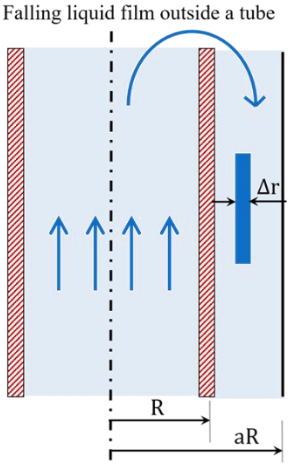
Momentum Transfer in Fluids
Prof. Sunando DasGupta
Department of Chemical Engineering
IIT Kharagpur
Week-02
Lecture-09

Welcome, once again. We will continue our lectures on Shell Momentum Balance and this would probably be the last class on shell momentum balance. In the next class I will show you the limitations of shell momentum balance. So far, we were dealing with very simple geometries in which there could be flow through a tube or flow along an inclined plate or you can have the last one in which we have initiated Couette flow and pressure gradient driven flow, all in a very compact a very simple situation where the geometry is easy to visualize. Because unless you cannot visualize the situation geometry completely you would not be able to choose the shell and the boundary conditions would also probably have to be looked at carefully. So, the prerequisite for choosing the shell is to have a very clear picture of what is happening in such a system. And we will see with increasing complexity of the geometry, this will become more and more difficult.

And secondly, our velocity in all the problems that we have discussed so far are, our velocities are a function of one of the coordinates. It could either be a function of x, y, r etc. But what if the velocity is a function of both x and y. It is a two-dimensional flow. What if it is a transient case in which velocity apart from being a function of y is also a function of x. For such situations, a simple shell momentum balance will not suffice.

So, we will see what could be the proper methodical a structured way to deal with such problems. So, the first problem that we deal with today is essentially one such situation in which you have flow of a liquid film outside of a tube. So, this part is the, this is the wall of the red crossed surfaces or the walls of the tube. The liquid travels in the upward direction. So, you have a pressure gradient which forces the liquid to move from the bottom of the tube, at toward near r equals 0 towards the top. So, the flow between r equals 0 that means, at the centre line and r equals capital R is taking place inside the tube.

Falling liquid film outside a tube



Whereas, when the liquid reaches the top of the tube, obviously it is going to spill over. So, it is going to spill and it is going to fall along the walls of the tube, along the outer walls of the tube. So, it is a unique situation because the moment it crosses the boundary at the top then it is a freely falling liquid. There is no imposed pressure gradient on this falling film. The pressure gradient is inside between r equals 0 to r equals capital R, where you need a pressure gradient in order to force the liquid to reach the top.

But once it reaches the top and starts its downward journey along the outer wall of the pipe, it is a freely falling liquid. And we need to figure out what is going to be the velocity distribution in this region, in the falling film which is located between capital R at this point and A r what is going to be the velocity distribution. We have done similar problems when there was flow inside a tube upon the action of gravity and an imposed pressure gradient. And what we have found in our previous class that it would give rise to a parabolic distribution of velocity inside the tube and it would, the volumetric flow rate which was calculated has given us the well-known Hagen Poiseuille equation, but this case is different. In this case, it is a freely falling liquid on the outside of the tube.

So, I guess the geometry of the problem is clear to you. So, it is as if the fluid starts coming from the bottom, reaches the top and then it is going to come along the sides of this. So, what is the imaginary shell that we are going to choose? Now, we can see on the outside of the tube in the falling film, the velocity is simply the nonzero component of velocity is v_z. So, v_z the downward acting velocity and this is the direction of z. So, z downwards is the is the positive direction.

So, v_z is the only nonzero component of the velocity, there is no v_r, but this requires some sort of an assumption over here. The with the region where it spills over near this region, the velocity profile is not going to be exactly the way I have drawn over here. There is going to be some 2-dimensional effect there would be some v, some v_r and the situation quickly stabilizes and it becomes a 1 D flow, but very near to the top we need to understand that the velocity is not going to be 1 dimensional. It could have a v_r component as well, but we are not considering that. We are going to analyse the situation where the velocity is fully developed, maybe after some region where you have only v_z and no v_r. And this kind of stabilization will come quickly if we are dealing with a liquid with a high viscosity.

So, for that case the order come is established fairly quickly and except for a small region near the top, the analysis that we are going to present would be valid everywhere beyond that region. So, this cautionary statement I wanted to explain to you. Now, since here v_z is a function of r only, in the r direction only. So, therefore, the smaller dimension according to our convention the smaller dimension should be equal to delta r. You can take any length of this imaginary shell having a thickness of delta r that is unimportant as long as our system does not get into the top portion where there could be 2 dimensional currents.

So, it is like a, it is if this is the tube through which the liquid is spilling over. I am thinking about just a clamp sleeve kind of thing which in which encompasses the whole tube and whose thickness is equal to delta r and this is at r, this point is at r. So, this point is at r plus delta r and it could be of any length L and across which we are going to write our shell momentum balance. So, this is what it looks like, r and r plus delta r. We have flow in at the top, flow out at the bottom. So, convective momentum comes in from the top, convective momentum out from the top and here I realize that v_z is a function of r only, v_z is not a function of z. So, it is fully developed flow in which the velocity is a function of r only.

So, if that is the case then v_z at z equals 0 must be equal to v_z at z equals L. So, under these conditions we know that the convective, net convective transport of momentum into the control volume would be equal to 0. So, the governing equation is again, once again the difference equation is, in minus out, momentum in, rate of momentum in minus rate of momentum out plus sum of all forces acting on the control volume would be equal to 0. v_z is not a function of z and of course, v_r v_θ etc. It is a cylindrical coordinate system. So, v_r v_θ would be equal to 0. So, the net convective momentum, as I said it is also going to be 0 since $v_z$is not a function of z.

So, what I get then is the shear stress and note the subscript of τ once again. I am stressing it again and again to impress upon you that it is very important that you realize which component of momentum is getting transported in which direction due to the molecular transport mechanism. So, it is the z component of momentum and since there is a variation in the, z component variation of velocity with variation in the z component of momentum. So, it is getting transported in the r direction. So, that is what we have decided. Now, what is this, what is the area on which this τ is acting on.

$$\tau_{rz}|_r 2\pi rL - \tau_{rz}|_{r+\Delta r} 2\pi(r+\Delta r)L + 2\pi\, r\, \Delta r\, L\rho g = 0$$

So, in term this one in term is simply going to be over here twice pi r L, that is the area, twice pi r times L is the area, this is shear stress, force per unit area and it goes out at r plus delta r, the convention is always it comes at smaller value leaves at higher value, that is the convention that is followed throughout this course. So, shear stress, the molecular momentum in, conductive momentum in minus conductive momentum out plus the only force acting in this case, it is as I said it is freely falling liquid. So, there is no imposed pressure gradient. So, only force which is acting on it is a gravity. So, these beings the volume this makes it mass the ρ makes it mass and it is simply mg.

$$\frac{d(r\tau_{rz})}{dr} = \rho g r$$

$$\tau_{rz} = \frac{\rho g r}{2} + \frac{C_1}{r}$$

So, this is my governing, this is my difference equation. So, what I do is I divide both sides by delta r take in the limit when delta r approaches 0, use the definition of the first derivative and what you get is d/dr of r τ_{rz} is equal to ρgr. Now, this then becomes my, I have to evaluate the first constant of integration C1, but how do I do this. If you recall in the case of flow inside a cylinder we have evaluated C1 by saying that at r equal to 0, τ_{rz} cannot be undefined. Which means which requires that C1 has to be equal to 0. But if you look at this problem, the governing equation which you have obtained in this case is valid for the region outside of the tube. It is not valid for any point inside the tube. So, the domain of applicability of the governing equation is to be, is to be studied carefully which simply tells you that I cannot use the condition at r equal to 0.

Whatever happens for any r, small r less than capital R, is beyond the domain of applicability of this governing equation. So, I have to figure out something else to evaluate C1. The next is, I can, what I can say is that at r equals capital R, the velocity of the fluid is going to be 0 which is just a no slip condition. So, I could use no slip condition. I cannot definitely use r equals 0 condition because that is not where my equation is going to be valid. And then when you think of the other end that means, over here at r equals A r, I have a liquid on this side and a vapor or air on the other side.

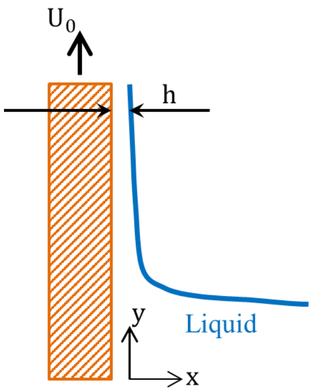
$$r = R,\ v_z = 0, \text{No Slip}$$

$$r = aR,\ \tau = \frac{dv_z}{dr} = 0, \text{No Shear}$$

So, it is a liquid air interface. If it is a liquid air interface then at r equals A r τ, which τ is equal to 0 and τ, the shear stress being equal to 0 would give me dv_z/dr at that point would be equal to 0. So, at r equals A r τ or dv_z/dr would be equal to 0. So, these are the two conditions no slip and no shear which would be applicable in this case. And once you assume that it is a Newtonian fluid, substitute that in here use the two boundary conditions.

$$v_z = \frac{\rho g R^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2 + 2a^2 \ln\left(\frac{r}{R}\right)\right]$$

This is the expression for velocity that you should get for a flow outside of a tube, where the film is formed and the downward moving film is going to have a no pressure gradient condition. Only gravity situation, and the velocity is, you cannot say that the velocity is going to be parabolic. So, it is going to be a complex function of r. So, here this is an example where you see that you have to mentally first imagine what is happening, what kind of a shell you are going to choose, what is the domain of applicability of the governing equation that you have obtained and you can yourself see that the situation is more complex than the ones that we have analysed before. So, this concludes one of the problems that I wanted to discuss. The second problem that I have is something which is industrially very relevant.



An upward moving belt, with a velocity equal to Uo , drags a viscous solution along with it creating a thin film of thickness h, as shown in the figure. Obtain an expression for the liquid velocity in the film, in terms of the relevant parameters.

Now, many of you have, I mean many of you are aware that when you want to coat something on a solid surface, a liquid or a solid surface, you dip the solid surface and then you pull it up. As you pull it up there would be a small thickness of the liquid made which will cling to the surface. So, you pull this up and then you dry it such that the surface is, being going to be coated with the liquid. So, this is very relevant in photographic film and other coating processes where this process, the dip coating is used in order to cover a solid with a liquid which would impart certain property, a photosensitive or other property to the surface. So, the situation here. the picture here depicts one such solid which is being pulled through a liquid bath and it moves up.

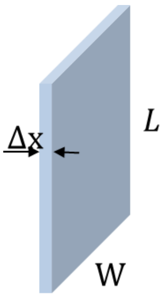
As it is moving up, it is going to drag some amount of liquid along with it. Now, if you think carefully, you would understand that the flow, the liquid very close to the surface will move along with the solid. Whereas, the fluid slightly near the interface will probably try to move down due to the application of gravity. And a balance between the two will leave a constant thickness on the plate, constant thickness of the liquid film on the plate. So, what are the factors intuitively if you think, what are the factors in which this would depend on, this is what I call about the relevant parameters.

Of course, the operational parameter, it will depend on the value of the velocity imposed on the solid. So, it is going to be a function of u_0. It is also going to be a function of the liquid property which is which is obviously, the most important one being the viscosity. And it is also going to be a function of whatever forces that are present in this system. There is no pressure gradient in this system.

It is just a freely upward moving film with part of it, which is moving downward resulting in a constant thickness of h, at some sections of the film. And so, once again when the solid starts to leave the liquid surface, there is a region which is not going to be stable, where the flow is not going to be one dimensional. Like the problem we discussed in the last class, where the liquid spills over from the top of a tube near that spilling over region, the flow is going to be complex, it is going to be two dimensional or more than one dimensional. Similarly, over here also there could be disturbances present in the region that I have circled over there. But in most of the cases, the viscosity brings order back to the system.

And for coating a solid with a liquid, one of the major parts, important parameter or property of a coating is that it is a high viscous material. So, we can safely assume that from this point onwards that it is one dimensional flow and we will restrict ourselves to that analysis of the two-dimensional analysis of the one-dimensional region only. The region where the flow is going to be one dimensional. And we understand that according to the coordinate system that we have chosen that is going to be the only nonzero component is going to be v_y and this v_y is going to be a function of x only, it is not going to be a function of either in either y or z which is the width of the film. And we will assume that the width of the film is too large as compared to the thickness of the ultimate film h which remains on the surface.

So, my v_x is going to be a function of x only. And it is easy to right now you must be feeling very comfortable about what is going to be the smaller dimension of the shell of course, it is going to be delta x because my velocity is a function of x. So, I am going to have a thin layer of the liquid near the falling film whose size could be anything in terms of the length and the width, but its thickness is going to be equal to delta x and we are going to make a shell momentum balance for that. So, as the downward acting gravity and the upward moving belt create a flow pattern in the thin film that I have, that I have discussed. So, part of it is going up, part of it is going down and the nonzero component of the velocity is v_y and as I said v_y could be a function of all these parameters, the geometric parameters, the operational parameters and the properties of the fluid.



The chosen shell will be of thickness Δx, as the velocity varies with x

So, this is once again going to be the shell of thickness delta x could be any width, any length does not matter because your v_y is not a function of w, not a function of L because it has reached the steady state, it has reached the fully developed fully developed condition it has reached the v_y is only a function of x. So, once again the equation that we have is rate of momentum in minus out plus sum of all forces acting to 0. Just to recap the it contains the momentum, contains convection and conduction those are the mechanisms by which the momentum can come into a control volume or leave a control volume and in two most commonly site used forces are gravity and the pressure though there could be additional forces due to electrostatic or other may other things. So, we understand that the convection, net convection would be equal to 0 since v_y is not a function of y. So, any momentum comes in through the top surface is going to be equal to the momentum, convective momentum which is going from the bottom surface.

$$\text{Rate of momentum IN - Rate of momentum OUT} + \sum F = 0$$

The net convective momentum will be zero

So, the conduction is, convection part in this case is going to be equal to 0. So, that leaves us with conductive transport of momentum. Conductive transport is essentially shear stress multiplied by the area. So, too we have to write that. So, shear stress, it is one, if you go back to this figure it is the y component of momentum getting transported in the x direction.

$$\tau_{xy}\big|_{x}LW - \tau_{xy}\big|_{x+\Delta x}LW - (\Delta x\, WL\rho)g = 0$$

Boundary conditions:

1. No slip: $v_y = U_o$ at $x = 0$
2. No shear the liquid-vapor interface i.e., $\tau_{xy} = 0$ at $x = h$

So, this is going to be τ_{xy} and the area on which it is acting is L times W this is at x and x plus delta x and then you have the gravity which is acting downwards and then the two boundary conditions are no slip at, if you go to this figure this is no slip over here at the liquid solid interface. And no vapor, no sorry, no slip at this condition and no shear at this condition. So, with these boundary conditions that is τ_{xy} is 0 at x equals h, one should be able to solve the governing equation. So, this is your governing equation and this governing equation can then be solved in order to obtain, this being the velocity, this being the velocity profile inside the falling film of liquid. Now, if you once again see carefully that the three could be velocity depending on where you are the, look the gradient of the velocity the velocity could be could vary with thickness.

$$v_y = U_o - \frac{\rho g h^2}{2\mu}\left[2\left(x/h\right) - \left(x/h\right)^2\right]$$

The velocity is sustained, the flow is sustained by something like a Couette flow without anything on this side. So, it is the motion of the plate is going to drag the fluid along with it whereas, the gravity will try to pull the liquid down and it is the complex combination of this as exemplified by this relation. The effect of gravity and the effect of upwardly moving plate, the downward acting gravity and the upward moving plate, these two will together balance what is going to be the velocity in this. And you can find, you can if this gives you a very good tool for the design engineer, a very good tool about how do I change this u how, for a specific fluid with a viscosity μ what should this value of u_0 be so as to have some sort of a v in it. And you would most likely like to have a situation in which the net velocity is going to be equal to 0 such that h remains a constant.

And a design engineer who likes to coat the surface with a thin layer of a photosensitive material would understand the physics of the process by looking at this. So, we have solved two problems, two problems using shell momentum balance. One is the flow, the spill over flow along the side of a tube. how do I find out the velocity distribution in that case? And you really have to have mentally imagine what kind of a situation is being depicted by the description of the problem. And there we could see that the velocity distribution you can have logarithmic terms as well. And the present one is about again, once again an industrially relevant situation in which a film is being pulled in a liquid where you would like to have a specific coating left on the surface.

So, this method is known as dip coating which is quite common in a number of applications including in semiconductor applications. Where you apply this technique in order to have specific thickness of the material on top of the surface, but this essentially gives us insights into the physics of the process as far as the fluid flow is concerned. So, that is more or less that I wanted to cover for shell momentum balance. And in the next lecture I will show you it will become difficult when the flow situation is more complex, when the geometry is complex how do we handle such situations that I will introduce in the next class. And it would give rise to theoretical fundamental concepts of equation of continuity which is nothing, but mass conservation equation and equation of motion which is again a Newton's second law for an open system.

I will not go into the detail derivation of them, but I will talk about the specific points the importance, the significance of each of these terms and briefly how it was developed and then start using those equations for solving our problems that would be the topic of next two classes. Thank you.