Momentum Transfer in Fluids
Prof. Sunando DasGupta
Department of Chemical Engineering
IIT Kharagpur
Week-02
Lecture-08

Good morning. We are going to continue our discussion on shell momentum balance. And in the previous classes we have seen how do we choose a shell, an imaginary shell through the control surfaces of which the liquid, the fluid can come in and leave. And while the liquid enters the shell it carries with it some momentum which is known as the convective momentum. And due to the presence of a velocity gradient from the top and the bottom surfaces of the imaginary shell, the shear stress or in other words the molecular transport of momentum is also taking place. So, these four terms, the convective momentum in and out, the molecular momentum or shear stress in and out, the algebraic sum of this plus all the forces, the body and the surface forces acting on the system at steady state would be equal to 0.

Then we divide both sides by the smaller dimension take the limit when that approach is 0. And what we get is from the difference equation a differential equation governing the process at hand. That governing equation would be integrated with appropriate boundary conditions, so that we can get a velocity profile which would be valid for every point in the flow field.

So, by inserting the correct value of the location, one would be able to obtain what is the velocity at that point. So, we will continue with this approach for a few more problems so as to get a clearer idea about how to handle such processes. And the physics of the process also starts to become interesting when you think of the interplay between the applied pressure gradient, the body force and something else where one of the bounding surfaces of the fluid is not fixed, but is having certain velocity of its own. So, how do these factors affect the velocity distribution and what kind of insight we can obtain out of this. So, the problem that we are going to solve in this class is essentially, it is a liquid between two parallel plates.

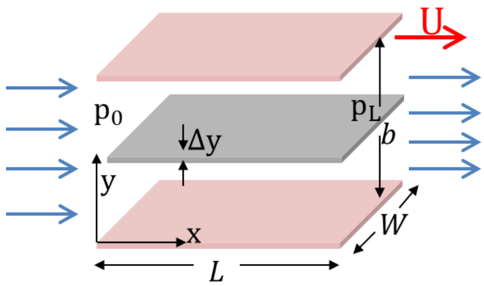
I will show you the image, the figure of this and it is separated by certain distance h and we have decided what is going to be the origin, it is on the bottom plate. Unlike previous cases the top plate moves with a constant velocity u while the bottom plate remains stationary. You need to, we need to figure out that what is going to be the velocity profile, if an unfavourable pressure gradient is imposed on the flow such that the net flow rate is 0. And we will also have to find out the maximum favourable pressure gradient that can be applied and we will discuss about that, but first I think we should consider what is termed as favourable pressure gradient and unfavourable pressure gradient. So, when you think of two plates separated by certain distance h and one of the plates, let us say the top plate is moving to the right with some velocity u.

So, due to viscosity that plate will try to drag the fluid along with it. So, there is going to be a net flow because of the motion of the top plate to the right, along with the top plate. Now, let us say you apply a pressure gradient. The pressure gradient is applied in such a way that the pressure on the left side is going to be more than the pressure on the right side. So, there is a pressure gradient from the left to the right.

This pressure gradient helps the flow. It creates more flow from the left to the right additional flow in addition to whatever flow that has been induced by the motion of the top plate. So, I call that as favourable pressure gradient. A favourable pressure gradient drives the fluid in the direction in which the top plate is also moving. So, in order to make a fluid move in a specific direction, you have to have continually decreased pressure as you move along.

Because only if the pressure is lower than the before, than the previous point only then the flow is going to take place. And what is going to be the unfavourable pressure gradient? Because of the movement of the top plate, I have velocity, imposed velocity initiated in the fluid from left to right. Now, let us say I apply a pressure gradient in such a way that the pressure on the right side is going to be more than the pressure on the left side. So, the pressure difference induced flow is going to be from right to left, whereas the plate motion induced flow is going to be from left to right. So, this pressure gradient which opposes the flow is called unfavourable pressure gradient in this case.

So, we will start with this description of the problem and see what is the problem, what are the information that we would get for a combination of both motion of the top plate and imposition of a pressure gradient in the flow field. So, that is going to be our today's problem. So, this is what I mean, that if you look at the image over here, the top plate the slight pink in colour moves to the right with some velocity u and the p_0 in key if it is a. So, if it is a favourable pressure gradient the p_0 is going to be more than p_L. So, that there would be a flow due to pressure alone from left to right.



Now, if let us say p_0 is equal to p_L and you have only the top plate moving, then also you are going to have a flow from left to right because the top plate will drag some liquid along with it because of viscosity. So, the x and y the coordinate system is marked over here. So, the it lies on the bottom plate and then when the length of which is equal to L, the width of the plate width of the plates w, and the separation between the two plates is b. Now, here we can see that the velocity is going to be principally in the x direction. So, the only non-zero component of velocity it is one dimensional steady flow is going to be v_x and as the figure suggests that this v_x is going to be a function of y.

So, if the velocity varies in the y direction, then when we choose the shell, imaginary shell for making the momentum balance, the thickness of that is going to be equal to delta y which is what has been shown over here the grey one for which the length is L the width is w. However, the thickness is delta y once again this delta y is chosen since the velocity v_x varies with y. So, and then we will use the equation, the fundamental statement, the momentum conservation for steady state case, the rate of momentum in minus rate of momentum out plus sum of all forces acting on the fluid inside the control volume would be equal to 0. Once again, we understand that since the velocity v_x is not a function of x and therefore, the convective momentum in through this face would be equal to the convective momentum out of this face. So, the v_x is a function of y only and the width of the surface is equal to the velocity v_x.

So, the width of the plate is very large as compared to the separation. So, v_x is not going to be a function of z as well. So, since v_x is not a function of x it is a constant at a given y if you change the y, v_x changes, but at a given y the velocity remains constant. So, this leads to our statement that the net convective momentum will be 0. Additionally, we also identify what is going to be the pressure gradient.

$$\text{Rate of momentum IN} - \text{Rate of momentum OUT} + \sum F = 0$$

$$\frac{dp}{dx} = \frac{p_L - p_0}{L - 0}$$

The pressure gradient is going to be p final which in this case is p_L minus initial divided by the spatial locations at L and 0. So, with this knowledge I will not get into this any further. So, we would be able to obtain what is going to be the governing equation which is d2v_x/dy2 is equal to 1 by μdp/dx. Once again, this expression is valid only for a Newtonian fluid. The shell momentum balance is exactly the one which we have done before.

$$\frac{d^2 v_x}{dy^2} = \frac{1}{\mu}\left(\frac{dp}{dx}\right)$$

Rate of momentum in minus rate of momentum out plus sum of all forces would be equal to 0. There is no convective momentum, only conductive momentum and if you think about the conductive momentum, it is the x component of momentum getting transported in the y direction. So, you know what is going to be the subscript of τ_{yx} which is applied on the area w times L in at y out at y plus delta y. So, two terms for the shear induced momentum transport and the there is no body force in the x direction, no gravity in the x direction. So, what you have in the x direction is surface force and the surface force is essentially the pressure gradient.

So, pressure gradient, the pressure multiplied by the area. So, p_L multiplied by the corresponding area which would be delta y times w and p_0 multiplied by the area the algebraic sum of these two and when you couple that with the Newton's law of viscosity this is the governing equation, this is the governing equation that you should get. I leave this to you to do it on your own and this is the form of the governing equation. Now, next we have to identify what are the boundary conditions and if you look at the figure that we have on the screen we are going to use two no slip conditions. One is v_x is equal to u, capital U the velocity of the top plate at y equals b.

$$\text{Boundary Conditions: } \left.\begin{aligned} &1.\ v_x = U \text{ at } y = b \\ &2.\ v_x = 0 \text{ at } y = 0 \end{aligned}\right\} \text{No Slip}$$

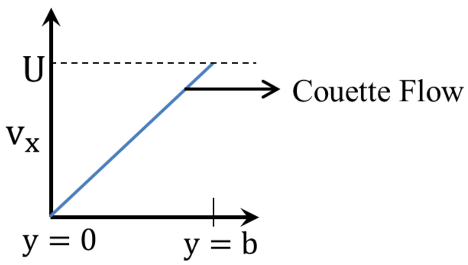
So, and so whenever the top plate moves it will drag the fluid along with it and at the right, at the liquid solid interface the velocity of the liquid particles would be the same as the velocity of the solid part. So, velocity of the solid. So, therefore, v_x is u at y equals b and the no slip condition also tells us that at the bottom plate v_x is going to be equal to, v_x is going to be equal to 0 because of no slip again. So, when you solve this, integrate, find out the governing, find out the boundary integration constants using the boundary condition, this is what you are going to get. It is interesting to note that you have two distinct terms, one is this the other is the one which contains the pressure gradient.

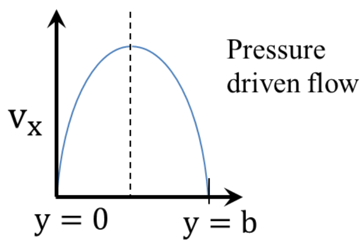
$$v_x = \frac{U}{b}y - \frac{1}{2\mu}\left(\frac{dp}{dx}\right)b^2\left[\frac{y}{b} - \left(\frac{y}{b}\right)^2\right]$$

So, the first part of the velocity distribution is purely due to the motion of the top plate which is which is denoted by capital U. The second part of the motion is caused by the pressure gradient. So, this is, as if it is a superimposition of two flows, one due to the motion of the top plate the second due to the imposition of pressure. So, if dp, if I set the imposed pressure gradient to be equal to 0 then v_x is simply going to be a linear distribution in terms of y. So, v_x and y are related in a linear fashion. This kind of flow initiated by the motion of one of the bounding surfaces in this case, one of the plates without any pressure gradient is known as Couette flow.

$$\text{If } \frac{dp}{dx} = 0 \quad v_x = \frac{U}{b}y \longrightarrow \text{Couette Flow}$$

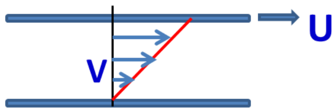
So, this Couette flow is quite common in certain applications which we will discuss subsequently. So, this is how the Couette flow would look like a linear distribution of velocity starting at y equal to 0 and becoming equal to capital U at the top plate. So, this is at y equals b, the velocity that is what it is going to be. Now, if you assume, now that the top plate has stopped moving only you have pressure gradient present in the system. So, if you look at the second part of the expression the second part of the expression is due to pressure driven flow and the pressure driven flow will give you a distribution something like this, with the maximum velocity being at the mid plane that is at y equals b by 2.





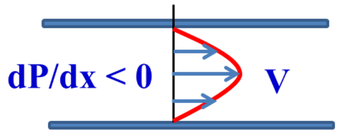
And in this case, both the plates are stationary that is why at y equal to 0 and y equal to b, the v_x is going to be equal to 0. So, that is what is pressure driven flow. So, a Couette flow looks something like this. Once again, no pressure gradient and only pressure gradient the velocity distribution would look like the figure at the bottom. And when you superimpose one on the other that means, when you do have pressure gradient and, in this case, it is favourable pressure gradient. So, whichever direction the top plate is moving the pressure gradient.

Couette flow (only motion of the top plate, no pressure gradient – linear velocity distribution



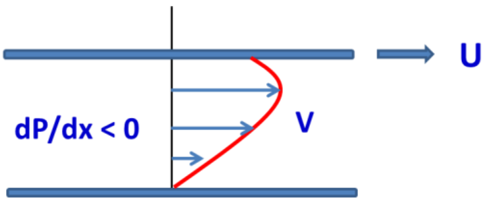
Couette Flow

Only pressure gradient driven flow – parabolic profile



Pressure gradient Driven Flow

So, the pressure keeps on decreasing as you move from towards the right. So, the pressure gradient also tries to move the velocity, move the fluid towards the right. So, this is called the favourable pressure gradient and the profile that I have drawn over there is essentially the Couette and pressure gradient driven flow when they are acting together. So, this is a superposition and another interesting thing is what is going to be the location of the maximum velocity. If you look at this one, the location of maximum velocity for Couette flow was here for the case of pressure driven flow the location of maximum velocity was at the centre line, but what if both are present.



Couette and Pressure gradient Driven Flow

So, if both are present, where are we going to have the maximum mean velocity. So, here as I have drawn it, is the maximum mean velocity could be somewhere over here, but the location, position of this maximum would depend on what is the velocity of the top plate and what is the pressure gradient that you have applied. If you apply a large pressure gradient, then the maximum is going to be more than the velocity of the top plate. If you just apply a small pressure gradient, then the velocity profile would probably look something like this where the top plate velocity will still be the maximum. So, the magnitude of the maximum velocity is a function of the velocity of the top plate and the impose and the value of the pressure gradient, favourable pressure gradient that has been applied.

So, the location of the maximum velocity is something interesting to look at and what if I have unfavourable pressure gradient and Couette flow in such a way that the top part of the fluid will move along with the plate towards the right and since it is unfavourable pressure gradient, the bottom part of the fluid will move due to pressure gradient to the left. So, there is going to be two flows, two flows superimposed on each other- Couette flow and pressure driven flow. So, this kind of situation may give rise to a condition where even though you have flow at the top and flow at the bottom, but the net flow across any cross section could be equal to 0. So, the 0 net flow that is something which you would like to look at. So, if the second part of the question is, obtain and sketch the velocity profile if an unfavourable pressure gradient is imposed on the flow such that the net flow is 0.

$$v_x = \frac{U}{h}y + \frac{1}{2\mu}Ay^2\left[1 - \frac{h}{y}\right] \qquad A = \frac{dp}{dx}$$

So, this is what I was talking about and we would like to see how it looks. So, this is the expression for velocity which you have obtained from shell momentum balance and the expression can be integrated in order to obtain what is the average velocity. So, since the velocity varies with y. So, therefore, it is from 0 to h v_xd y would give me the area averaged velocity and note here is that I have taken unit width basis 1. If it is not unit width then I could have it this from 0 to w and dy dz where z is the width of the plate.

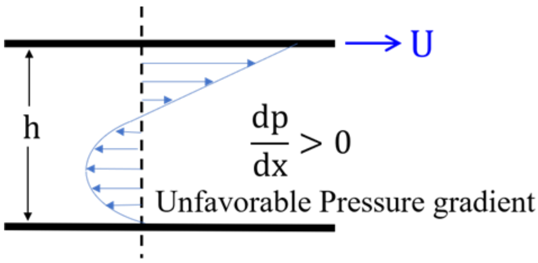
For net flow to be zero, $\int_0^h v_x\, dy = 0$ (Avg velocity = 0)

So, along the flow area which is w times delta w times delta y along that area how v_x changes. So, the expression for v_x is plugged in here and it is equated to 0 which essentially means the average velocity for such a case of an unfavourable pressure gradient would be equal to 0. When you perform that you would see that the pressure gradient is simply going to be equal to 6μu by h square that is the expression for the unfavourable pressure gradient that you have to apply in order to obtain a 0 net flow when you have Couette flow and unfavourable pressure gradient together. And you also note that this dp/dx is positive. So, this dp/dx is positive which means that as the flow progresses, the pressure over here is going to be more than the pressure over here and so on.

$$A = \frac{6\mu U}{h^2} = \frac{dp}{dx}$$

$$v_x = \frac{U}{h}y + \frac{3U}{h^2}y^2\left[1 - \frac{h}{y}\right]$$

So, with in the pressure is going to force the fluid to move from the right to the left and the after you plug in the value this would be the expression for the velocity and this is how it looks like. So, near the top it is Couette flow at that point it is more like Couette flow. Over here it is pressure gradient driven flow and in order to have the have the 0 net flow the area under the curve over here and over here must be equal. So, essentially what we are saying is that whatever be the flow rate in the top part of it and the flow rate in the bottom part they are equal, but opposite in direction. So, this is what it should look like when you have Couette and unfavourable pressure gradient acting together.



Now, then comes the last part of this problem, find the maximum favourable pressure gradient that can be applied so that the maximum velocity in the fluid will be the velocity of the top plate and sketch the velocity profile for this case. So, if I draw it like this, if it is Couette flow, the velocity is simply going to be linear. So, your v max is over here and which is equal to the velocity of the top plate. If you apply a small value of the favourable pressure gradient this is still my capital U. So, instead of this being a straight line, it is going to be slightly deviated from straight line, but it will still the maximum is still going to be over here.

If we apply a large pressure gradient then this being capital U it will go it will look something like this. So, in between these two, I have a situation in which if you apply just a slightly more pressure gradient it is the maximum in velocity is going to be more than capital U. So, if you apply a lower pressure then it would look something like this, where the maximum is going to be still going to be the velocity at the top plate. So, what is that transition pressure which would give you the last point of the maximum on the plate. Any more pressure gradient and the maximum will lie somewhere in between the two plates, not at the top plate. So, you have the expression for the velocity and since it is going to be maximum at the top.

$$\text{Here, } \left.\frac{dv_x}{dy}\right|_{y=h} = 0$$

So, therefore, dv_x/dy at y equals h must be equal to 0. You can intuitively also arrive at the same conclusion because as our discussion suggests over here, if this is my capital U then it should look something like this. Because if it is rightly just vertical over here any increase in dp/dx would force the maximum to move somewhat over here. So, this is the condition where I have the maximum still on the top plate and this is the condition where the maximum shifts from the top plate to the right. And if you look carefully, the boundary condition over here is that the profile, the velocity profile approaches the top plate with a condition where dv/dx for dy at y equals h to be equal to 0.

$$A = \frac{-2\mu U}{h^2} = \frac{dp}{dx}$$

So, when you plug that condition, the expression for force the pressure gradient that you would get the expression for favourable pressure gradient that you would get is minus twice μu by h square. Note the minus sign which denotes that you have the favourable pressure gradient. So, this is a unique problem and as I mentioned any dp/dx larger than minus 2 μu by h square. This would result in a velocity which is going to be greater than the velocity of the top plate, greater than the velocity of the top plate in between 0 and h. So, this is a unique problem which shows us so many things about the interplay of the pressure gradient and the Couette flow.

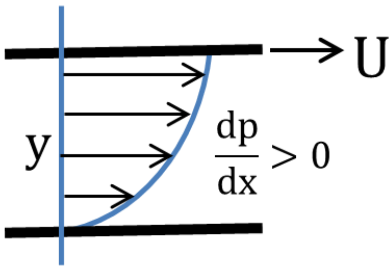
Now, let us think about Couette flow once again. The top plate is moving and it is imparting a flow due to viscosity in the fluid below it, in the fluid which is situated in between the two plates. So, as the top plate moves, what it is going to do is that it because of liquid viscosity the flow will start in the direction of the motion of the top plate. Now, viscosity always resists motion. So, viscosity will, in order to make the top plate move some external agency must apply a force on the top plate. So, the top plate will only move if a force is being applied on it.

$$\text{As any } \frac{dp}{dx} \text{ larger than this would result in a velocity} > U \quad \text{between } y = 0 \rightarrow h$$

Now, you can always calculate that force by evaluating what is τ, what is τ at the top plate that means, at y equals capital H and multiply it with the area of the plate in contact with the liquid which would simply be w times L. And this τ is essentially if I use Newton's law of viscosity, it is going to be viscosity multiplied by velocity gradient evaluated at the top plate at y equals H. So, that multiplied by w times H would give me the force needed to make the top plate move. So, this would be μtimes $dv_x$by d y at y equals 0 multiplied by w times L.

This is the force needed to move the top plate. So, interest and the faster the velocity of the top plate as the velocity increase, as the u becomes larger and larger this velocity gradient term it keeps on increasing because your separation is fixed at H. So, if your separation is fixed at H any increase in the top velocity would essentially enhance the velocity gradient. So, if the velocity gradient increases then the force will also increase. So, the question then comes is what is going to be the force when you have a condition that the velocity, the maximum is going to be at the top plate. Now, the condition that we have used for getting the maximum at the top plate is d v_x/ d y at y equals H is equal to 0.

So, interestingly then that and the profile would look something like this. So, the blue one is the profile the it approaches the top plate with in a vertical fashion dp/dx is greater than 0. So, the force necessary to move the top plate under this condition what is going to be its value? Its μ times velocity gradient, but the velocity gradient at that point is 0 that is being shown over here. So, essentially that means, that for that special condition you do not require any force to move the top plate under that condition. So, for a situation in which the maximum will lie, the last point of maximum for which the velocity in the flow field is maximum at the top plate would require that the top plate will move on its own without having any force to be applied on it.



So, we have solved an interesting problem where we have seen that the motion of the top plate can initiate a flow which is Couette flow, which is termed Couette flow. The velocity profile in that case is going to be linear. If I have a favourable pressure gradient then the velocity profile would not be linear, but it is going to have a maximum at the centre line. When we superimpose the two together what we have is Couette flow plus favourable or unfavourable pressure gradient. The unfavourable pressure gradient can give rise to a situation in which the net flow rate could be 0. And the net flow rate could be 0 essentially you can derive what is the value of the pressure gradient that is to be applied in order to have 0 net flow.

All these are going to be a function, the expressions are going to be a function of the physical property mainly viscosity of the fluid and the geometrical parameters. For example, what is the separation between the two plates H and it is also going to be what is the magnitude of the velocity u with which the top plate is moving. And we have also analysed an interesting case, what is going to be the force needed to move the top plate with a constant velocity when the last point of the maximum still lies on the top plate for a specific pressure gradient. Any pressure gradient more than that will cause that to shift from there.

So, that is all for this class. Thank you and we will continue; we will have one more class on the shell momentum balance and then we will see that how we will feel the need to have a more generalized approach rather than a shell momentum balance. Thank you.