Momentum Transfer in Fluids
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Week-02
Lecture-07

Good morning, once again. We will continue with our discussion on the shell momentum balance and as before, this portion you can find in the prescribed textbook of Bird, Stewart and Lightfoot. So, we have already defined in the previous class what is a shell, how do we define it, the fact that we have to use the smaller dimension as the direction in which the flow is changing. So, if the velocity is a function of x, then it is the dimension, a smaller dimension of the imaginary shell is going to be delta x. The length and width could be anything. So, on that shell we made the momentum balance and there are two mechanisms by which the momentum can come into the shell. One is due to the real flow, which we call as the convective transport of momentum. The other is, since there is a variation in velocity on the top surface, let us assume on the top surface of the of the imaginary shell.

So, there is going to be a transport of momentum. The viscous transport of momentum, also called the molecular transport of momentum in a direction perpendicular to that of the flow. So, that is known as the conductive transport of momentum. So, the convective due to actual flow, conductive due to the presence of a velocity gradient. One is in the direction of flow, the other is in a direction perpendicular to that of the flow. And the common forces which we, one encounters in fluid mechanics would be the body force, which acts on every point of the fluid inside the imaginary shell equally and this it could also be an electrostatic force, and the second one is the applied, the surface force.

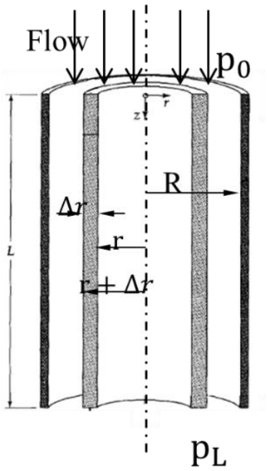
One of the examples of surface force very common example of surface force is the applied pressure gradient. That there is an external agency, most likely a pump which creates a pressure gradient that sustains the flow. So, these are the two different types of forces which will get into that equation and then since it at steady state the sum, algebraic sum of rate of momentum in minus rate of momentum out plus sum of all forces acting on it would be equal to 0. That is what we have covered in the last class. Then from the difference equation we obtained the differential equation which is also known as the governing equation.

Shell MM Balance

At steady state: Rate of MM IN – Rate of MM OUT + Sum of all forces acting on it = 0

The governing equation after incorporation of the after-Newton's law of viscosity which relates shear stress with the velocity gradient, the proportionality constant being a material property viscosity μ, one can then use the boundary conditions of no slip and no shear at the liquid vapor interface in order to obtain the in order to obtain the unknown unknown current unknown integration constants. So, one is the no slip that is the relative velocity at the solid liquid interface would be equal to 0. The other is the no shear that the shear at the liquid vapor interface is equal to 0 or in other words τ should be equal to 0. With this and the first problem that we have solved is flow along an inclined plane. The one that we are going to solve in this class is going to be the flow of a through a circular tube.

Now, all of us are aware of flow through a circular tube or a pipe. So, if you look at the figure on the left then you have flow coming from the top into this and the flow is going out of the tube at let us say at z equals L. So, this z direction is along the centre line down and r is in the radial direction. So, you can imagine that there is going to be a one-dimensional flow situation, there is going to be a nonzero value of v_z and there is not going to be any v_r. There is just not going to be any velocity in the r-direction and since this is cylindrical coordinate system, there is also the possibility of a v_θ , that is the velocity in the theta direction which is also not present. So, I have v_z as the only nonzero component of the velocity and if you think again this v_z is going to be a function of r only.



So, if it is near the wall, the velocity is going to be smaller, if it is at near the centre line, the velocity is going to be larger. So, the velocity is definitely a function of r, it is steady state. So, it is not going to be a function of time and it is not going to be a function of z or of theta. So, since the velocity is changing with r and r alone. So, in order to make a shell across which we are going to make a momentum balance, I have to choose a cylindrical shell inside the pipe, inside the tube inside, the circular tube whose longitudinal length could be L.

However, the smaller dimension is going to be delta r. As I have shown in over here because this delta since v_z is a function of r. So, my shell, smaller dimension of the shell is going to be delta r. So, I have a shell inside the tube with having the dimension same length, but the thickness is going to be equal to delta r. So, I hope that this is very clear to you how we have we choose the smaller dimension of the shell and then make the balance. So, next is, we are going to write the convection, the rate of momentum in by convection and in order to write the rate of momentum in by convection first I have to find out what is the area that we are talking about.

So, this area is going to be the annular area, twice pi r times delta r. So, this is this area which is twice pi r times delta r. What you are seeing is just a sectional view of the tube along with the imaginary shell. So, the flow from the top is entering into the shell into the cylindrical shell whose top area perpendicular to the direction of flow is twice pi r delta r. It enters with a velocity v_z which is at z equals 0. That means, over here. So, area multiplied by velocity would give you the volumetric flow rate meter cube per second.

Volumetric flow rate twice pi r delta r times v_z multiplied by ρ, the density would give you the mass flow rate. The mass flow rate again multiplied with another velocity, with the velocity at that location would give you the rate of momentum in by convection, ok. So, area, velocity, density velocity again. So, meter square, meter per second, kg per meter cube. This will be the mass flow rate and then the mass flow rate is to be multiplied by v_z. This is the in term and the out term when I wrote rate, I should probably have said I think you understand that it is the net rate of momentum.

Rate of momentum in by Convection:

$$2\pi r\ \Delta r v_z|_{z=0}\rho\ v_z|_{z=0} - 2\pi r\ \Delta r v_z|_{z=L}\rho\ v_z|_{z=L} = 0$$

So, this is the in term, this is the out term and the out term would remain exactly the same except z equals 0. The velocities are to be evaluated at not at z equals 0, but at z equals L. We also realize that, this is a one-dimensional flow. So, v_z is a function of r only. It is not a function of z. So, if it is not a function of z then v_z at z equals 0 would be equal to v_z at z equals L. So, under these conditions there is not going to be any net rate of momentum to the control volume due to convection.

So, first these two terms will cancel out each other and therefore, I do not have any contribution of moment, contribution in momentum by convective flow. So, what is by conduction? In order to write the conduction, I first have to find figure out what is going to be the subscripts of τ. Now, this is the z momentum getting transported in the r direction. z momentum getting transported in the r direction. So, it is going to be τ_{rz} and if you think of the annular area, the inside area the τ_{rz} is acting on this area. So, what is this area? It is simply going to be twice pi r times L, that is the annular area, inside area of the tube on which τ is acting and going out through this phase where the length remains the same except the radius is going to be r plus delta r.

Net Rate of momentum in by Conduction along with surface and body forces:

$$L[\tau_{rz}|_{r=r}2\pi r - \tau_{rz}|_{r=r+\Delta r}2\pi(r+\Delta r)] + 2\pi r\Delta r p_0 - 2\pi r\Delta r p_L + 2\pi r\Delta r L\rho g = 0$$

So, thus the shear stress is multiplied by twice pi r times L. So, everything is evaluated at r equals r and the other one is going to be the shear stress evaluated at r plus delta r that means, on the outer edge of the imaginary shell and the area would simply be equals twice pi twice pi times r plus delta r multiplied by L. So, this is the in term this is the out term and so, that is what the shear stress is all about. So, I think it is very clear to all of you at this point that the shear stress how to evaluate the shear stress, the area on which it is acting on and so forth. Now, if you look at the figure once again the pressure at the top is equal to p_0, the pressure at the bottom is p_L.

So, there is a difference in pressure at the top of the shell and at the bottom of the shell. So, one has to then write the force difference due to the applied pressure and the pressure is acting on the annular area at the top. So, what is the annular area of the top? It is twice pi r times delta r at the top and the same at the bottom. Over the top the pressure is p_0 pushing the fluid downwards in the plus z direction, the force at the bottom is going to be p_L multiplied by area and since it is trying to push the liquid up. So, that is why it comes with a minus sign.

So, the net force and here I have an imposed pressure gradient. So, therefore, I need to take into account the surface forces in this. In this case and of course, there is going to be gravity it is acting downward. So, the component of gravity is simply g and what is the volume contained in it, in the shell multiplied by ρ multiplied by g would give me the force due to gravity the body force. So, this is my area, this is the density, this is the density and I have the g. So, this essentially then constitutes my complete governing, a complete difference equation, I have already identified that the convective shear stresses do not play any role since the velocity is not a function of z.

$$\lim_{\Delta r \to 0}\left[\frac{r\tau_{rz}|_{r+\Delta r} - r\tau_{rz}|_{r}}{\Delta r}\right] = \left(\frac{p_0 - p_L}{L} + \rho g\right) r$$

$$\frac{d(r\tau_{rz})}{dr} = \left(\frac{p_0 - p_L}{L} + \rho g\right) r$$

So, they simply cancel out, but these are the terms which will remain in here. So, with this, I then take the τ containing terms on one side divide both sides by delta r and take the limit when delta r approaches 0. So, this difference equation then gets converted into a differential equation which we will subsequently call as the governing equation. And if you look at the right-hand side, I have the imposed pressure rather imposed pressure gradient p_0 minus p_L divided by L, pressure difference per unit length and I have the ρ g also r on the right-hand side. So, the difference equation then gets converted and this becomes my governing equation for the flow of a fluid through a circular tube.

What are the boundary conditions? At r equals 0 that means, at this point the velocity is going to be maximum, right. Our intuitively it tells us that the τ is going to be maximum, sorry velocity is going to be maximum and τ_{rz} will be 0. Since τ is proportional to the velocity gradient. So, for the case of maximum, the velocity gradient is going to be 0 there. So, therefore, my τ is going to be 0 and the other boundary condition is the no slip condition that is at r equals r where the liquid encounters the walls of the tube, there is going to be no slip condition. So, at r equals capital R the velocity is going to be 0.

Boundary Condition: $r = 0, \tau_{rz} = 0$

Boundary Condition: $r = R, v_z = 0$

So, the no shear in this case comes from our physical understanding that the velocity has to be maximum at the centre line and the other one is the no slip condition. Now, in order to make it more compact no, additional understanding is required. This is just to make this part look more compact, I define a capital P which is p_z minus ρ g z. So, what is going to be my p_0. P at z equals 0 is simply going to be p_0 and p_L is going to be p at l minus ρ g L. This is only to make this term more compact nothing else. So, this capital P are defined in terms of the actual pressure and the ρ g z nothing more than that. So, when you, if you can continue without doing this and you will be right as well, but to make it more compact, the equation would look like the pressure the $d\tau_{rz}/dr$ would be this.

Define: $P = p_z - \rho g z;\ P_0 = p_0\ \&\ P_L = p_L - \rho g L$

So, once you integrate this, you are going to use the first boundary condition and use Newton's law and the second boundary condition. So, you should be able to evaluate what is c1 and what is c2, I am not doing all these steps in here they are done quite well in Bird's Lightfoot. Take a look at it, try to do it yourself, this would be a good exercise starting with the governing equation with the help of these two boundary conditions whether and how you arrive at the velocity distribution. Now, if you look at the velocity distribution in this case, this capital

P are what we have defined which have embedded in it the effect of applied pressure and that of gravity as per my definition of capital P which contains the pressure and the gravity term together. So, that is what I meant by making it look more compact nothing else. So, I leave this to an exercise so that you can figure out what is going to be the velocity.

$$v_z = \frac{(P_0 - P_L)}{4\mu L} R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]$$

So, it is going to be the applied pressure gradient pressure difference divided by length, but the most important thing is the form of this, which would tell me that the profile of the velocity in the tube is going to look something like a parabola. So, this would look like a parabola with the maximum at the central line. So, which is v_z is going to be equal to the maximum, is going to be equal to maximum and the no slip would tell me that v_z over here is going to be equal to 0. So, this is the profile and for any flow in a tube a cylindrical system the velocity will obviously, be the parabolic distribution. And once again by putting r equal 0 you can get an expression for what is v_z max the maximum velocity at the central line.

$$v_{z,\,max} = \frac{(P_0 - P_L)}{4\mu L} R^2$$

And therefore, you can also write the v_z in terms of v in terms of v max. The next important thing what we have done before is finding out the average velocity. In order to find the average velocity, you have to integrate the velocity over the entire cross section and then divide it with the cross section. So, it is going to be 0 to 2 pi r dr $d\theta$, this 0 to 2 pi r dr $d\theta$ is the area. So, $v_z$times r dr $d\theta$ integrated over the possible limits of 0 to 2 pi and 0 to r.

$$\langle v_z \rangle = \frac{\int_0^{2\pi} \int_0^R v_z \, r \, dr \, d\theta}{\int_0^{2\pi} \int_0^R r \, dr \, d\theta} = \frac{(P_0 - P_L)}{8\mu L} R^2$$

This essentially is average velocity, is the area averaged flow velocity. So, this area that we talk about are the area in a direction perpendicular to the direction of flow. So, if this is the v_z then the area is pi r square. So, it is the area, is the averaged, is the velocity averaged over the cross-sectional area that is what we have over here and this is going to be the expression for the average velocity. Now, all these you should try to do yourself by plugging in the expression of v_z in here and try to see if you arrive at the same result.

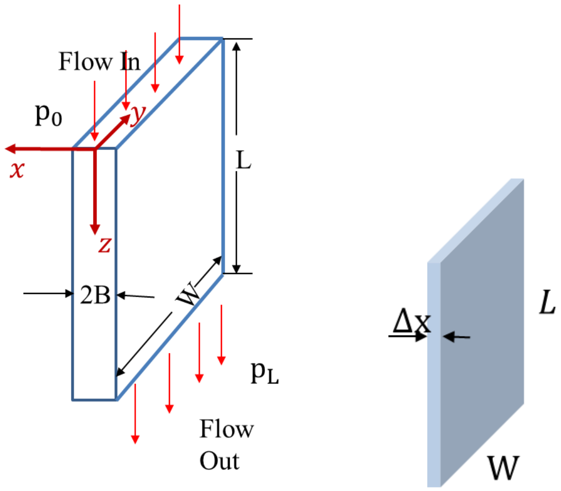
So, the volumetric flow rate would simply be equal to the average velocity multiplied by the flow area. When you do that, this is the expression for the volumetric flow rate because of an applied pressure gradient and flow assisted by gravity which was our case. This equation has a special name, it is called Hagen–Poiseuille equation. And what are the assumptions that we have chosen for this one? The assumptions are that it is laminar flow, it is one dimensional flow, only nonzero component being v_z it is a steady state flow. So, it does not, the velocity does not vary with time.

$$\text{Volumetric flow rate, } Q = \langle \pi R^2 \rangle \langle v_z \rangle$$

$$Q = \frac{\pi(P_0 - P_L)}{8\mu L} R^4 \text{ ; Hagen Poiseuille Equation}$$

So, all these together would give us this expression for Hagen–Poiseuille equation which is extremely useful important you will see its application later on as well. But one of the common examples of the use of this is if you have used a capillary viscometer for trying to find out what is the average viscosity during your school days. What you have done is you have chosen a capillary and you let a fluid liquid flow through it, collect the amount in a beaker figure out what is going to, what is your q, the volumetric flow rate amount collected per unit time. You know what is the radius of the capillary that was handed to you and you also know that in absence of any applied pressure gradient this is nothing, but gravity this p_0 minus p_L is due to gravity only and you know the length of the tube. So, through a large number of experiments and calculating that you would be able to obtain what is the unknown μ.

So, this is one of the simplest ways to measure the viscosity of any fluid. So, next I am going to quickly give you some pointers about a problem which I will leave to you as an exercise. So, I will give you the answer for, but let me first describe the problem to you. What you see on the left is a narrow slit, what is the significance of a narrow slit? That means, the separation between the two plates which is 2b is very small as compared to the width or length of it. So, what means it means is that if you take, if you take 2 pages, 2 A4 size papers and bring them very close to each other with leaving a small gap in there.



So, the gap thickness is essentially very small in compared to the width or the length of the A4 size paper. What it means is that, for most of the flow area, the flow and you let a liquid flow through them. Most of the flow area, the flow is going to be one dimensional. It is going to be a function of its distance from the side walls, but it is not going to be a function of these open ends since they are very long in comparison to twice b, it is not also going to be a function of the length of the two papers. So, for evaluating the flow in a narrow slit when you have a pressure difference imposed of them in terms of p_0 and p_L.

So, this is the flow in and out and its laminar flow. How do I find out what is the velocity distribution? That is first part of the problem. So, we have to find out what is the velocity profile, the average velocity and the relation between the average velocity and the maximum velocity. So, that is what the problem is all about. Now, in order to solve the problem, I first have to choose my shell.

So, the velocity is a function, in this figure velocity is a function of x only, it is not a function of y or of z. So, therefore, my shell, chosen shell should have the dimensions, it could be any length and width. So, it could be L and W and the smaller dimension is going to be equal to the delta x. Since velocity so, the nonzero component of velocity v_z is a function of x only and v_z is not a function of time x or y. So, this understanding would let us draw the shell in this way.

So, it is delta x any length L and any width W and then make the balance of momentum for this shell. We understand that the fluid is entering through the top, it is leaving through the bottom. Since it is one dimensional flow there is not going to be any flow on this face or at the back face on this side or this side. So, my velocity is going to be a function of x only.

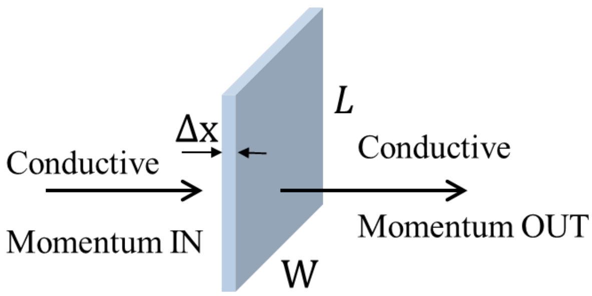
$$\text{Rate of momentum IN - Rate of momentum OUT} + \sum F = 0$$

So, let us write the equation. So, once again the shell momentum balance for one dimensional case is rate of momentum in minus rate of momentum out plus sum of all forces acting on it would be equal to 0. And over here I realize that I have gravity force and I also have pressure force because there is some imposed pressure on this system. We also understand since velocity is not a function of z. So, v_z at z equals 0 would be equal to v_z at z equals L. What this tells me is that the convection in and convection out will be equal and in my equation.

So, rate of momentum in the convection part need not be taken into account because the in and the out will cancel each other. With this what I am and I have mentioned that v_z is not a function of z, it is not a function of y because of this geometric reason and it is fully developed, that means the flow is completely developed and we will talk more about fully developed later on what exactly is fully developed. It is fully developed when the flow does not change with the direction of flow. The flow velocity can still be different because as in this case v_z is in this case is a function of x. So, velocity changes with x, but velocity does not change with z the flow direction.

So, whatever be the profile of the velocity with x will be maintained at all values of z. So, if it starts if it is parabolic, it will remain parabolic throughout the length of the pipe or the tube or the slit and that is what is known as the fully developed flow. So, once again the pictorial description of this and after for cancelling the convective terms, I have my conductive terms. So, what is conductive? z momentum being carried in the x direction. So, the convention that we have chosen would be would fix the subscript of τ as τ_{xz}.

The area on which it is acting on is L times W. So, I have this area and the one out would be everything remaining of the same except for x everything is going to be evaluated at x plus delta x now. The convective part is 0. So, the conductive part the molecular transport part or viscous transport part the it is called in different names. So, it is simply going to be τ multiplied by the area on which it is acting on.



I have some pressure force acting on it. This is the area over which the pressure is acting on. So, W times delta x is the area on which the pressure p_0 is acting at the inlet. The one, the pressure at the outlet is p_L. So, that is why I simply substitute p_0 by p_L in here the area remains the same.

What is the volume of it? It is delta x W L is the volume. This ρ multiplication with ρ converts it to mass and this is mass times m times g m g. So, that is essentially the downward acting gravity force in this case. So, what you now, you understand what you have to do. You have to cancel the terms which are there. For example, you can cancel W from both sides and then divide it by the smaller dimension which is delta x.

$$\tau_{xz}|_x LW - \tau_{xz}|_{x+\Delta x} LW + p_0 W\Delta x - p_L W\Delta x + (\Delta x\, WL\rho)g = 0$$

Take in the limit when delta x tends to 0 and with that you can convert this difference equation into the governing equation governing differential equation and then you solve the governing equation with appropriate boundary conditions and what could be the appropriate boundary conditions if I go back to this figure. What are the boundary conditions? Now, what I have in here is a flow contained between two solid plates. There is no liquid vapor interface, but I have two liquid solid interfaces. One is at x equals plus b and the other is at x equals minus b. So, my boundary condition for this case would be the no slip condition at the plus x and no slip condition at minus x such that v_z the velocity in the z direction is 0 at x equals plus b and x equals minus b.

So, these are the two no slip conditions which are to be used in order to solve this governing equation and you should get, once again you should do it on your own and check if you are getting the correct results. So, your v_z is going to be the applied pressure gradient whereas before as we have done with the cylindrical tube problem this p_0 and p_L capital $P_0$and P_0 contains embedded in them the effect of gravity. You could leave the limit as before you could leave it the gravity separate also it will not change anything, it will be equally correct. But the important part is the nature of the variation of velocity and you can see from here is that it is going to be parabolic in nature. So, in between the two plates the velocity is going to be parabolic in nature with the maximum velocity at the center line and the 0-velocity due to no slip at the side walls.

$$v_z = \frac{B^2}{2\mu}\left(\frac{P_0 - P_L}{L}\right)\left[1 - \left(\frac{x}{B}\right)^2\right]$$

So, this is what the distribution suggests that it should be. Once you have the point velocity, we could find out what is the average velocity and from here you can also say that v_z max is whatever be the v_z at x equals 0. So, that means, right at the center line the velocity is going to be the maximum and the average velocity can be obtained as I have explained when the velocity is integrated over the flow area and divided by the flow area itself. So, the average across the flow area when the flow area is simply going to be equal to this area. This is the flow area W times delta w times delta w times 2 b. So, the flow area is twice b W because this is the width and this is the gap through which the flow is taking place.

$$\langle v_z\rangle = \frac{2}{3} v_{max}$$

So, this is essentially equals to the flow area. So, in order to obtain the average velocity v_z you have to integrate the, in order to calculate the v_z you have to integrate this over dx and dy. dx and dy with the limits of x from plus minus v with the limits of y between 0 to w. Once again you should try that and the final result, I am writing over here is that the velocity, the average velocity is going to be two-third of v max. So, this is an exercise which I leave for you. So, before I leave, we have seen how a simple momentum balance in an imaginary shell can give rise to compact analytical expressions for velocity, the average velocity, the maximum velocity and the flow rate. For two cases one is quite common which is flow through a circular tube and the second is an application example where the flow is in between a narrow slit.

So, in both cases the velocity distribution turned out to be parabolic in nature and you can have relations, one of the very important relations in fluid mechanics that we have derived in today's class is the Hagen–Poiseuille equation. So, in the next class we will see slightly more complicated problem, more complicated geometries to be handled and more conceptual problems and we would slowly start to realize that the shell momentum balance will not work in every case and we need to have something more general than a shell momentum balance to tackle problems of varying increasing complexity. So, that would be the topics of my subsequent classes. Thank you.