reasons why our approximation should give a wrong answer, except for the fact that we have a truncation error; except for the fact that we are approximation the derivatives with finite difference approximation of first order or second order, and in the process of doing this approximation, we are neglecting certain terms and is it because of the combination of this approximation that exactly causing is.

So, this one question and the other question is that we have to ask is that is there a combination which will give us a proper solution? So, we have to look at what kind of combination and is it all question of accuracy or is there something more to this and is there any guarantee? For example, that we are solving the right equation is a solution - the computed solutions - not good because we are not solving the right equation or is it not good because we are solving it only approximately, because any numerical computation, any computer based solution with finite precision of accuracy is always going to be approximate. So, is it because of those kind of consideration? What is a guaranty that computed solutions will approach or will be equal to the exact solution of the governing equation? So, these are the question that we have in mind and these are the question that will try to address through formal procedure of the analysis of the discretized equation.

So, now, we can begin to understand, begin to see some points that we have to consider. Firstly, we are solving the partial differential equation in an approximate way, in the approximate way by writing a approximate form, of the, of the derivative and we solving it in a discrete way, we are not looking a solution which is continuous; we are looking at only discrete points and we are doing method of solutions which has finite precision.

So, we can see that the computed solution is different from the exact solution in three different ways - one is at instead of looking at phi of x t, we are looking at phi of x i t n. So, it is at a discrete point, and second thing is instead of getting a exact solution at x i t n, we only getting an approximate solution which satisfies only the discretized form or the approximate form of the governing equation. On top of that, we are getting a solution which is not even an exact solution of the approximate formal equation, because we are not doing the exact arithmetic, we have only finite precision arithmetic.

So, there are these kinds of errors, this kind of discrepancies may be playing have a good solution, because of which, we are getting a wrong answer. So, let us therefore, try to answer question related to this. So, this kind of analysis, this analysis of the condition of a computed solution approaching the exact solution partial differential equation is put up in three different stages using three conditions known as the consistency condition and the stability condition and the convergence condition.

When we say consistency condition, we are talking about is the discretized equation a very good replica of the actual partial differential equation. So, that is, can we claim that the discretized equation would approach the exact equation if we make a discretization very small? So, we are saying that we are not looking at phi of x y, we are looking at phi of x i t n. So, that is at discrete points.

So, if we had the capacity to make delta x very small delta and delta to very small, so that we are approaching almost a continuous variation of x i and t n. So, in that case, we can claim that our discretized equation, the approximate form of the equation that we are solving. Can we say that this approximate form will approach the partial differential equation? So, this is known as the consistency equation, consistency between the partial differential equation or the governing equation and the discretized equation.

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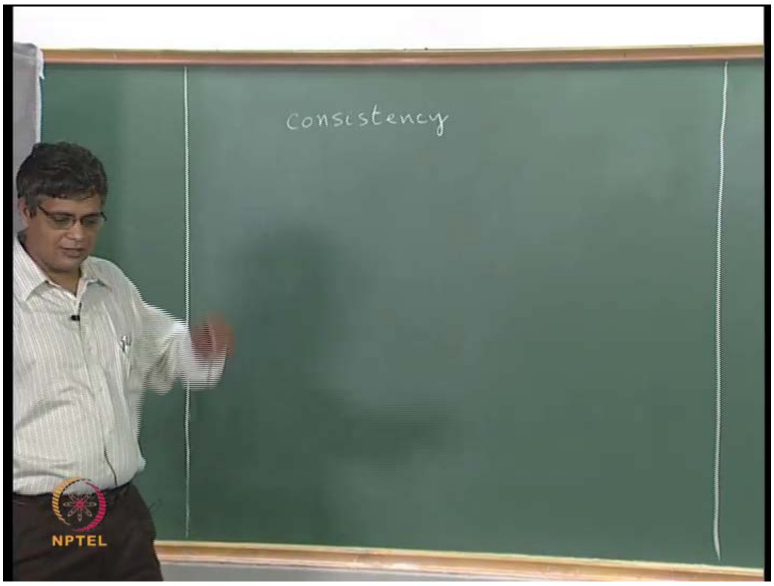

So, we will put this as consistency where we are looking at the assurance that the discretized equation, where is the same as partial differential equation in the limit as delta x and delta t tending to 0. The next question is about the solution method. If provided, we have this consistency condition satisfied; provided we can show that we are solving the correct equation which has the property of going to very small delta t and approaching the exact solution for very small delta t and delta x.

Now, this point is important because we saw in the examples, especially with the FTFS method and the FTCS method which is central in space methods. No matter how small that delta t was; we were still getting a wrong answer. So, even if those methods are such that even if we make delta t and delta x very small in the computed solution, even then the computed solution will not give us a proper solution, which compares favorably with the exact solution. So, it is not that if in all conditions if we can make delta t delta x and ah other discretization very small, we cannot guarantee that the solution would be automatically approach this.

So, that is why this query about whether or not are the discretized equation would approach the exact equation for very small delta x and delta t is a justified query. So, we will have to considerate that and that is why is consistency condition is. Now, once we have satisfied ourselves that we are solving an equation which is consistent, then the second question comes are we solving it properly? Are we solving it properly in the sense that does the discretization, does the discretized equation have a property of amplifying errors.

Again when we go back to the example of FTFS and CT and FTCS and even FTBS, we saw that there are, it, the solutions seems to be getting out of control, out of bounds. The expected variation of phi was between 0 and 1, and the computed variation of phi was something like plus or minus 400 within a few time steps, within 5 time steps in case of FTS method for certain conditions.

So, it seems to be amplifying errors, whereas, the true solution denotes something which is nicely being uniformly being carried forward in the x direction. So, the true solution of the governing equation is not going to does not have this feature of amplifying errors, but the discretized form, at least in some discretization seems to be having this nature of amplifying errors.

So, this brings us to the second condition stability. So, does the discretized equation have the property of stability, whereby errors that introduced from whatever reason; from finite precision arithmetic or from rounding of errors or from boundary connections or from truncation error, errors related to discretization, all this kind of errors or even simple typographically mistakes, those kind of things. Such errors do they have the property of getting amplified or is a schemes stable in such a way that those errors will be suppressed and attenuated, and finally, as time progresses we get to stable solutions.

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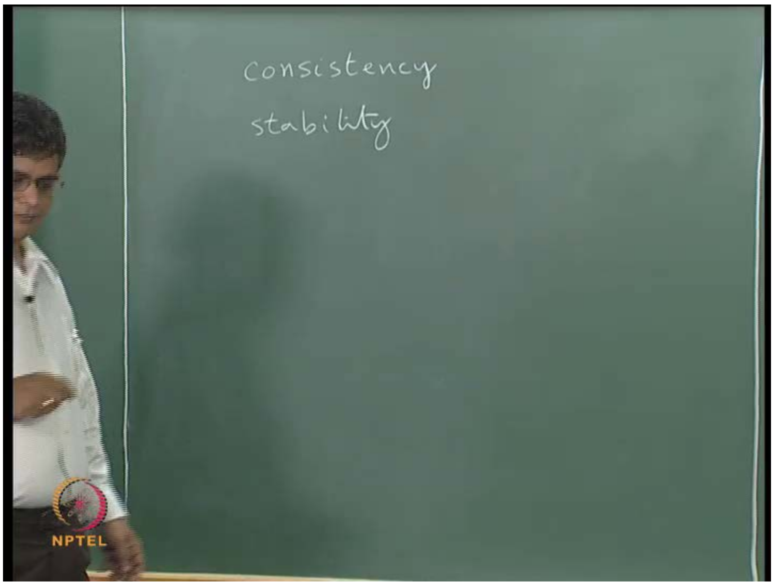

So, the second question then therefore is stability. So far what we have been looking at is whether the discretized equation is very good approximation of the partial differential equation, and we are looking at whether the discretized equation, the computed solution of the discretized equation has a property of amplifying or attenuating the errors. We have not said about the comparison between the computed solution and the exact solution. So, we are talking only about the method of solutions.

So far in the consistency and stability conditions, but now is the time for us to look at the equivalence between the computed solution and the exact solution of the governing equation at those discrete points at which we are looking at it. This condition where we are saying that the computed solution at the discrete points would match with the exact solution at the same discrete points is known as the convergences conditions, convergence.

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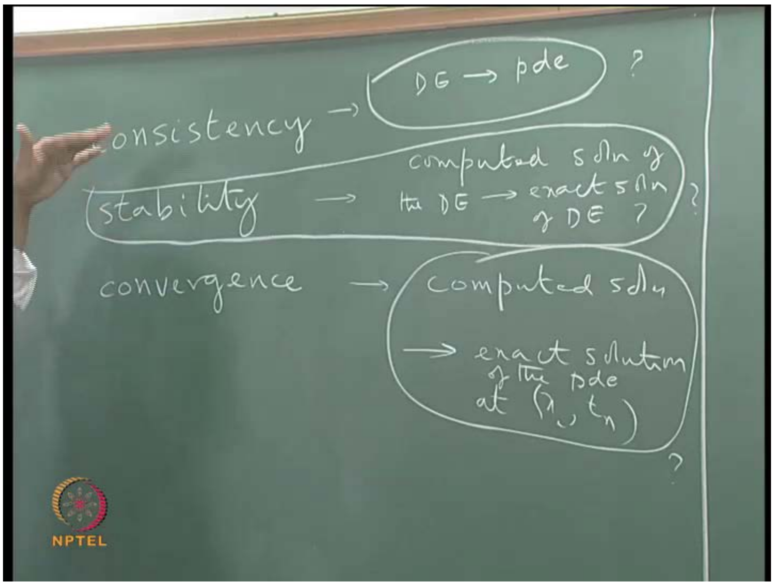

So, if you have scheme which is convergent, then we can say that the computed solution would approach the exact solution at x i t n. So, the convergence criteria is about the computed solution and the exact solution. Now, how is it different from stability? Stability is about, is only about the discretized equation. So, the computed solution of the discretized equation – DE: discretized equation - approaches the exact solution of the discretized solution. So, this stability condition refers to it poses the query on whether or not the computed solution with all those errors and amplification features; whether or not it would be an exact solution of this. And convergence is talking only about the computed solutions and the exact solution of the partial differential equation.

Consistency is not about solution, it is only about whether the discretized equation approaches the partial differential equation, the governing equation. So, there are three different conditions; we are addressing three different issues. Ultimately, we want the convergence; ultimately, we want the computed solution to match with the exact solution of the partial differential equation at the points x i and t n, but in order to get here, we have to pass these to hurdles. We to make sure that we are getting this computed solution in the right way by ensuring that we are solving the right kind of equation and the equation that we are solving has a right kind of stabilization or attenuation properties so that errors do not amp get, get, amplified and spoil the solution.

So, this condition, this triple criterion here will, if we can satisfy this condition, - the consistency condition and the stability condition and convergence condition - then in such a case we can say we can have confidence in our computed solutions. We can have confidence that the computed solution will approach the exact solution of the partial differential equation at those grid points we have computing. It does not mean the computed solution is error free, because of any computation, we must have some delta x and some delta t; so, that means that there is some discretization errors; there is an error of approximation that is there, but with this, if we satisfy the consistency condition and stability condition, then we know that this error is contained bounded; it does not get amplified and it does not gives us any problem.

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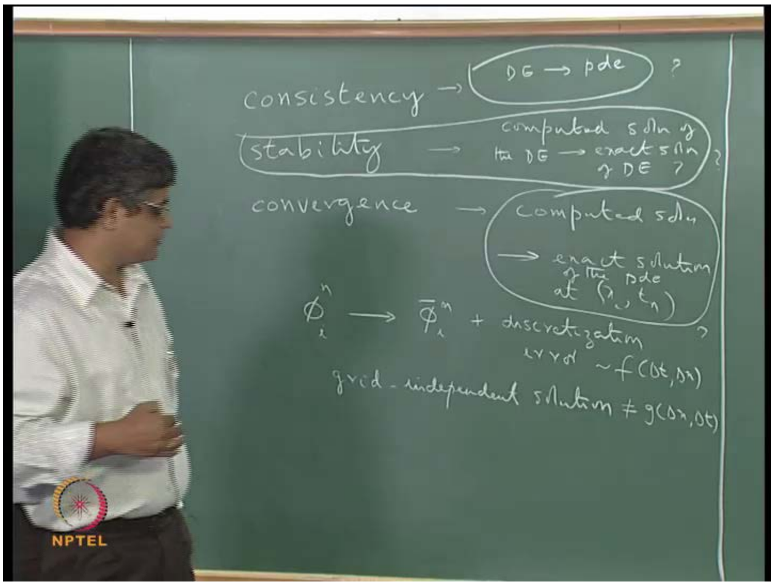

So, in that sense, if you are able to satisfy the convergences condition while simultaneously satisfying this. Then we can say that the computed solution at space point i and time point n will approach the exact solution which I am denoting by this, at that particular point subject to plus discretization error, or we can call this as truncation error, which depends only on delta t and delta x and these are in our control. Once we have computed solution which is not really sensitive to delta t and delta x in terms of delivering a proper solution through the stability, then this is in our control and we can reduce delta t and delta x to as low as possible as we want and then we can make our computed solution approach the exact solution.

So, and we can minimize the error as to as low as what we desire as it is required for the solution. So, in, in this, under this condition, we can say that a grid independent solution. So, what we mean by grid independent solution? A grid a solution which does not depend on either delta x or delta t so which does not vary with successive reduction of delta x and delta t.

So, let us say that we square dart and we have a 20 by 20 grid; so, that means that same length, we have 20 intervals in that so that gives you a certain delta x. You make it 40 by 40, you are reducing delta x by half; you make it 80 by 80, 160 by 160, 1000 by 1000, and what you will see is that as u make the grid finer and finer, obviously, the that computation cost increases because the matrix a that leads to the differential, equation, algebraic equations that becomes bigger and bigger, but if we did not have any contents on, on, the computing resources, then you can make it as high as possible and does you keep on increasing the grid size or the number of grid points, and as you keep on decreasing the delta x and delta t, then the parameter that is of interest to you. For example, you are looking at the pressure gradient, pressure drop in, in, that particular that for given flow rate, so, in a such a case, you will find that after some number of grid points, the computed pressure grading does not really change much to you reduce the spacing by another half; it does not give you any significant variation in the parameter of interest.

So, at that point, you can say that my computed solution which is of interest to me, which I want to get from my c f d calculation is no longer sensitive to the delta x and delta t. So, that is what I call as a grid independent solution. So, and grid independent obviously means not only delta x, but also delta t if you are looking at time dependent problem.

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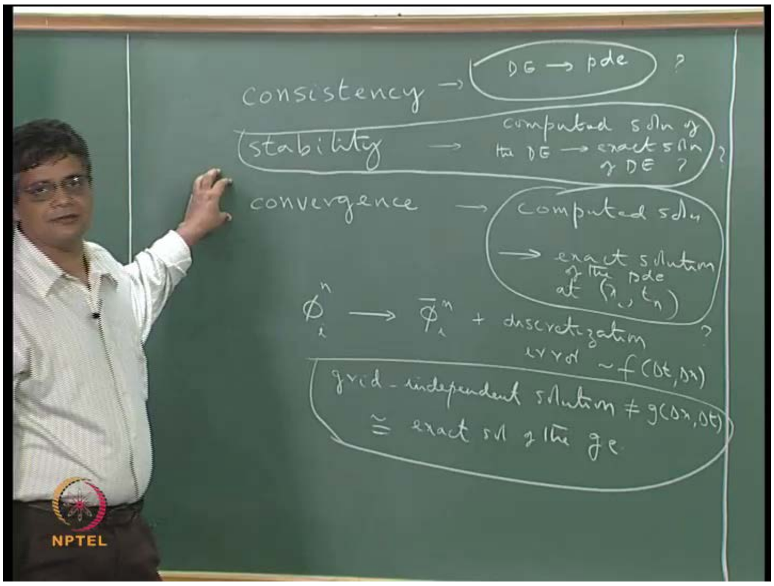

And we can say that grid independent solution is almost equal to the exact solution of the governing equation. So, the satisfaction of these things is as powerful as this; that if we are able to generate grid independent solution, then we can rest assured that it almost as good as the exact solution of the governing equation.

So, this is the kind of power, that is, there in these three conditions that if you are able to demonstrate the consistency, stability and convergence of a numerical solution procedure, then we can claim that the grid independent solution which we obtain, from the computed, from the procedure is, is, the exact solution of the governing equation, but the caveat is that if we want have this, this is the desired condition this the desired guarantee that we would like to have. In order to do this, we need to prove these three; these three are not related in a way.

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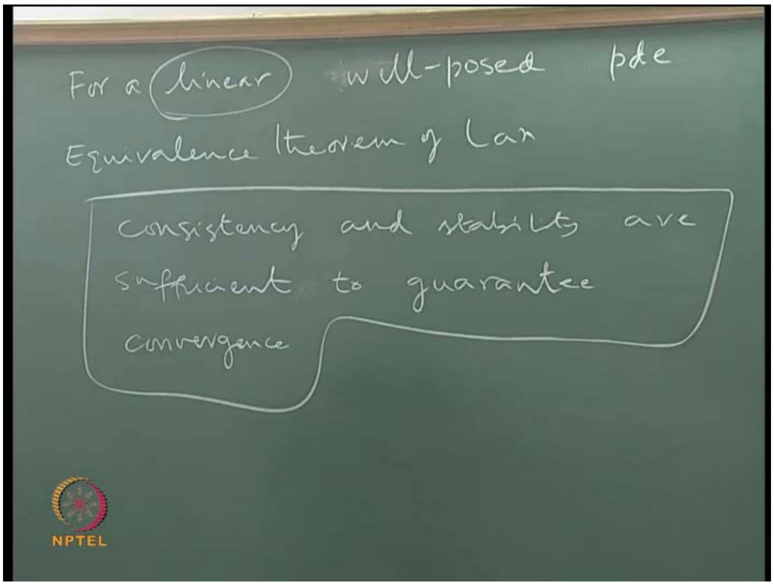

And these three for the case of linear problem, for a linear partial differential equation, then there is a theorem called the equivalence theorem of Lax. Lax is obviously a mathematician, and it is not the laxity on the part of Lax, he is actually made a very nice theorem, which is, for a linear well posed mathematical problem of in the form partial differential equation, the equivalence theorem says that consistency and stability are sufficient, sufficient, to guarantee convergence. This in a way bit of layman speak, but the point is that if we are able to show the consistency of a discretization and the stability, stability, of the computational form computed solution of this, then we can rest assured that we will have convergence condition is satisfied and convergence is guaranteed, and under those condition, we can say that the difference between the computed solution and the exact solution has a discretization error, which is, which goes to 0 in the limiting case as this thing goes to 0 and in the limiting case as the nth ending infinity.

So, in that senses, the two will approach. So, this is a very useful thing, but like all mathematical theorems, it is also not of a great practical value from the fact that this is limited only to a linear equation. So, when we have linear partial differential equation, then this theorem works. For a non-linear problem which is what we normally have in our c f d, then this theorem does not work; obviously, it is only for the linear thing,but in such a case, we have to do a local linearization; we can convert the non-linear problem into a linear problem, and then, for that linearized problem, we can check for consistency using this.

So, in that senses, it is, it is not something we can brush aside as being useless, but at the same time, it, it, tells us that for a under linearized condition, if you are able to check for consistency and stability, then we can get the guarantee of convergence, and while this theorem has those kind of limitations. Experience has shown that if you apply standard methods for the analysis, consistency and stability and if you follow these as practical guidelines for determining the parameters that we are going to employee like delta t and delta x, then in most of the cases, we do get a good solution, but it is not guaranteed and we also have to worry about other reasons for getting a non convert solution and solve it, but in general, this is a good point and it is a it is an indication of why and how problems may arise.

So, let us, let us take it for granted that this is a useful thing for us, useful pointer in terms of what kind of approximation that we have to, that we are allow to make for derivatives so as to get a proper solution, and let us investigate this further, and let us investigate how we can prove the consistency and stability of our discretization, because now the problem reduces to how we can demonstrate consistency. If there is method of which we can demonstrate the consistency of a method and, if we can, if they is procedure by which we can demonstrate the stability, then we can apply these methods to each discretization that we propose and see whether the discretization satisfies these conditions. If it does, then we can proceed with the solution procedure; otherwise, if it does not satisfy, in such a case we have look for alternate ways of discretization.

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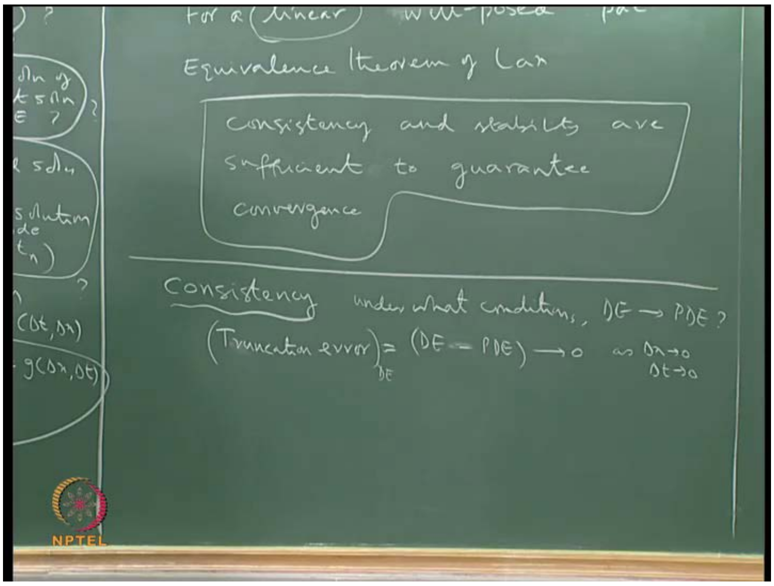

So, now, let us consider how we can do consistency. We have said consistency is a condition which, which, tries to look at whether or not the discretizated equation will approach partial difference equation under the condition of delta x and delta t tending to 0 and any other. If you have a three dimension equation, then delta y tending 0 and delta z tending to 0 and so on. And so, we are looking at, we have to see under what conditions the discretized equation tends to the partial differential equation.

So, when you pose a question like this, then we can see, we can start by examining what is the different between the discretized equation and partial differential equations. We know that the, the, two differ only by the fact that the discretized equation is in approximation of the partial derivatives of the derivatives in appearing in the equation and each derivative has approximations which is given by the Taylor series approximation.

So, if you go back to the Taylor series expansion and then look at error that as arisen in making particular approximation, then we can say that the difference between the discretized differential equation and partial differential equations is the truncation error, that is, resulting from the approximation of each derivative that appears in the partial differential equation.

So, if you have five derivatives, then each derivatives is approximated in a particular way using a finite different approximation and that each derivatives will have its own truncation error, and the combination of the all these truncation errors is the difference between the discretized equation partial differential equation.

So, if we can show that the truncation error goes to 0 as delta t delta x delta y delta z tend to 0, then we can claim that the discretized equation will approach partial differential equations. So, we can say that the truncation error we, we, are saying that truncation error, we are calling it truncation error, because in writing a finite difference approximation, we are truncating the Taylor series which has infinite number of terms to first three terms or first four terms and so on. So, rest of the terms is neglected; so, the neglected part is a truncation error.

So, the truncation error of the discretized equation between the discretized equation and partial differential equation which we are calling as say DE minus PDE; this should tend to 0 as delta x tends to 0 and delta d tends to 0 in that case.

So, the consistency condition does not say that the two must be equal, it only says that it has the property that the difference will tends to 0 in the limit as delta x tends to 0 delta t tends to 0. We are not claiming that there is a, there will not be any difference. We are only saying that solution method as the capability of reducing the error to as low as is required as is possible. So, it only that the difference will tend to 0 as delta x is made to go to 0 and delta t is made to go to 0. So, then in order to verify the consistency, we have look at the truncation error.

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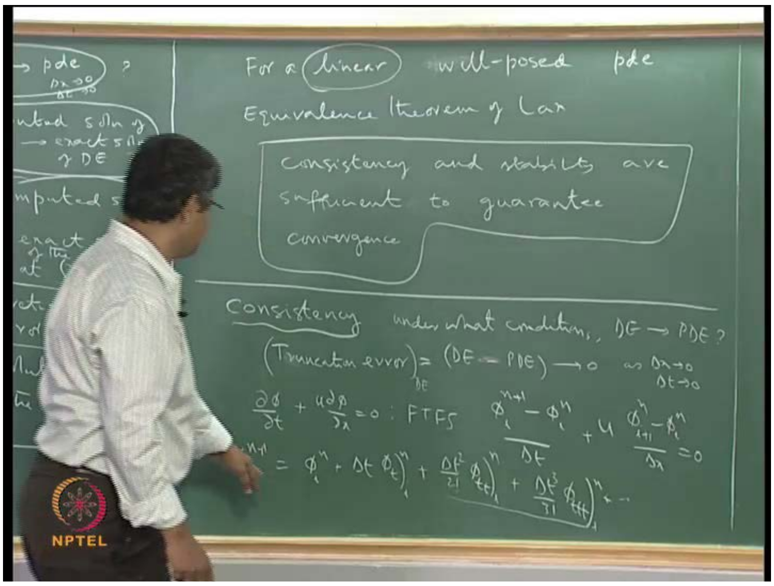

So, let us do that for the simple case, the method is very general. We have considered dou phi by dou t plus u dou phi i dou x equal to 0, and in the, let us consider case of FTFS, that is, forward in time and forward in space. Then why would be consider this? Because this has given as an immediately non sequential answer for whatever values of delta t and whatever values of Courant number that we have taken, and not only that, it has really blown up very fast as compared to FTBS.

So, we can take this as in example and see whether the FTFS approximation for this particular equation has the, will satisfied the conditional consistency.

So, when we write the FTFS approximation, we are writing this as phi i and plus 1 minus phi i n by delta t plus u is being taken as constant. In the example, we have taken it as a 1 meter per second phi i plus 1 n minus phi i n by delta x equal to 0. Now, this is an approximation, first order approximation and that results from the expansion of phi i n plus 1 as phi i n plus delta t phi t at i and n - where phi subscript t is actually dou phi dou t. So, delta t square by factorial 2 phi t t indicating the second derivative delta the delta t t cubed by factorial 3 phi t t t i and n so on.

So, when we make use of the first three terms, first two terms in this, then we get approximation for phi t which is dou phi by dou t as as this particular thing. So, we can say that the truncation error resulting from this approximation is this whole thing, and similarly, the truncation error resulting from this approximation is the same very similar to this except that instead delta x, we have delta t; we have delta x, and instead of derivative with respect to time, we have derivative with respect to space.

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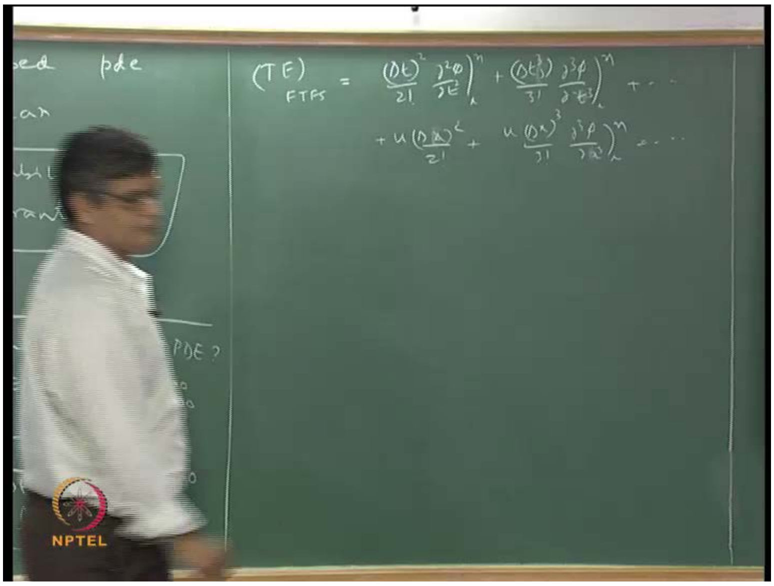

So, we can say that the truncation error from the FTFS scheme for this will be given by delta t square by factorial 2 dou square phi dou t square at i n plus delta t cubed by factorial 3 coming from, we, if we take the first two terms in the time derivatives plus u delta x square by factorial 2 plus u delta x cubed by factorial 3 dou x cubed i n plus so on. So, this is the difference between the discretized equation and the partial differential equation with either plus or minus depending on whether we do DE minus PDE or PDE minus DE, but it varies like this, and in this, u is constant. And we are looking what happens to the truncation error as in the limit as delta x tending to 0 and delta y tending to 0.

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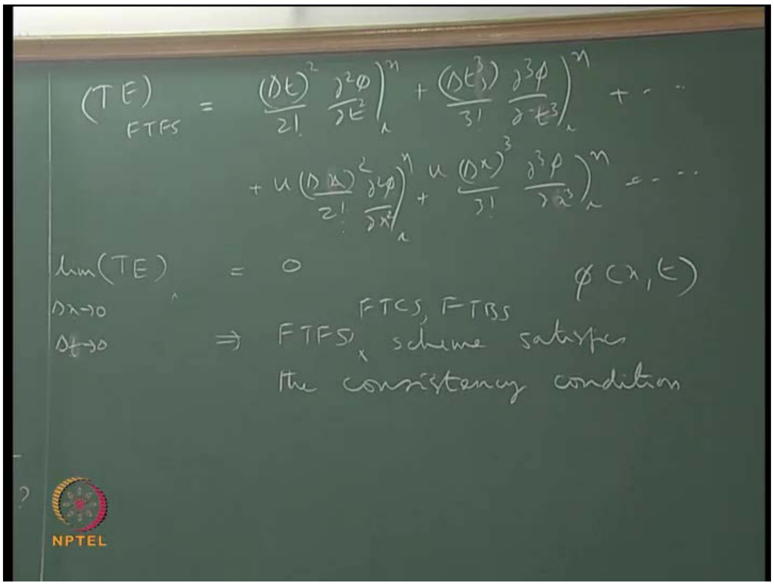

So, we can see that, we can look at each term here delta t goes to 0 as this goes to 0; this thing goes to 0; this thing goes to 0 and these also goes to 0, go to 0 so that in the limiting case of delta x tending to 0 and delta t tending to 0, the truncation error goes to 0, and we can see that if we make delta t very small, then this term which is constant, see for a given functional variation of a phi with respective t and x, then the derivative - the second derivative - at a particular point with respect to time and third derivative and and, and the space derivatives these have a fixed value for a given phi of x t. So, these will not change within the exact case with changes delta x and delta t.

So, once these are fixed, then this thing can made as small as possible as delta t and delta x tends to 0 and they can be made go to 0 for very very small values within machine error and so on. So, we can say that the limit of the truncation error in these cases is equal to 0; so, that means that the FTFS scheme satisfies the consistency condition.

So, we can see that FTFS scheme has a truncation error which reduces to 0, which goes to in the limiting case delta x and delta t tending to 0, and we can also show similarly that FTCS scheme and FTBS scheme all the three schemes that we have employed for this equation they, they go to, they satisfy the consistency condition.

So, at least from these examples, we can confidently say that the approximations that we have put here are consistent and we can also confidently say that consistency condition alone does not guarantee us a proper solutions, because with even though the satisfies the consistency condition, we saw that the computed solution seemed to be not so good. So, we have to look at the second aspect of a stability to see whether it satisfies the stability condition, but when look at the this consistency condition like this and we see that it almost looks like every schemes we consider must obviously satisfy the consistency condition.

So, is it that consistency condition superfluous by the very fact that way using the Taylor series expansion for writing the approximations and using the same analysis for looking at consistency is it superfluous or other cases were the equations are not consistent.

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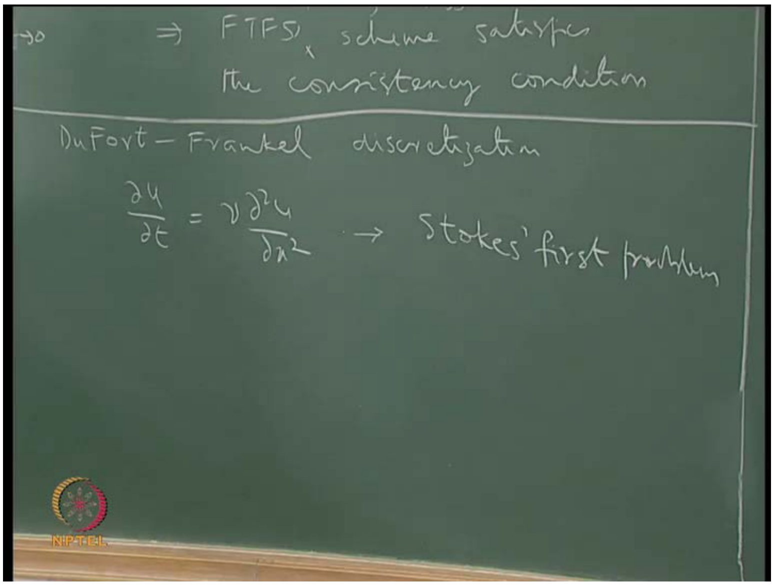

The most famous of this those inconsistent cases is probably the Dufort Frankel scheme discretization for a different problem, for a problem which is which can be written as dou u by dou t equal to nu dou square u and dou x square. So, this is for written for different cases, but in this case u here represents the velocity and t is obviously time and nu can kinematic velocity. One can see this as a fluid mechanics problem and this is also known as the Stokes first problem which arises, for example, that when we consider the case of infinite expanse of fluid, that is, a liquid and we have an infinitely long plate thin plat which is submerged in this fluid and its horizontal at a particular depth, and at time equal to 0, you suddenly said this infinitely long wide thin plat in to motion in the horizontal direction, that is, in its length direction at uniform velocity of a capital U.

So, because the fluid that we are considering is a real fluid with real viscosity and because of the no slip condition, the fluid which is adjacent to the plate will start moving. So, we will find that because of the plate movement in this direction, the fluid which is above it also starts moving, above it also starts moving and above it also starts moving. So, the variation of the development to the velocity profile with respect to time is encapsulated in this equation is given by this equation and this is the known as Stokes first problem.

So, this is not any artificial problem, this is a problem which is quite a fluid mechanic problem, and not only that, we can see that, this is, this is also in a way a subset of the generic scalar equation. If you at substitute u for phi here like this, then it becomes like the accumulation diffusion problem. So, the transient diffusion problem and it is also a transient fluid conduction problem which is given by the same equation. There are number of cases which are also given by, by this and it also simple equation.

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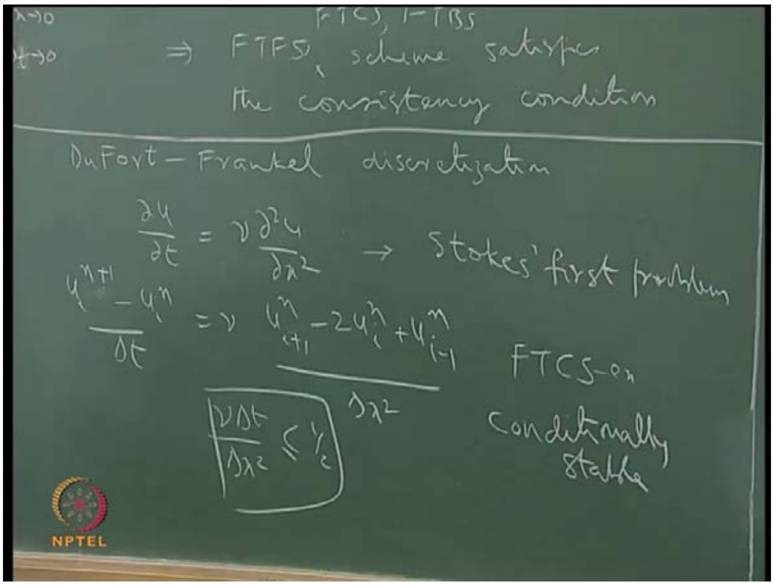

Now, we can write many kinds of finite difference approximations for this and we can for example, write u i n plus 1 minus u i n by delta t equal to new times u i plus 1 n minus two u i n plus u i minus n by delta x square. So, this, this is forward in time and center in space.

We know that this is and this is explicit. So, our first intuitive way of writing a final deferential approximation is like this, because this is self starting method as part initial condition we know u i n. So, we can compute the u i n plus 1, and this is first order accurate but we are ok with that first order accurate given that this is a self staring problem and we can start implementing it straight away and it is quiet easy have a second order approximation for this without too much of complexity.

So, this is first order in time and second order in a space, and it gives us a nice and easy expression in terms of an explicit method for the calculation of u i n plus 1. So, the solutions seems to be nice and smooth in terms of implementing, but when we actually try it and having seen the difficulty with another simple linear wave equation that we have considered, we should be suspicious as to whether or not to get a proper solution, and in fact, one will show later on that the solution that will get from this FTCS explicit equation for this is only conditionally stable; in the sense that only for certain, certain, range of values of delta t and delta x can we hope to get a good solution, and in other cases, we will not get a good solution.

So, in that sense, this is a only we can, which only conditionally stable and the stability condition will be that nu delta t by delta x square must be less than or equal to half. So, for a given new which is the kinematic viscosity and delta x for a given grid, delta t must be less than the value given by this. If it is more than that, then we will get a proper solution; so, that means that we are limited in how fast we can go forward in time because the time slip is limited by this.

So, it would be nice to have something which is unconditionally stable, something which will allow us to choose any values of delta t and delta x and it will also be nice to have not a first order accurate, but is second order accurate approximation (( )). So, with that, if you word to say that let me not take a first order accurate thing and I will make it second order accurate.

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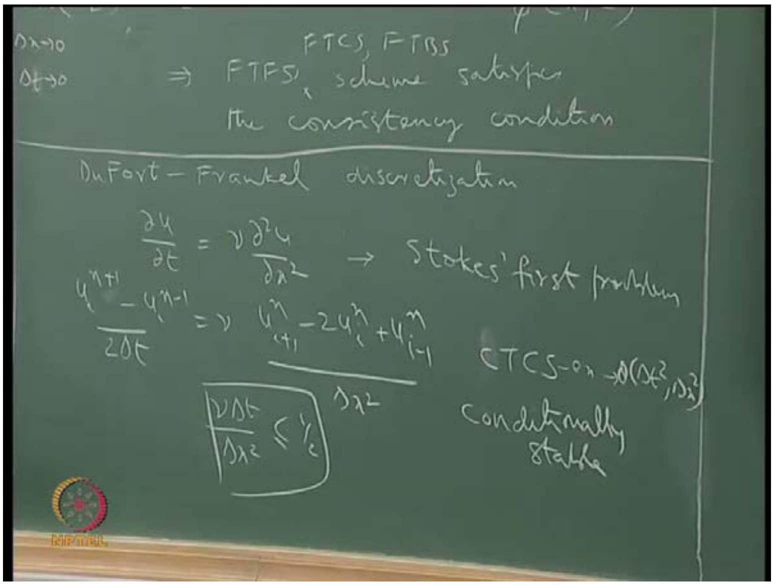

Then I can, for example, I can write this as u i n plus 1 minus u i n minus 1 by 2 delta t it will make this as central in time and central in space explicit, still explicit. So, this is second order accurate in time and second order accurate in space.

So, that is much better than FTCS scheme because that is only first order accurate in time, but if you do this, if try this, then you will find that it is no longer conditionally stable. In fact, it is unconditionally stable; that means that no matter how small delta t and delta x that you choose, you will always get ah an unstable E solution and we have actually got something like that in the case of FTFS scheme for the linear wave equation.

So, we have seen that kind of thing. So, a straight forward a naive approach to improving the accuracy, of, of a scheme which seems to be mathematically ok. It has actually is spoiled the conditional stable condition and is made it into unconditionally unstable condition.

So, if you want to have more higher accuracy and so on, if you want second order accurate thing, then this is obviously not correct. So, this is where Dufort and Frankel have stepped in and they have said that let us not go for a fully, if you look at the strokes first problem, FTFS scheme which is only first order accurate is conditionally stable; central in time and central in space is second order accurate, but it has become unconditionally unstable. Therefore, if you want increase the accuracy of time, then we are having to compromise on the overall solution itself. This is where Dufort and Frankel have stepped in 1953 and proposed a small modification to the way this CTC scheme is implemented. We have a central in time, central in space as given by u i of n plus 1 u i of n minus 1 2 delta t, which makes it central in time and central in space, which is given like this.

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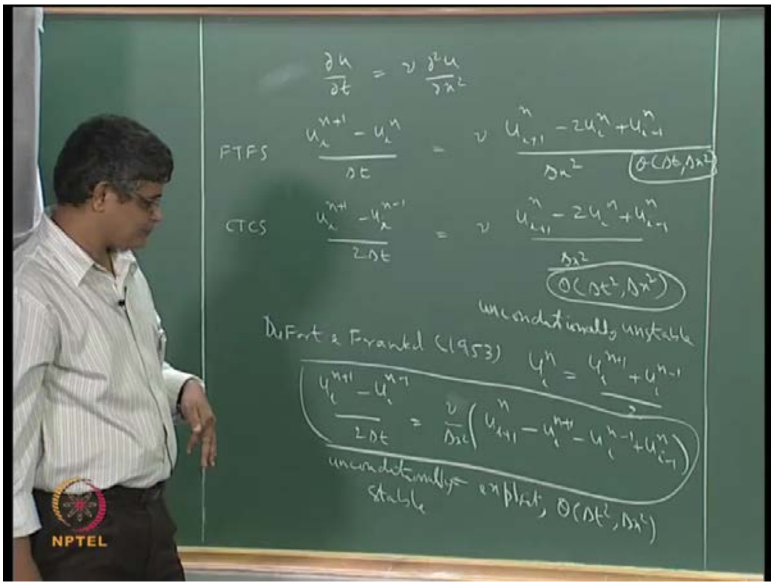

And this is unconditionally unstable; so, we cannot hope to get a reasonable solution with this. So, what Dufort and Frankel have suggested in 1953 was to replace this u i n as u i n plus 1 plus u i n minus 1 by 2.

So, it is taken as the average of the previous time step in the coming time step, and then, if you now substitute this into this, then this two and this two will cancel out and we will have u i n plus 1 minus u i n minus 1 by 2 delta t equal to alpha nu by delta x square times u i plus 1 n minus u i in plus 1 minus u i n minus 1 plus u i minus 1 n. So, this is another way of calculating this u i n plus 1, and this is a second order of approximation, it being a central order approximation. So, the overall scheme has not compromised on the second order accuracy of either delta t or delta x. So, the overall scheme is delta t square and delta x square.

So, we have retain the second order accurate nature of this particular discretization, and not only that, if you look at the terms here, u i n plus 1 is appearing here; otherwise, it is n minus 1 which is known; u i plus 1 n which is known; u i is value that we have actually seeking.

So, in a way this is nothing new and you have n minus 1 here and n here. So, in that sense, this can be evaluated explicitly. So, not only has this retained the character of second order approximation here, this is also an explicit method which is therefore easy to solve, which is easy to compute and the difference that this makes as compared to the standard CTS schemes is, whereas, this is unconditionally unstable. This is unconditionally stable; that means that you are free to choose whatever value of delta t and delta x that we want to get and it will not amplify errors, which is very rare for an explicit method.

So, in that sense, this is unconditionally stable and, we are, we have gotten rid of the conditional stable restriction with the FTFC scheme and we have completely overturned this unconditionally unstable thing while retaining the same ease computation in the form of explicitness and the same accuracy of computation in this. So, in that sense, this simple modification that is made here is a brilliant modification and it has gives us a lot of advantages. Therefore, it is always nice to explore what approximations we can make in order to improve the solution.

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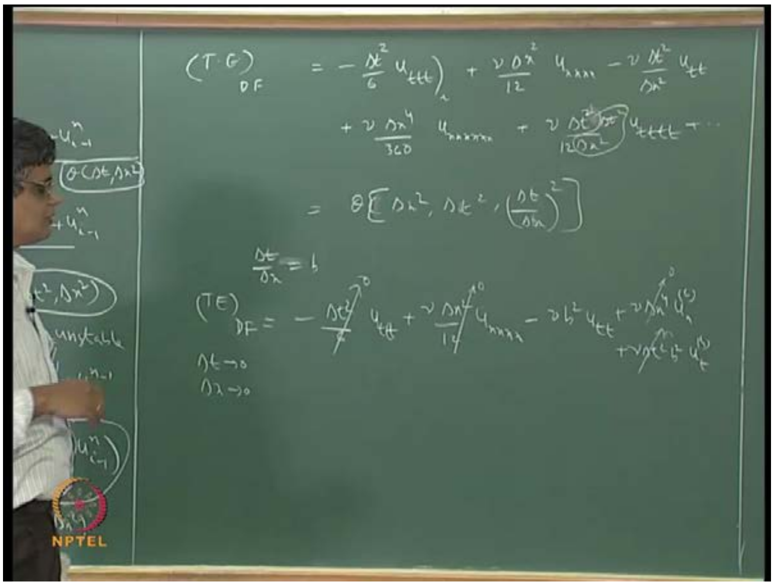

But it does not stop there; the story does not stop there. If we were to look at the truncation error of the Dufort Frankel error, we will see that it is of this particular form, the third derivative times delta t square plus nu delta x square by 12 u x x x x, that is a fourth derivative minus nu delta t square by delta x square times u t t all derivatives are evaluated it a x plus nu delta x 4 by 360 times u x x x x x x, that is a sixth derivative plus nu delta t square t 4 by delta x square by 12 delta x square fourth derivative of time and so on like this. The leading terms in the truncation error or second order thing here and delta x square here and this is delta t by delta x square.

So, we can say that order of accuracy of the Dufort Frankel scheme delta x square and delta t square and delta x by delta t delta t by delta x whole square. Will this goes to 0? As a delta t and delta x go to 0 is not necessary that as delta x and delta t tend to 0, then these two terms goes to 0, but this term did not go to 0. For example, they can be fixed ratio in which these two are varying, while E both of them are going to 0, this need not goes to 0, and therefore, this thing will not go to 0.

So, whereas, this goes to 0 and this also written as delta t square by delta x square like this will also go to 0. So, for example, if you say that delta t by delta x is a constant b, then we can write the truncation error of Dufort Frankel scheme as minus delta t square by 6 derivative plus nu delta x square by 12 fourth derivative minus nu b square second derivative of time plus nu delta x 4 sixth derivative with respect to x plus nu delta t square b square tends fourth derivative with respect time.

So, in this as delta t tends to 0 and delta x tends to 0, this will be 0; this will be 0; even this will be 0, because even though b is constant, this is 0 and this is 0 and so on. So, the rest of terms will also go to 0, but we are left with is this particular term. So, from that point of view, the Dufort Frankel scheme does not have a truncation error which goes to 0, but which goes to nu tends and b square tends u delta t u second derivative of u so that the Dufort Frankel scheme is an approximation not of this, but this equation dou u by dou t plus nu b square dou square u by dou t square equal to new dou square u square by d x square.

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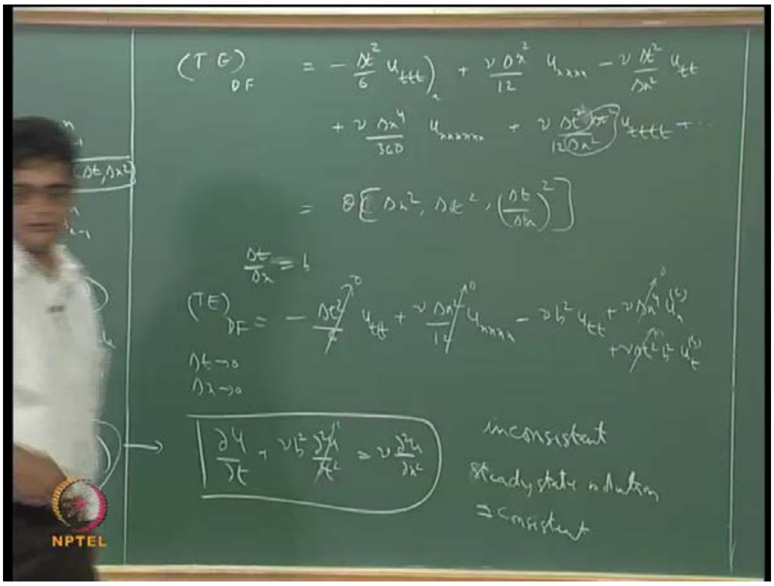

So, the Dufort Frankel scheme is a proper approximation of this, this, particular equation probably this equation, and therefore, it approaches this partial difference equation and not this partial difference equation.

So, in that sense, it is inconsistent. So, the Dufort Frankel approximation with all the desirable features in terms getting as second order accurate solutions in an easy way and without any restriction in delta t and delta x is actually inconsistent, but what we can see also is that it is consistent when you are looking at time dependent solution. If we are looking for a steady state solution, this term will go to 0 anyway.

So, the study state solution obtained by, by, this method will be consistent solution. So, in that sense, if you are looking at a study state solution from this transient solution which is very commonly done in CFD solutions, because that is probably one of the ways overcoming the inherent nonlinearity contained in the Navier-Stokes equations. So, in such a case, the final solution is not arising from an inconsistent discretization, but the way through that steady state solution is definitely going through an inconsistent approximation of the governing equation. So, whatever error that arises from, from, this term which depends on what the value of b is with respect to and what the value of second derivative with respect to these other terms.

So, depending on the magnitude of this as opposed to the magnitude of these two at the particular time and space will actually determine the accuracy of the time dependent solution. So, even though it has all desirable features, the solution that is the transient solutions that we get from this need not necessarily be conversing towards the exact solution of this partial differential equation. So, that is what we mean by lack of consistency.

So, it is not that every scheme that we do is consistent and it is also that arbitrary approximation that we make. For example, say that u i n is a average of a the two things here can lead to inconsistent solutions, inconsistence formulation, and we have to be aware of that; we have to do consistency of our governing discretization, discretized equation. In order to see that in the limiting case, the assurance that we are actually solving an equation which resembles closely the partial deferential equation is very important.