Momentum Transfer in Fluids
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Week-02
Lecture-06

Good morning. In today's class, I am going to start with a new topic. It is called shell momentum balance. And this shell momentum balance, this part of the course is taken from the textbook Bird, Stewart, and Lightfoot. So, what is a shell that we have to decide first. Whenever we are making a balance of the forces on a small control volume of a fluid, we first need to decide what is the direction in which the velocity is changing. So, if the velocity of the fluid flowing between two plates, both are stationary, and if this is the x direction let us say.

So, the first thing as I mentioned, is going to decide what is going to be my shell dimension. So, let us say that we have two plates and a fluid is situated in between them and the fluid is flowing. This is the x direction and this is the y direction. So, obviously, you can well imagine that v_x is going to be a function of y. The closer the moving layer is toward the walls of the two plates, the velocity will be minimum, will be smaller and at the centre line the velocity is going to be maximum.

So, under these considerations, since v_x is going to be only a function of y, it is called one dimensional flow. And in one dimensional flow the shell to be chosen for any balancing of momentum we will choose a shell whose dimensions are going to be the length. This is the width of the of the entire system, this essentially is the width of the volume that we are going to choose and this one is going to be my smaller dimensions delta y recognizing the fact that v_x is a function of y only. So, whenever you have to choose a shell, always choose the shell the smaller dimension of the shell is the direction in which the velocity is changing. So, since v_x is a function of y alone. So, therefore, my shell would be of any length L.

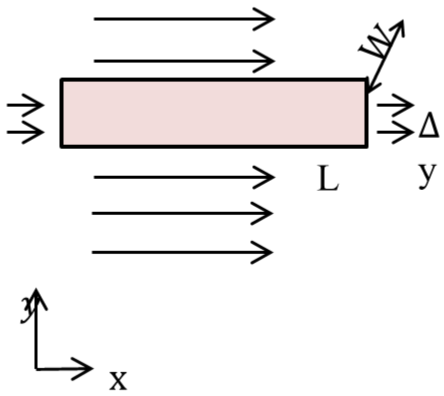
So, this could be the length L and its width could be W. However, the thickness of the imaginary shell across which we are going to make a balance of momentum will be the direction in which the velocity is changing with position. And also, this is a case of steady state flow. So, therefore, v_x is not a function of time. So, v_x is a function of y and therefore, that is the method by which one should assign the smaller dimension of the chosen imaginary shell of fluid across which the fluid is moving. And of course, if we write something like a Newton's second law of motion for an open system at steady state where the velocity is not a function of time, the rate of momentum in, the MM here refers to momentum.

Shell MM Balance:

At steady state: Rate of MM IN – Rate of MM OUT + Sum of all forces acting on it = 0

So, the rate of momentum in minus rate of momentum out plus sum of all forces acting on the shell would be equal to 0. Had it not been a steady state, then the fluid may be accelerating, and therefore, I will have additional term on the right-hand side. So, here we have just momentum in and out of the shell because of the flow. And we will recognize what are the mechanisms by which the fluid can move in and out of the shell and the mechanism with which the momentum gets transferred in the liquid contained in the imaginary shell. So, this is what the shell momentum balance is all about. Whenever we write the physics of the problem in terms of these equations, the momentum in, momentum out, plus forces, we need to first identify what are the inflow terms of momentum in such a system.

In here, you can see that we have some momentum coming to the control volume. The control volume has the dimension, smaller dimension as y, the length could be anything any length L and width could be the system width W. So, this is called the convective momentum. The convective momentum is because of flow. So, the mass of fluid which enters the control volume will carry some momentum with it because of its velocity. So, let us first see what is going to be the momentum in. First if you look at delta y times W, this is the area available for flow.

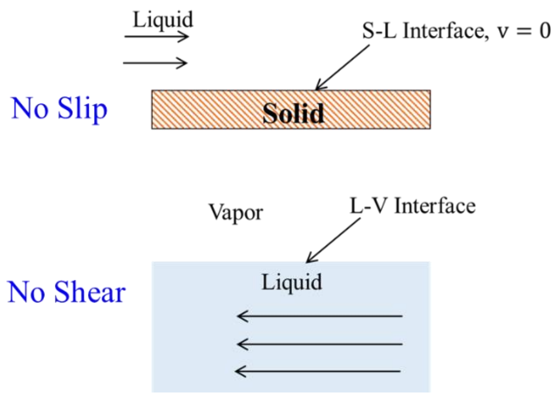


$$\text{IN: } \rho \Delta y W v_x|_{x=0}\ v_x|_{x=0}$$

So, this area is W times delta y. So, w times delta y being the area of flow you multiply that with velocity. So, this becomes the volumetric flow rate at x equals 0, with units of meter cube per second, and you multiply it with the density then this becomes the mass flow rate. So, this entire thing up to this point becomes the mass flow rate of liquid coming in to the control volume because of flow. Now, when you multiply that with velocity, this becomes the momentum in, the momentum into the control volume by convection at this point.

$$\text{OUT: } \rho \Delta y W v_x|_{x=L}\ v_x|_{x=L}$$

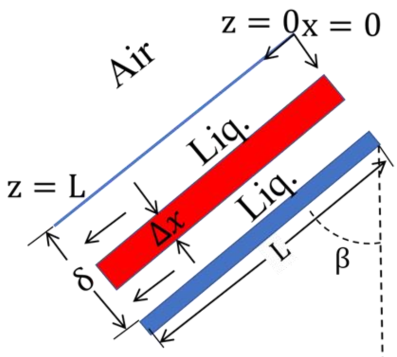
Now, in a similar fashion one can write what is the out one. Everything remains same except that instead of x equals 0, where x equals 0 is this point and x equal to L is over here. So, this is going to simply become the velocity at x equals L. So, these are the in and out term of momentum by convection and by convection, I mean when we have actual flow to the control volume. And any flow carries some momentum represented by its velocity. So, this momentum into the control volume and out of the control volume, these two expressions are I hope they are clear to you now. Any equation governing equation which comes out of this would require certain boundary conditions.



Relative velocity: at S-L interface, $v_{rel} = 0$

Shear at L-V interface = 0, $\tau = 0$

I have discussed the boundary conditions with you before, one is going to be the no slip condition in which at the solid liquid interface there is not going to be any velocity, v equals 0 at the solid liquid interface. So, the relative velocity at the solid liquid interface is going to be 0. Similarly, when we talk about the liquid vapor interface, as I mentioned in the previous class, because of the significant difference in viscosity of the liquid and the vapor, the liquid or the vapor cannot transfer any momentum to each other at the liquid vapor interface and the shear at the liquid vapor interface τ is therefore, going to be 0. So, these two are the major boundary conditions which we encounter in fluid mechanics. Now, we are going to write a shell momentum balance and try to solve a problem. The figure on the left side of your screen is essentially a liquid film of thickness delta falling along an inclined plane represented by the blue box which is at an angle beta with the vertical.



So, the liquid and air, we have a liquid air interface at this point and we have a solid liquid interface at this point and the motion of the liquid is in this direction. And we realize that the velocity in the z direction, that is this is the z direction, velocity in the z direction is going to be the only nonzero velocity in this system. There is no v_x there is no v_y . I only have v_z which is in this direction. So, this v_z we realize, it is of going to be a function of x only and therefore, our shell, assumed shell is going to have the smaller dimension as delta x as has been shown over here. And it could be of any length L and any width W that is not going to affect our calculation. And we will assume that it is a Newtonian fluid and v_z is the nonzero and v_z is not a function of time since it is a steady state. So, with this the volume of the shell we know that it is going to be L times W times delta x and we are going to write the momentum equation that is rate of momentum in minus rate of momentum out plus sum of all forces acting on the system at steady state must be equal to 0.

At Steady State/flow:

$$\text{Rate of MM IN} - \text{Rate of MM OUT} + \Sigma F = 0$$

So, we are going to write the individual terms for convection for conductive flow and then identify the forces. As we have discussed in the previous class, the previous slides, w times delta x is the area, multiply the area with the velocity you get the volumetric flow rate. The volumetric flow rate multiplied by the density would give you the mass flow rate, mass flow rate multiplied by the velocity would give you what is the rate of momentum which is coming in by convection to the control volume. In a similar fashion, the nonzero component z momentum, the z momentum out by convection would be exactly identical with the previous expression, except that velocities are now evaluated at z equals L. So, in is at z equals 0, out is going to be at z equals L and therefore, the velocities are to be evaluated at each of these points.

$$\text{Rate of z-momentum in by conv.: } \rho(W\Delta x v_z|_{z=0})v_z|_{z=0}$$

$$\text{Rate of z-momentum out by conv.: } \rho(W\Delta x v_z|_{z=L})v_z|_{z=L}$$

So, the convection part is taken care of. So, what is going to be the mass in by conduction that is essentially shear stress. As I have told you before that this τ has two subscripts. The first one is the direction in which you have the flow which obviously, is in this case the z direction. And because of a variation in velocity in the z direction, some momentum is going to get transported in a direction perpendicular to the flow. So, that direction is x.

So, τ_{xz} the significance of this is the z momentum getting transported in the x direction because of the presence of viscosity. This is the shear stress and it acts on an area which is, if this is my shell, it acts on this area. So, this being the shell with thickness being equal to delta x, the flow is over it, the momentum due to conduction is going to be exerted in the x direction. The area on which this act on must be equal to L times W. So, that is why I have written over here, that this being the area, this being the shear stress, this z is the direction is the momentum, and x is the direction in which the momentum is getting transported. So, this is nothing but shear stress, shear stress acts on the lateral top area. So, this is evaluated at x equals x that means, over here and it is going to go out of this through the thickness at x plus delta x.

$$\text{Rate of momentum in by cond.: } (LW)\tau_{xz}|_{x=x}$$

$$\text{Rate of momentum out by cond.: } (LW)\tau_{xz}|_{x=x+\Delta x}$$

So, the rate of momentum out by conduction, which is shear stress, it is also known as molecular transport of momentum. So, it is going to be, everything will remain the same, the area will remain the same, τ_{xz} is going to be evaluated at x plus delta x. So, that is the only difference that we have between the in and out by conduction. So, this is something which we have to understand is that the convection is on the area W times delta z, conduction is on the area W times L and the nomenclature of τ, I have already explained to you. The next one is going to the only force that you have in this is because of the inclination of the plate.

$$\text{Body Force: } (LW\Delta x)\rho g \cos\beta$$

So, the inclination of the plate and the film associated with it, which is flowing along it. So, of course, this being the volume and this being the density. So, this is mass and this is the component of the gravity and therefore, this is going to be simply, if you look at the figure, it is going to be g cos β. So, the body force acting on the fluid situated, located inside the control volume would simply be L W delta x ρg cos β. So, my starting equation which is similar to Newton's law of motion for an open system at steady state, the rate of momentum in minus out plus all forces acting on it would be equal to 0.

$$\text{Rate of momentum IN - Rate of momentum OUT} + \sum F = 0$$

This is a freely falling film. So, there is no imposed pressure gradient on the fluid. Therefore, we do not have any pressure force, which is a surface force acting on it. The only force acting on this is the gravity force. So, I write all these terms.

$$LW\tau_{xz}|_x - LW\tau_{xz}|_{x+\Delta x} + W\Delta x\rho v_z^2|_{z=0} - W\Delta x\rho v_z^2|_{z=L} + LW\Delta x\rho g\cos\beta = 0$$

So, this is my conductive or molecular transport of momentum in and out, convective or the momentum due to the actual flow in and out, and the force which is acting on the system. Now, you can see that I would get, I could get rid of this L times w I mean not L times w, w everywhere and after cancelling from both sides what we would get is, we also understand that v_z at the start of my discussion, I wrote that v_z is not a function of time, it is not a function of z. Therefore, v_z at z equals 0 must be equal to v_z at z equals L. So, the film at any x location is falling with a constant velocity, v_z is not a function of z. So, once you do that then you understand that this term and this term will cancel each other.

$$\text{As } v_z|_{z=0} = v_z|_{z=L} \Rightarrow \tau_{xz}|_{x+\Delta x} - \tau_{xz}|_x = \Delta x\rho g\cos\beta$$

So, therefore, the sorry this term and this one the two terms which you have now marked with an x they cancel each other. Since v_z at z equals L is equal to v_z at z equals 0. So, therefore, in this imaginary shell there is not going to be any contribution from convective transport of momentum, in is equal to out. So, I am left with three terms, the two terms for the two conductive or molecular transport of momentum and one which is for the body force. So, with these three my equation therefore, reduces through the one which I have shown over here. Then what I am going to do is I am going to divide this side by delta x and take the limit as x tends to 0.

$$\lim_{\Delta x\to 0}\frac{\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_x}{\Delta x} = \rho g\cos\beta$$

Once I do that I am essentially using the definition of the first derivative and the definition of the first derivative leads us to the governing equation for the flow and momentum transfer for a film which is falling along a solid plate at steady state with known thickness delta, and realizing that only velocity, non-zero velocity component being v_z and this v_z is not a function of time, it is not a function of z, it is not a function of y, it is only a function of x, which is called one dimensional flow. So, for one dimensional steady state flow, the governing equation for this is what I have shown in the bracket over here where τ_{xz} is the shear stress and the z and x the two subscripts I have already explained, z is the direction of flow and x is the direction in which the momentum is getting transported because of viscosity. So, with this governing equation I can integrate it once. Once I have the governing equation, then I can obviously integrate it once C_1 being the constant of integration. So, this constant of integration will have to be evaluated using physical boundary conditions. So, we identify what are the physical boundary condition that can be applied in this case.

$$\text{Governing Equation: } \frac{d(\tau_{xz})}{dx} = \rho g \cos\beta$$

$$\tau_{xz} = \rho g \cos\beta \, x + C_1$$

As I have mentioned before, there is no shear at the liquid air interface due to the marked significant difference in viscosity, that is a very common acceptable practice. So, with this, we know that at x equals 0, τ is 0. So, this leads to C_1 equals 0.

$$\text{No shear at L-V interface: } x = 0, \tau_{xz} = 0 \Rightarrow C_1 = 0$$

So, I have successfully evaluated the integration constant using a physical boundary condition which we knew before. The second boundary condition then I am going to put the expression for Newtonian fluid in terms of τ.

$$\tau_{xz} = \rho g \cos\beta \, x$$

$$\text{or, } -\mu\frac{dv_z}{dx} = \rho g \cos\beta \, x \text{ ; (Newtonian Fluid)}$$

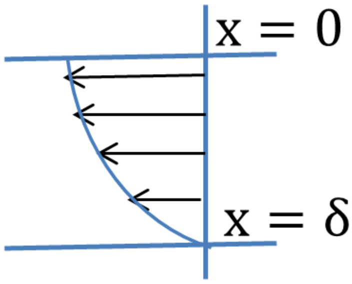
So, this τ is essentially expressed in the form of a velocity gradient and viscosity with a minus sign and you know what is the significance of the minus sign is that the momentum always is getting transported from higher velocity to lower velocity from a layer moving with a higher velocity to a layer moving with this lower velocity. That is why we have this minus term. Integrate it once again and you get this C_2 constant of integration and of course, this C_2 is to be evaluated using the second boundary condition at the liquid solid interface, which essentially tells us that at the liquid solid interface we will invoke no slip boundary condition, that is at x equals delta, that is on the solid plate over here, the velocity is going to 0. This would let us evaluate the expression for C_2.

$$\text{or, } v_z = \frac{-\rho g \cos\beta}{\mu} \frac{x^2}{2} + C_2$$

$$\text{No Slip Boundary Condition: } x = \delta, v_z = 0 \Rightarrow C_2$$

Once you substitute that C_2 back into the governing equation what you get is the expression for velocity of a falling film along an inclined plate in terms of the properties. The properties are the ρ, the density, μ, the viscosity, the component the gravitational constant and the angle of inclination, the thickness of the film, but most importantly is it tells us how v_z varies with x and this you could clearly identify that this is going to give rise to a parabolic distribution of velocity.

$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu} \left\{1 - \left(\frac{x}{\delta}\right)^2\right\}$$



So, this parabolic distribution it would look something like this. So, when you, on the solid plate the velocity is going to be 0 over here at this point the profile would approach the interface with a 0 slope. So, it would look, the distribution will look something like this. So, that is the first, our first exercise of converting a physical problem into the basic equation and from the difference equation dividing both sides by the smaller dimension that is the thickness of the control volume and in the limit when delta x in this case approaches 0, I can use the definition of the first derivative and get our governing equation which would be subsequently integrated and with appropriate boundary conditions of no shear and no slip to get a concise expression, closed from expression of velocity distribution in a falling film.

$$v_{z,Max}\Big|_{x=0} = \frac{\rho g \delta^2 \cos\beta}{2\mu}$$

So, well you can also see that the maximum value of velocity in the falling film is going to be at x equals 0, at this point. So, when you set x equals 0 what you would get is this is the expression for the maximum velocity at the top layer. So, once you have the maximum velocity and the point velocity, engineering application would require that you get what is the average velocity because it is not the point velocity which would be relevant in engineering applications you would like to know how much of fluid is flowing rather than what is the velocity of the fluid at any given point. Both are important, one from an application point of view, the other is from an understanding point of view. So, the average velocity, the definition is that you integrate the velocity v_z which we understand is a function of x over the entire area, through which the flow is taking place. So, this is essentially the flow area and the flow area would consist of w times delta when you integrate them.

$$\text{Average Velocity: } \langle v_z \rangle = \int_0^W \int_0^\delta \frac{v_z dxdy}{W\delta}$$

$$\langle v_z \rangle = \frac{\rho g \delta^2 \cos\beta}{3\mu}$$

So, this is your delta x and this is your width w. So, if we call it as w. So, its integration of the velocity which is a function of x over the cross-sectional area across which the flow is taking place. So, once you plug in the expression of velocity in here and perform the integration, this is what you are going to get as the average velocity for such a system. So, for a falling film along a solid, the average velocity, the expression for average velocity and the expression for point velocity can be obtained by a simple shell balance of momentum.

So, a shell balance of momentum identifying that, I am going to choose the shell as the smaller dimension of the shell, I am going to choose the smaller dimension of the shell as the one in which the velocity is changing. In this specific application if you look at the figure once again the velocity is changing with x, not with y or with z. So, the smaller dimension of my imaginary shell is going to be delta x. This is extremely important. So, I am saying it time and again to make sure that you are you are completely familiar with your choice of the shell. The length L or the width w is immaterial because v_z is not a function of z it is not a function of y.

So, whatever length dimensions that you choose for x and for z and for y is unimportant, but your choice of delta x is crucial that is going to be the smaller dimension in the limit you would like that delta x to approach 0 such that you can write the differential equation which is going to be the governing equation. And once you have the average velocity then you could find out what is the volumetric flow rate, what is going to be the flow per unit time the meter cube per unit time. Since it is volumetric flow rate, it is going to be meter cube per time which would simply be the average velocity multiplied by the flow area. The average velocity expression is known to you and the flow area is the area which is perpendicular to the flow direction. So, the area perpendicular to the flow direction is w times delta therefore, your volumetric flow rate would simply be equals v_z times w times delta.

$$\text{Volumetric flow rate, } Q = \langle v_z \rangle . W\delta$$

So, with this, these are the three major results of this class. The expression for point velocity, the maximum velocity and the velocity in terms of v_z max. I have discussed them extensively. The average velocity, the flow rate these I have this.

$$v_z = \frac{\rho g \delta^2 \cos\beta}{2\mu} \left\{1 - \left(\frac{x}{\delta}\right)^2\right\}$$

$$v_{z,Max} = \frac{\rho g \delta^2 \cos\beta}{2\mu}$$

$$v_z = v_{z,Max} \left\{1 - \left(\frac{x}{\delta}\right)^2\right\}$$

Average velocity based on flow area:

$$\langle v_z \rangle = \frac{\int_0^W \int_0^\delta v_z \, dx \, dy}{\int_0^W \int_0^\delta dx \, dy}$$

$$\langle v_z \rangle = \frac{\rho g \cos\beta}{3\mu}$$

$$Q = W\delta \langle v_z \rangle = \frac{W\delta \rho g \cos\beta}{3\mu}$$

So, this is essentially the summary of what we have covered in this class and the last thing that remains is what is going to be the force exerted by the fluid on the plate. So, the force in this direction is first of all we identify that this is going to be the shear force, shear stress exerted by the moving fluid on the plate. So, the shear stress is simply minus μ times velocity gradient dv_z dx evaluated at x equals delta.

$$F_z = \int_0^L \int_0^W \left(-\mu \frac{dv_z}{dx}\right)\Bigg|_{x=\delta} dydz$$

$$F_z = (\rho g \delta \cos\beta)LW$$

So, if you see look at the geometry, the shear stress is acting at the liquid solid interface that means, at x equals delta and the area and the length scales the area in this case is going to be dy, where y is this direction and dz, where dz varies from 0 to L, whereas, y varies from 0 to w. So, once you have the expression for v_z available to you when you put that over here dv_z dx over here perform the integration what you get is the expression for the force and that would contain the property ρ it is going to contain g $\cos\beta$ L and w. So, this is what we have covered in today's class. How to choose a shell? Write the convective and conductive momentum flow in and out of the shell, identify the forces body or surface. Major body force is gravity, possible surface forces could be a pressure difference and imposed externally imposed pressure difference which in this specific case we did not have.

Write the difference equation, divide both sides, cancel out the convective term if possible if v_z is not a function of z which was in this case. So, I am going to be left with only the conductive term and the force term, in this case only the gravity term divide both sides by delta x take in the limit when delta x approaches 0 that is my definition of the first derivative differential equation, get the integration constants using the appropriate boundary condition and the rest follows after that. So, that is more or less of what I wanted to cover and I hope it is clear to you and in the next class we will see applications of this shell momentum balance to other geometries to other applications which we commonly encounter for flow of fluid through conduits. Thank you.